Geometric actions for gravity

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In all these cases effective field theories provide the most convenient method for working out the consequences of symmetries and the general principles underlying quantum field theory.

Weinberg QFT II

- Suppose G group of symmetries known but not necessarily fundamental theory
- construct model that can be quantized & has G as global symmetry group

(Noether charges, current algebras)

$$S[g] = -\int d^4x \, \mathrm{Tr}\left[(\partial_\mu g\, g^{-1})(\partial^\mu g\, g^{-1})\right]$$ (+ Poincaré invariance)

$$S[g] = \int dt \, \mathrm{Tr}\left[(\dot g\, g^{-1})(\dot g\, g^{-1})\right]$$ (particle action)

· global right inv. $g \to g h_R$, $K = d(g \cancel{h_R})\, \cancel{h_R^{-1}}\, g^{-1}$ RI Maurer-Cartan form

Exercise : express Lie group G & algebra $\mathfrak{g}$ theory in local coordinates

g^i "Euler angles" arbitrary $\qquad e_\alpha$ basis of $\mathfrak{g}$

generators of right / left translations = left / right invariant vector fields

$$g\, \overbrace{\frac{d}{dt} h_R(t)\Big|_{t=0}}^{X} \Big/ \overbrace{\frac{d}{dt} h_L(t)\Big|_{t=0}}^{X}\, g \qquad \vec{L}_\alpha = L_\alpha{}^i(g)\, \frac{\partial}{\partial g^i} \Big/ \vec{R}_\alpha = R_\alpha{}^i(g)\, \frac{\partial}{\partial g^i}$$

$$[\vec{L}_\alpha , \vec{L}_\beta] = f^\gamma_{\alpha\beta}\, \vec{L}_\gamma \ , \qquad [\vec{R}_\alpha , \vec{R}_\beta] = - f^\gamma_{\alpha\beta}\, \vec{R}_\gamma \ , \qquad [\vec{L}_\alpha , \vec{R}_\beta] = 0$$

" frames $e_\alpha{}^\mu \partial_\mu$, structure functions $\pm f^\gamma_{\alpha\beta}$ "

left / right invariant MC forms $\quad \theta = g^{-1} dg \;/\; K = dg\, g^{-1}$

$$\theta = e_\alpha L^\alpha{}_i\, dg^i \quad / \quad K = e_\alpha R^\alpha{}_i\, dg^i$$

$L^\alpha{}_i L_\beta{}^i = \delta_\alpha^\beta = R^\alpha{}_i R_\beta{}^i$

$L^\alpha{}_i L_\alpha{}^j = \delta_i^j = R^\alpha{}_i R_\alpha{}^j$

$$d\theta + \tfrac{1}{2}[\theta,\theta] = 0 \quad / \quad dK - \tfrac{1}{2}[K,K] = 0$$

" coframes $e^a{}_\mu dx^\mu$ "

Adjoint representation $\quad \mathrm{Ad}_g e_\alpha = g\, e_\alpha\, g^{-1} = e_\beta R^\beta{}_i L_\alpha{}^i$

$$\boxed{S[g^i] = \int dt\ \tfrac{1}{2}\ g_{ij}\ \dot{g}^i \dot{g}^j}$$

$g_{\alpha\beta}$: Killing metric $\quad g_{ij}(g) = g_{\alpha\beta} R^\alpha{}_i R^\beta{}_j$

geodesic flow on G

$$\ddot{g}^i + \tfrac{1}{2}\Gamma^i{}_{jk}\, \dot{g}^j \dot{g}^k = 0$$

Euler-Arnold equation

Global sym & Noether charges

$$\delta_X g^i = L_\alpha{}^i X^\alpha = \xi^i_X \quad \text{Kof of } g_{ij}$$

$$Q_X = g_{ij}\, \xi^i_X\, \dot{g}^j$$

Theorem (Arnold) geodesic flow on G $\Longleftrightarrow$ $\boxed{\dot{\pi} = -\mathrm{ad}^*_{g^{-1}\pi}\,\pi}$, $\pi \in \mathcal{G}^*$

Proof = Hamiltonian analysis $\{g^i, p_j\} = \delta^i_j$, $\{g^i, g^j\} = 0 = \{p_i, p_j\}$

$$p_j = \frac{\partial L}{\partial \dot{g}^j} = g_{\alpha\beta} R^\alpha_i R^\beta_j \dot{g}^i$$

$$\Longleftrightarrow \boxed{\begin{array}{c} R_\beta{}^j p_j \\ \shortparallel \\ \pi_\beta \end{array}} = g_{\alpha\beta} R^\alpha_i \dot{g}^i \qquad \Longleftrightarrow \qquad \dot{g}^i = R_\alpha{}^i g^{\alpha\beta} \pi_\beta$$

Non-Darboux coordinates $\boxed{\{\pi_\alpha, \pi_\beta\} = f^\gamma_{\alpha\beta}\,\pi_\gamma}$ KKS bracket

$$\{g^i, g^j\} = 0 \qquad \{g^i, \pi_\alpha\} = R_\alpha{}^i \qquad \pi_\alpha e^{*\alpha} \in \mathcal{G}^*$$

$$S_H = \int dt \, [\, \pi_\alpha R^\alpha{}_i \, \dot{g}^i - H \,] \qquad , \qquad H = \frac{1}{2} \pi_\alpha g^{\alpha\beta} \pi_\beta$$

$$\boxed{\dot{\pi}_\alpha = \{\pi_\alpha, H\} = f^\gamma{}_{\alpha\beta} \, \pi_\gamma \, g^{\beta\delta} \, \pi_\delta}$$

$$\dot{g}^i = \{g^i, H\} = R_\alpha{}^i \, g^{\alpha\beta} \, \pi_\beta \quad \Longleftrightarrow$$ definition of momentum

Noether charges $\quad Q^{\bar{\pi}}_X = \langle \pi, \mathrm{Ad}_g X \rangle$

Remarks: 1) More general Hamiltonians "inertia operators"

$A : \mathfrak{g} \to \mathfrak{g}^*$ invertible $\qquad H = \frac{1}{2} \pi_\alpha (A^{-1})^{\alpha\beta} \pi_\beta$

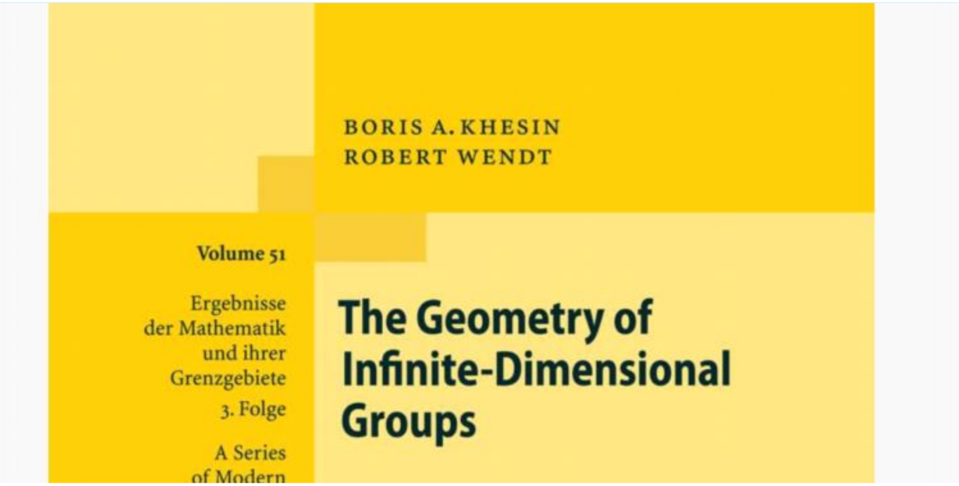
$X^\beta \mapsto A_{\alpha\beta} X^\beta$

$SO(3)$: Euler top

34 I. Preliminaries

Group	*Metric*	*Equation*
SO(3)	$\langle\omega, A\omega\rangle$	Euler top
$SO(3) \ltimes \mathbb{R}^3$	quadratic forms	Kirchhoff equation for a body in a fluid
SO(n)	Manakov's metrics	n-dimensional top
Diff(S^1)	L^2	Hopf (or, inviscid Burgers) equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa–Holm equation
Virasoro	$\dot{H}^1$	Hunter–Saxton (or Dym) equation
SDiff(M)	L^2	Euler ideal fluid
SDiff(M)	H^1	averaged Euler flow
$\mathrm{SDiff}(M) \ltimes \mathrm{SVect}(M)$	$L^2 + L^2$	Magnetohydrodynamics
Maps(S^1, SO(3))	H^{-1}	Heisenberg magnetic chain

Table 4.1: Euler equations related to various Lie groups.



2) Bi-Hamiltonian integrable systems

$$\{\pi_\alpha, \pi_\beta\}_0 = f_{\alpha\beta}^{\gamma}\, b_\gamma$$ 2nd compatible Poisson bracket

$b_\gamma \cancel{(x)} \in \mathfrak{g}^*$ fixed

Digression: Scalar sector of $\mathcal{N}=4$, $d=4$ SUGRA

$SL(2,\mathbb{R})/SO(2)$ symmetric space

$sl(2,\mathbb{R})$: $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$g = e^{\psi(E_+ - E_-)} \underbrace{e^{\phi/2\, H}\, e^{\chi E_+}}_{g_B} \qquad g^i = (\psi, \phi, \chi), \quad g_B^i = (\phi, \chi)$$

Borel gauge

$$P_\mu = \frac{1}{2}\left[(\partial_\mu g\, g^{-1}) + (\partial_\mu g\, g^{-1})^T \right]$$

$$\underset{\substack{\uparrow \\ SO(2) \\ \text{local}}}{g \to k(x)}\; g\; \underset{\substack{\uparrow \\ SL(2,\mathbb{R}) \\ \text{global}}}{h} \quad \Rightarrow \quad P_\mu \to k\, P_\mu\, k^{-1}$$

$$S[g_B] = -\int d^4x \,\mathrm{Tr}\, P_\mu P^\mu = -\frac{1}{2}\int d^4x \left[\partial_\mu\phi\,\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\,\partial^\mu\chi\right]$$

$SL(2,\mathbb{R})/SO(2)$

axion - dilaton

$$\delta_{E_+} g_B = g_B E_+ \quad \Leftrightarrow \quad \delta_{E_+}\chi = 1, \quad \delta_{E_+}\phi = 0$$

$$\delta_H g_B = g_B H \quad \Leftrightarrow \quad \delta_H \phi = 2, \quad \delta_H \chi = -2\chi$$

$$\delta_{\mathfrak{so},E_-} g_B = \delta k\, g_B + g_B E_- \quad \Leftrightarrow \quad \delta_{\mathfrak{so},E_-}\phi = 2\chi, \quad \delta_{\mathfrak{so},E_-}\chi = e^{-2\phi} - \chi^2$$

compensating gauge transformations

"hidden symmetries", non-linear realizations

Geometric actions

$\boxed{\text{No Killing metric ?}}$

- use fixed coadjoint vector $b_\alpha \in \mathfrak{g}^*$ to built first order action

$$S_b[g; b, Y] = \int dt \left[\langle b, \frac{dg}{dt} g^{-1} \rangle - \langle b, \mathrm{Ad}_g Y \rangle \right]$$

$$= \int dt \left[b_\alpha R^\alpha{}_i \, \dot{g}^i - b_\alpha R^\alpha{}_i L_\beta{}^i Y^\beta \right]$$

$$\begin{cases} \delta_X g = g X \\ \delta_X g^i = L_\alpha{}^i X^\alpha \end{cases} \Longleftrightarrow \frac{dX}{dt} = [X, Y]$$

cf. boosts

- first order presymplectic kinetic term $\begin{cases} a = \langle b, K \rangle & \text{1-form} \\ \sigma = da = \langle b, \frac{1}{2}[K, K] \rangle & \text{2-form} \end{cases}$

- no path integral quantization because of gauge invariance

$\delta_{\epsilon(t)}\, g = \epsilon(t)\, g$ little algebra $\mathcal{H}_{b_0}$, $ad^*_{\epsilon(t)}\, b = 0$

- are these all gauge transformations?

- How many models $S_G[g; b, y]$ to study?

$$\begin{cases} b' = Ad^*_{h^{-1}}\, b \\ y' = Ad_h\, y \end{cases} \qquad S_G[g; b', y'] = S_G[g'; b, y]$$

$g' = h g k$ field redefinition $\Rightarrow$ QM equivalent

- only 1 representative needed per partition of $\mathfrak{g}^*, \mathfrak{g}$ into $\boxed{\text{coadjoint orbits, conjugacy classes}}$

Classification for Virasoro group:

Lazutkin & Pankratova, Kirillov, Witten, Balog et al.

Constrained Hamiltonian analysis (purely algebraic)

$\{g^i, p_j\} = \delta^i_j$, $\quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{g}^i} = b_\alpha R^\alpha{}_i$ primary constraint

$p_i \leftrightarrow \pi_\alpha = R_\alpha{}^i p_i \quad \boxed{\phi^b_\alpha = \pi_\alpha - b_\alpha \approx 0}$

$S^\#_G[g,\pi,\omega; b, Y] = \int dt \left[\pi_\alpha R^\alpha{}_i \dot{g}^i - H_Y - \omega^\alpha \phi^b_\alpha \right], \quad H_Y \approx \langle \pi, \mathrm{Ad}_g Y \rangle = \pi_\alpha R^\alpha{}_i L_\beta{}^i Y^\beta$

linear in π

Dynamics: $\quad \dot{g}^i = \{g^i, H_Y + \omega^\alpha \phi^b_\alpha\} = L_\alpha{}^i Y^\alpha + R_\alpha{}^i \omega^\alpha$

no 2nd order Lagrangian

$\dot{\pi}_\alpha = \{\pi_\alpha, H_Y + \omega^\alpha \phi^b_\alpha\} = \pi_\gamma f^\gamma{}_{\alpha\beta} \omega^\beta$

Noether charges: $\quad Q^\pi_X = \langle \pi, \mathrm{Ad}_g X \rangle, \quad \{Q^\pi_{X_1}, Q^\pi_{X_2}\} = Q^\pi_{[X_1, X_2]}$

Secondary constraints ?

$$\dot\phi^b_\alpha \approx 0 \Leftrightarrow \boxed{\underbrace{b_\gamma f^\gamma_{\alpha\beta}}_{C_{\alpha\beta}} u^\beta = 0} \quad (*)$$ No, only restrictions on Lagrange multipliers

complete set of null eigenvectors $e_a{}^\beta$ of $C_{\alpha\beta}$: $u^\beta = e_a{}^\alpha u^a$

arbitrary

adapted basis: $e_a{}^\beta,\ e_A{}^\beta,\ e^a{}_\beta,\ e^A{}_\beta$

$$e_a{}^\beta e^b{}_\beta = \delta_a^b, \quad e_A{}^\beta e^b{}_\beta = 0, \quad e_A{}^\beta e^B{}_\beta = \delta_A^B, \quad e_a{}^\beta e^a{}_\gamma + e_A{}^\beta e^A{}_\gamma = \delta^\beta_\gamma$$ orthonormality / completeness

$$f^c_{ab} = 0, \quad C_{ab} = 0 = C_{aB}, \quad C_{AB} = b_c f^c_{AB} + b_C f^C_{AB}$$ invertible

subalgebra

$$(C^{-1})^{AB} C_{BC} = \delta^A_C \qquad (*) \Leftrightarrow u^A = 0, \quad u^a$$ arbitrary

$\phi^b{}_a \approx 0$ first class $\qquad \phi^b{}_A \not\approx 0$ second class

solve 2nd class constraints & work with Dirac brackets

$$S_G[g^i, \pi_a, \pi_A, \omega^b, \omega^B; b, \gamma]$$

(π_A, ω^B) : auxiliary fields $\rightarrow$ solve in the action $\qquad \pi_A = b_A, \quad \omega^B = 0$

$$\boxed{S_G^R[g^i; \pi_a, \omega^b; b, \gamma] = \int dt \left[a^R_i \dot{g}^i - H^R_\gamma - \omega^a \phi^b{}_a\right]}$$

$$a^R = (\pi_a R^a{}_i + b_A R^A{}_i)\, dg^i, \qquad \sigma^R = d a^R$$

Dirac brackets :

$$
\begin{array}{c|cc}
 & g^i & \pi_b \\
g^i & & \\
\pi_a & &
\end{array}
\left(\begin{array}{cc}
C_{AB} R^A{}_i R^B{}_j & -R^b{}_i \\
R^a{}_j & 0
\end{array}\right)
\qquad
\begin{array}{c|cc}
 & g^k & \pi_c \\
g^i & & \\
\pi_b & &
\end{array}
\left(\begin{array}{cc}
R_C{}^j (C^{-1})^{CD} R_D{}^k & R_c{}^j \\
-R_b{}^k & 0
\end{array}\right)
=
\left(\begin{array}{cc}
\delta_i^k & 0 \\
0 & \delta_c^a
\end{array}\right)
$$

$$\sigma^R = \tfrac{1}{2}\sigma^R{}_{ij}\, dg^i dg^j + \sigma^R{}_i{}^b\, dg^i d\pi^b \qquad \{g^j, g^k\}^* = R_C{}^j (C^{-1})^{CD} R_D{}^k, \quad \{g^j, \pi_c\} = R_c{}^j$$

$$\{\pi_b, \pi_c\}^* = f^d{}_{bc}\,\pi_d = 0$$

- $\widehat{\mathrm{Diff}(S^1)}$ typical little groups $U(1)$, $SL(2,\mathbb{R})$ $\Rightarrow$ at most 3 π_a's

g^i : ∞ dimensional, at most 3 gauge invariances

Unconstrained model : $\boxed{\text{drop all constraints} \quad \phi^b_\alpha \approx 0}$

$$S^U_G[g^i, \pi_\alpha; y] = \int dt \left[\pi_\alpha R^\alpha{}_i \, \dot{g}^i - H_y\right]$$

$$\dot{g}^i = \{g^i, H_y\} = L_\alpha^i \, y^\alpha \iff \frac{dg}{dt} g^{-1} = \mathrm{Ad}_g \, y$$

$$\dot{\pi}_\alpha = \{\pi_\alpha, H_y\} = 0$$

$\boxed{\text{contains all orbits}}$

conserved charges : $\boxed{Q^\pi_x \, , \quad \pi_\alpha}$

level sets $\pi_\alpha = b_\alpha$ Hamiltonian reduction $\longrightarrow$ do previous analysis

Gravity models : • AdS_3 gravity group : $\widehat{Diff(S^1)} \otimes \widehat{Diff(S^1)}$

general solution with Brown-Henneaux (FG) type boundary conditions

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - \left(r\,dx^+ - \frac{8\pi G\ell}{r}\,b^-\,dx^-\right)\left(r\,dx^- - \frac{8\pi G\ell}{r}\,b^+\,dx^+\right)$$

$x^\pm = \frac{t}{\ell} \pm \varphi$, $\boxed{b^\pm(x^\pm + 2\pi) = b^\pm(x^\pm) \text{ arbitrary periodic functions}}$

conformal transformations $x^\pm \to f^\pm(x^\pm)$, $f(x^\pm + 2\pi) = f^\pm(x^\pm) + 2\pi$

residual diffeomorphisms

$$\tilde{b}^\pm = Ad^*_{f^{-1}}\,b^\pm = (\partial_\pm f^\pm)^2\,b^\pm \circ f^\pm - c^\pm S_{x^\pm}[f^\pm]$$

$$c^\pm = \frac{3\ell}{2G} \qquad S_x[f] = \frac{1}{24\pi}\left[\partial_x^2(\ln \partial_x f) - \frac{1}{2}(\partial_x \ln \partial_x f)^2\right]$$ Schwarzian derivative

$G = \mathrm{Diff}(S^1)$: $f(\varphi)$, $f(\varphi+2\pi) = f(\varphi)+2\pi$, $f'(\varphi)>0$,

$(f\,h)(\varphi) = (f\circ h)(\varphi)$, $e(\varphi)=\varphi$

$\mathfrak{g}$: $\mathrm{diff}(S^1)$ $\xi(\varphi)\frac{\partial}{\partial\varphi}$, bracket : $-[\xi,\eta]$

LI MC $\Theta = \frac{1}{f'(\varphi)}\,\delta f(\varphi)\,\frac{\partial}{\partial\varphi}$ / RI MC $K = (\delta f\circ f^{-1})(\varphi)\,\frac{\partial}{\partial\varphi}$

Adjoint : $\mathrm{Ad}_f\,\xi = \frac{1}{(f^{-1}(\varphi))'}\,\xi(f^{-1}(\varphi))$, $-\mathrm{ad}_\eta\,\xi = \eta\,\xi' \underset{=}{-} \eta'\xi$

$\mathfrak{g}^*$ $\mathrm{diff}(S^1)^*$ $b(\varphi)\,d\varphi^2$ quadratic differentials $\langle b,\xi\rangle = \int_0^{2\pi} b(\varphi)\,\cancel{d\varphi^2}\,\xi(\varphi)\,\cancel{\frac{\partial}{\partial\varphi}}$

Coadjoint : $\mathrm{Ad}_f\,b = (f^{-1}(\varphi))'^{\,2}\,b(f^{-1}(\varphi))$, $-\mathrm{ad}_\eta\,b = \eta\,b' + \underline{\underline{2}}\,\eta' b$

centrally extended group $\hat{G} = G \times \mathbb{R}$

$$(g_1, m_1) \cdot (g_2, m_2) = \left(g_1 \cdot g_2\,,\; m_1 + m_2 + \boxed{}(g_1, g_2)\right)$$

group 2-cocycle

$$\mathrm{Ad}_{(g,m)}(X, n) = \left(\mathrm{Ad}_g X,\; n - \langle S(g), X\rangle\right)$$

Souriau cocycle
$S(g) : G \to \mathfrak{g}^*$

$$\mathrm{ad}_{(X,m)}(Y, k) = \left([X, Y],\; \omega(X, Y)\right)$$

Lie algebra 2-cocycle

$$\mathrm{ad}^*_{(X,m)}(b, c) = \left(\mathrm{ad}^*_X b + c\, s(X),\; 0\right)$$

infinitesimal Souriau cocycle

MC : (θ, θ_π) , (K, K_π)

$$S_{\hat{G}}[(g,\mu);(b,c)] = I_G[g;b] + c\int[-\langle S(g),\theta\rangle + \Theta_\pi]$$

Virasoro group $\widehat{\mathrm{Diff}}(S_1)$, $\quad \Xi(f_1,f_2) = -\frac{1}{48\pi}\int_0^{2\pi} d\varphi\, \ln(\partial_\varphi f_1\circ f_2)\,\partial_\varphi(\ln \partial_\varphi f_2)$

Bott-Thurston

$$S_{\widehat{\mathrm{Diff}}(S_1)}[f;b,c] = \int dt\,d\varphi\,\left[b(f)\,f'\dot{f} + \frac{c}{48\pi}\frac{\dot{f}''}{f'}\right]$$

Hamiltonian $Y = \partial_\varphi$ in order to reproduce Gibbons-Hawking term

CS $\longrightarrow$ cWZW $\longrightarrow$ Liouville (chiral bosons) reduction

$$\boxed{S_{\widehat{\mathrm{Diff}}(S_1)}[f;b,c,Y] = 2\int dt\,d\varphi\left[b(f)\,f'\partial_- f + \frac{c}{48\pi}(\ln f')'(\partial_- \ln f')\right]}$$

Coadjoint representation & semi-direct product groups

adjoint $\quad \mathfrak{g} : [e_a, e_b] = f^c{}_{ab}\, e_c \qquad (\mathrm{ad}\, e_a)^b{}_c = f^b{}_{ac} \iff \mathrm{ad}\, e_a(e_b) = f^c_{ab}\, e_c$

coadjoint $\quad \mathfrak{g}^* : \langle e^b_*, e_a \rangle = \delta^b_a \qquad (\mathrm{ad}^* e_a) = -(\mathrm{ad}\, e_a)^T \iff \mathrm{ad}^* e_a(e^b_*) = -f^b{}_{ac}\, e^c_*$

group $\quad \mathrm{Ad}_g\, e_a = g\, e_a\, g^{-1}, \qquad \mathrm{Ad}^*_g = g\, e^b_*\, g^{-1}$

semi-direct product $\quad \boxed{G \ltimes_\sigma A \;:\; (f,\alpha)\cdot(g,\beta) = (f\cdot g,\ \alpha + \sigma_f(\beta))} \qquad \sigma$: representation

$ISO(3)$, $ISO(3,1)$, $\qquad A$: abelian ideal

BMS_3, BMS_4 ...

$$\mathfrak{g} \oplus_\Sigma A \;:\; [(X,\alpha),(Y,\beta)] = ([X,Y],\ \Sigma_X \beta - \Sigma_Y \alpha)$$

$$\mathrm{Ad}_{(f,\alpha)}(X,\beta) = (\mathrm{Ad}_f X,\ \sigma_f \beta - \Sigma_{\mathrm{Ad}_f X}\, \alpha)$$

$$\mathrm{ad}_{(X,\alpha)}(Y,\beta) = ([X,Y],\ \Sigma_X \beta - \Sigma_Y \alpha)$$

dual space $\quad \mathfrak{g}^* \oplus A^* \qquad \langle (j,p), (X,\alpha) \rangle = \langle j, X \rangle + \langle p, \alpha \rangle$

terminology $\quad j$: angular momentum $\qquad p$: linear momentum $\qquad$ BMS: add "super"

X : inf. rotation $\qquad \alpha$: inf. translation

ingredients $\quad \boxed{\times : A \oplus A^* \to \mathfrak{g}^* \quad : \quad \langle \alpha \times p, X \rangle = \langle p, \Sigma_X \alpha \rangle}$

change in angular momentum due to a translation

$$\sigma^* : G \times A^* \to A^* \quad : \quad \langle \sigma^*_f\, p, \alpha \rangle = \langle p, \sigma_{f^{-1}}\, \alpha \rangle$$

coadjoint representation

$$\mathrm{Ad}^*_{(f,\alpha)}(j,p) = \left(\mathrm{Ad}^*_f\, j + \alpha \times \sigma^*_f\, p,\ \sigma^*_f\, p\right)$$

$$\mathrm{ad}^*_{(X,\alpha)}(j,p) = \left(\mathrm{ad}^*_X\, j + \alpha \times p,\ \Sigma^*_X\, p\right)$$

Geometric action for semi-direct product groups

$$S = \int \Big[\underbrace{\langle \pi, K \rangle}_{\langle j - \alpha \times p,\, K_g \rangle + \langle p,\, d\alpha \rangle} - d\omega \, \langle \pi, \mathrm{Ad}_g Y \rangle$$

$$\langle \pi, \mathrm{Ad}_g Y \rangle = \langle j, \mathrm{Ad}_g Y \rangle + \langle p, \nabla_g \beta - \Sigma_{\mathrm{Ad}_g Y}\, \alpha \rangle$$

- $\widehat{BMS}_3 = \widehat{\mathrm{Diff}}(S_1) \ltimes \widehat{\mathrm{Vect}}(S_1)_{ab}$

$$S_{\widehat{BMS}_3}[f, \alpha; p, j, c_1, c_2] = S_{\widehat{\mathrm{Diff}}(S_1)}[f; p, c_1]$$

$$+ \int d\varphi\, dt \left[\dot{f} f' \left(p'\alpha + 2p\alpha' - \frac{c_2}{24\pi}\alpha''' \right) \circ f + p \frac{d\alpha}{dt} \right]$$

Hamiltonian $\quad Y = (0, \partial_\varphi) \qquad H = \int_0^{2\pi} d\varphi \left[f'^2 p(f) + \frac{c_2}{48\pi} \frac{f''^2}{f'} \right]$

Asymptotically flat metrics with Bondi-Sachs type boundary conditions

$$ds^2 = 2\left[\, 8\pi G\, p\, du - dr + 8\pi G\,(j + u p')\, d\varphi \right] du + r^2 d\varphi^2$$

$p = p(\varphi)$, $j = j(\varphi)$

finite BMS_3 transf.
$$\begin{cases} \tilde{p} = (f')^2\, p \circ f - c_2\, S_\varphi[f] \\ \tilde{j} = (f')^2 \left[j + \alpha p' + 2\alpha' p - \frac{c_2}{24\pi}\, \alpha''' \right] \circ f - c_1\, S_\varphi[f] \end{cases}$$

same actions as previously constructed from

$CS_{ISO(2,1)} \longrightarrow cWZW \longrightarrow BMS_3$ inv chiral boson action

G. Barnich, H. A. Gonzalez, and P. Salgado-Rebolledo, "Geometric actions for three-dimensional gravity," *Class. Quant. Grav.*, vol. 35, no. 1, p. 14003, 2018, doi: 10.1088/1361-6382/aa9806.

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4d gravity at $\mathcal{J}^+$ (future null infinity)

What is the asymptotic symmetry group ?

- $BMS_4^{Glob} = SL(2,\mathbb{C}) \ltimes C^\infty(S^2)$ (BMS, 1962)

Lorentz transf as globally well-defined conformal transf of S^2 — supertranslations

(BT, 2010)

applications in celestial holography

- $\boxed{BMS_4^{Ext} = Diff(S^1) \times Diff(S^1) \ltimes \text{Laurent series in } z, \bar{z}}$ $\rightarrow$

- $BMS_4^{Gen} = Diff(S^2) \ltimes C^\infty(S^2)$ (CL, 2014)

applications for soft theorems

⋮ more general corner groups

Geometric action for BMS_4^E

$$I^U_{\mathrm{BMS}^E_4}[f,\bar f,T,J,\bar J,P;Y_0,\bar Y_0,T_0]=$$

$$\int dtdzd\bar z\ \Big(\Big[\Big(\bar J+\Big(\frac{1}{2}T\partial P+\frac{3}{2}\partial TP\Big)\Big)\circ(f,\bar f)\Big]\Big[f'\bar f'\dot f-(f')^2\bar f'Y_0\Big]+\text{c.c.}$$

$$+P\dot T-[P\circ(f,\bar f)](f'\bar f')^{\frac{3}{2}}T_0\Big)\,.$$

$$H_{(0,0,1)}=\int dudzd\bar z\,P\Big[(f'\bar f')^{\frac{1}{2}}\circ(f^{-1},\bar f^{-1})\Big]=\int dudzd\bar z\,[P\circ(f,\bar f)](f'\bar f')^{\frac{3}{2}},$$

reproduces ~~flux~~-balance equations

the Poisson brackets $\{\pi_i,\pi_j\}=f^k_{ij}\pi_k$ read explicitly

$$\{\bar J(z),P(w,\bar w)\}=\Big[\frac{3}{2}\partial_w\delta(z,w)+\delta(z,w)\partial_w\Big]P(w,\bar w),$$

$$\{J(z),P(w,\bar w)\}=\Big[\frac{3}{2}\partial_{\bar w}\delta(\bar z,\bar w)+\delta(\bar z,\bar w)\partial_{\bar w}\Big]P(w,\bar w),$$

$$\{\bar J(z),\bar J(w)\}=[2\partial_w\delta(z,w)+\delta(z,w)\partial_w]\bar J(w),$$

$$\{J(\bar z),J(\bar w)\}=[2\partial_{\bar w}\delta(\bar z,\bar w)+\delta(\bar z,\bar w)\partial_{\bar w}]J(\bar w),$$

$$\{\bar J(z),J(w)\}=0,$$

$$\{P(z,\bar z),P(w,\bar w)\}=0.$$

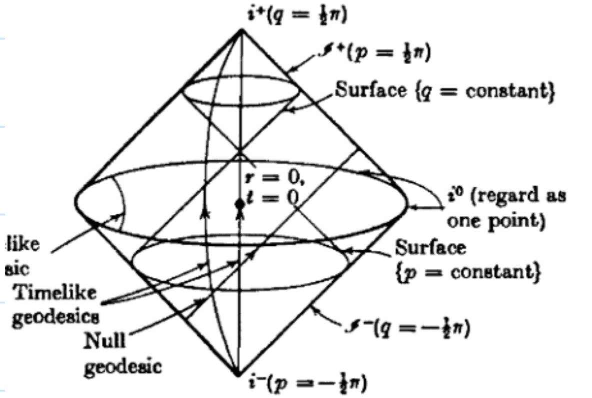
KKS Poisson bracket have appeared in the context of celestial holography

Details :

- technical control of coadjoint representation of $BMS_4^{\text{Gbb, Ext, Gen}}$
- identification of BMS_4^* in the data of non-radiative asymptotically flat spacetimes (B.R, JHEP 2021)

Open questions :

- classification of coadjoint orbits of BMS_4
- interaction with radiative Dof at $\mathscr{I}^+$
- connection with spatial infinity i^0



(Hawking & Ellis)

Coadjoint representation of BMS_4: general structure

2d conformally flat S aim: unified description for sphere & punctured plane

$$ds^2 = -2(P\bar{P})^{-1} d\zeta\, d\bar{\zeta} \quad \begin{cases} \zeta' = \zeta'(\zeta) \quad \bar{\zeta}' = \bar{\zeta}'(\bar{\zeta}) & \text{conformal coordinate trsf.} \\ & x = (\zeta, \bar{\zeta}) \\ P'(x) = P(x) e^{-E(x)}, \quad \bar{P}'(x) = \bar{P}(x) e^{-\bar{E}(x)} & \text{complex Weyl rescaling} \begin{cases} E_R & \text{standard Weyl} \\ iE_I & \text{local rotation} \end{cases} \end{cases}$$

zweibeins $\quad ds^2 = e^a_\mu dx^\mu \eta_{ab} e^b_\nu dx^\nu \quad \eta_{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad e_1^\mu \frac{\partial}{\partial x^\mu} = P \frac{\partial}{\partial \zeta} \quad e_2^\mu \frac{\partial}{\partial x^\mu} = \bar{P} \frac{\partial}{\partial \bar{\zeta}}$

conformal fields $\quad \phi'_{h,\bar{h}}(x') = \left(\frac{\partial \zeta}{\partial \zeta'}\right)^h \left(\frac{\partial \bar{\zeta}}{\partial \bar{\zeta}'}\right)^{\bar{h}} \phi_{h,\bar{h}}(x) \qquad P'(x') = P(x) e^{-E(x')} \frac{\partial \zeta'}{\partial \zeta}$ combined trsf.

weighted scalars $\quad \eta'^{s,w}(x') = e^{w E_R(x')} e^{-is E_I(x')} \eta^{s,w}(x)$

Held, Posadas, Newman JMP 1970

Lorentz group & sphere

interpolation $\quad \eta^{s,w} = P^h \bar{P}^{\bar{h}} \phi_{h,\bar{h}}$

D'Hoker, Phong Rev. Mod. Phys. 1988

map $\quad s = h - \bar{h}, \quad w = -(h + \bar{h}) \qquad h = \frac{s-w}{2}, \quad \bar{h} = -\frac{s+w}{2}$

covariant derivative $\quad \nabla : \quad \Gamma^{\xi}_{\xi\xi} = -\partial \ln(P\bar{P}) \qquad \Gamma^{\bar{\xi}}_{\bar{\xi}\bar{\xi}} = -\bar{\partial} \ln(P\bar{P})$

$$\Gamma'^{\xi'}_{\xi'\xi'}(x') = \Gamma^{\xi}_{\xi\xi}(x)\frac{\partial\xi}{\partial\xi'} + \frac{\partial\xi'}{\partial\xi}\frac{\partial^2\xi}{\partial\xi'\partial\xi'} + 2\partial' E_R(x')$$

introduce Weyl connection $\quad D: \quad W'(x') = \frac{\partial\xi}{\partial\xi'} W + 2\partial' E_R(x'), \quad \bar{W}'(x') = \frac{\partial\bar{\xi}}{\partial\bar{\xi}'}\bar{W}(x) + 2\bar{\partial}' E_R(x')$

$$\underbrace{D\phi_{h,\bar{h}}}_{(h+1,\bar{h})} = [\nabla + hW]\phi_{h,\bar{h}}, \quad \underbrace{\bar{D}\phi_{h,\bar{h}}}_{(h,\bar{h}+1)} = [\bar{\nabla} + \bar{h}\bar{W}]\phi_{h,\bar{h}}$$

$\partial, \nabla, D \equiv \partial_\xi, \nabla_\xi, D_\xi$

$\bar{\partial}, \bar{\nabla}, \bar{D} = \partial_{\bar{\xi}}, \nabla_{\bar{\xi}}, D_{\bar{\xi}}$

weighted scalars $\quad \eth\eta^{s,w} = P^{h+1}\bar{P}^{\bar{h}}\nabla\phi_{h,\bar{h}}, \qquad \bar{\eth}\eta^{s,w} = P^{h}\bar{P}^{\bar{h}+1}\bar{\nabla}\phi_{h,\bar{h}}$

$$= P\bar{P}^{-s}\partial(\bar{P}^{s}\eta^{s,w}) \qquad = \bar{P}P^{s}\bar{\partial}(P^{-s}\eta^{s,w})$$

Weyl covariant $\quad \underbrace{\mathscr{D}\eta^{s,w}}_{[s+1,\,w-1]} = P^{h+1}\bar{P}^{\bar{h}}D\phi_{h,\bar{h}}, \qquad \underbrace{\bar{\mathscr{D}}\eta^{s,w}}_{[s-1,\,w-1]} = P^{h}\bar{P}^{\bar{h}+1}\bar{D}\phi_{h,\bar{h}}$

$$= \left(\eth + \left(\frac{s-w}{2}\right)PW\right)D\phi_{h,\bar{h}} \qquad = \left[\bar{\eth} - \left(\frac{w+s}{2}\right)\bar{P}\bar{W}\right]\phi_{h,\bar{h}}$$

$$[\mathscr{D}, \bar{\mathscr{D}}]\eta^{s,w} = -\frac{s}{2}R_s\eta^{s,w} - P\bar{P}\left(\frac{s-w}{2}\bar{\partial}W + \frac{s+w}{2}\partial\bar{W}\right)\eta^{s,w} \qquad R_s : \text{ scalar curvature}$$

Ingredients

(super-)translation	$\mathcal{T} : [0,1]$	$\tilde{\mathcal{T}} : (-\tfrac{1}{2}, -\tfrac{1}{2})$	real
(super-) rotation	$\mathcal{Y} : [-1,1]$	$\tilde{\mathcal{Y}} : (-1,0)$	$\bar{\mathcal{D}} \mathcal{Y} = 0 \Leftrightarrow \bar{D} \tilde{\mathcal{Y}} = 0$
	$\bar{\mathcal{Y}} : [1,1]$	$\tilde{\bar{\mathcal{Y}}} : (0,-1)$	$\mathcal{D} \bar{\mathcal{Y}} = 0 \Leftrightarrow D \tilde{\bar{\mathcal{Y}}} = 0$
(super-) momentum	$\mathcal{P} : [0,-3]$	$\tilde{\mathcal{P}} : (\tfrac{3}{2}, \tfrac{3}{2})$	real
(super-) angular momentum	$\mathcal{J} : [-1,-3]$	$\tilde{\mathcal{J}} : (1,2)$	$\mathcal{J} \sim \mathcal{J} + \mathcal{D} \underset{[-2,-2]}{\mathcal{L}}$, $\tilde{\mathcal{J}} \sim \tilde{\mathcal{J}} + D \underset{(0,2)}{\tilde{\mathcal{L}}}$
	$\bar{\mathcal{J}} : [1,-3]$	$\tilde{\bar{\mathcal{J}}} : (2,1)$	$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar{\mathcal{D}} \underset{[2,-2]}{\bar{\mathcal{L}}}$, $\tilde{\bar{\mathcal{J}}} \sim \tilde{\bar{\mathcal{J}}} + \bar{D} \underset{(2,0)}{\tilde{\bar{\mathcal{L}}}}$

In all relations, weights/dimensions are such that Weyl connection drops out!

$\mathcal{D} \to \eth \qquad D \to \partial$ simplest description in terms of conformal fields

$\mathfrak{bms}_4$ algebra $\quad [(\mathcal{Y}_1, \bar{\mathcal{Y}}_1, \mathcal{T}_1), (\mathcal{Y}_2, \bar{\mathcal{Y}}_2, \mathcal{T}_2)] = (\hat{\mathcal{Y}}, \hat{\bar{\mathcal{Y}}}, \hat{\mathcal{T}})$

$$\hat{\mathcal{Y}} = \mathcal{Y}_1 \eth \mathcal{Y}_2 - \mathcal{Y}_2 \eth \mathcal{Y}_1 \qquad \hat{\mathcal{T}} = \mathcal{Y}_1 \eth \mathcal{T}_2 - \frac{1}{2} \eth \mathcal{Y}_1 \mathcal{T}_2 - (1 \leftrightarrow 2) + \text{c.c.}$$

subalgebra $\mathfrak{g}$ $\quad (\mathcal{Y}, \bar{\mathcal{Y}}, 0) \qquad (\tilde{\mathcal{Y}}, \bar{\tilde{\mathcal{Y}}}, 0)$

(Lorentz, Witt⊕Witt)

representation of $\mathfrak{g}$ on $\eta^{s,w}$ $\qquad$ on $\phi_{h,\bar{h}}$

$$\mathcal{Y} \cdot \eta^{s,w} = \mathcal{Y} \eth \eta^{s,w} + \frac{s-w}{2} \eth \mathcal{Y}\, \eta^{s,w} \qquad \tilde{\mathcal{Y}} \cdot \phi_{h,\bar{h}} = \tilde{\mathcal{Y}} \partial \phi_{h,\bar{h}} + h \partial \tilde{\mathcal{Y}}\, \phi_{h,\bar{h}}$$

$$\bar{\mathcal{Y}} \cdot \eta^{s,w} = \bar{\mathcal{Y}} \bar{\eth} \eta^{s,w} - \frac{s+w}{2} \bar{\eth} \bar{\mathcal{Y}}\, \eta^{s,w} \qquad \bar{\tilde{\mathcal{Y}}} \cdot \phi_{h,\bar{h}} = \bar{\tilde{\mathcal{Y}}} \bar{\partial} \phi_{h,\bar{h}} + \bar{h}\, \bar{\partial} \bar{\tilde{\mathcal{Y}}}\, \phi_{h,\bar{h}}$$

$$\mathcal{E}_{\times\alpha} = (\mathcal{Y}, \bar{\mathcal{Y}}) \cdot \mathcal{T}^{[0,1]} \qquad \mathcal{E}_{\times\alpha} = (\tilde{\mathcal{Y}}, \bar{\tilde{\mathcal{Y}}}) \cdot \tilde{\mathcal{T}}^{(-1/2,-1/2)}$$

action of inf rotation on translations

bms_4^* dual space $([\mathcal{J}], [\bar{\mathcal{J}}], \mathcal{P})$ $([\tilde{\mathcal{J}}], [\tilde{\bar{\mathcal{J}}}], \tilde{\mathcal{P}})$

$(0,0)$; $[0,-2]$

pairing $\langle([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P}); (\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})\rangle = \int_S d\mu\, [\bar{\mathcal{J}}\mathcal{Y} + \mathcal{J}\bar{\mathcal{Y}} + \mathcal{P}\mathcal{T}]$, $d\mu(\zeta,\bar{\zeta}) = \frac{iC}{P\bar{P}} d\zeta \wedge d\bar{\zeta}$

$\langle([\tilde{\mathcal{J}}],[\tilde{\bar{\mathcal{J}}}], \tilde{\mathcal{P}}), (\tilde{\mathcal{Y}}, \tilde{\bar{\mathcal{Y}}}, \tilde{\mathcal{T}})\rangle = \int_S d\tilde{\mu}\, [\tilde{\bar{\mathcal{J}}}\tilde{\mathcal{Y}} + \tilde{\mathcal{J}}\tilde{\bar{\mathcal{Y}}} + \tilde{\mathcal{P}}\tilde{\mathcal{T}}]$ $d\tilde{\mu} = iC\, d\zeta \wedge d\bar{\zeta}$

assumption : pairing annihilates total $\eth, \bar{\eth}$ $(\partial, \bar{\partial})$ derivatives

non-degenerate $\rightarrow$ integrations by parts

$$ad^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{J} = \bar{\mathcal{Y}}\bar{\eth}\mathcal{J} + 2\bar{\eth}\bar{\mathcal{Y}}\mathcal{J} + \underbrace{\eth(\mathcal{Y}\mathcal{J})}_{= ad^*_{\bar{\mathcal{Y}}}\mathcal{J} \sim 0} + \underbrace{\tfrac{1}{2}\mathcal{T}\bar{\eth}\mathcal{P} + \tfrac{3}{2}\bar{\eth}\mathcal{T}\mathcal{P}}_{\alpha_X \mathcal{P}} ,$$

$$ad^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{P} = \underbrace{\mathcal{Y}\eth\mathcal{P} + \tfrac{3}{2}\eth\mathcal{Y}\mathcal{P} + c.c.}_{\mathcal{E}^*_X \mathcal{P}}$$

work out formulas for the group ✓

Realization on the sphere

stereographic coord. $\xi = \cot\frac{\theta}{2}\, e^{-i\phi}$ $\quad ds^2 = -2(P_S\bar{P}_S)\, d\xi\, d\bar{\xi}$ $\quad P_S = \frac{1}{R\sqrt{2}}(1+\xi\bar{\xi})$

globally well-defined conf. coord. trsf $\quad \xi' = \frac{a\xi+b}{c\xi+d}\,,\quad ad-bc=1,\quad a,b,c,d\in\mathbb{C}$ $\quad \frac{\partial\xi}{\partial\xi'} = (c\xi+d)^2$

compensating Weyl trsf. $\quad e^{E_R(x')} = \frac{1+\xi\bar{\xi}}{|a\xi+b|^2+|c\xi+d|^2}$ $\quad e^{iE(x')} = \frac{\bar{c}\bar{\xi}+\bar{d}}{c\xi+d}$ $\quad w$: boost weight

Pairing $\quad \langle K^{s,-w-2}, \eta^{s,w}\rangle = \frac{1}{4\pi R^2}\int_{S^2}\frac{i\, d\xi \wedge d\bar{\xi}}{P_S\bar{P}_S}\,\overline{K^{s,-w-2}}\,\eta^{s,w}$ $\quad C=(4\pi R^2)^{-1}$

assumptions ✓

$$\frac{1}{4\pi R^2}\int_{S^2}\frac{i\, d\xi\wedge d\bar{\xi}}{P_S\bar{P}_S} = \frac{1}{4\pi}\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi = 1$$

adjoint repres. group

$$\eta'(x') = e^{\Xi_R(x')}\, e^{i\Xi_I(x')}\, \eta(x)$$

$$\beta'(x') = e^{\Xi_R(x')}\left(\beta - (\eta\,\bar\partial\alpha - \tfrac{1}{2}\bar\partial\alpha\,\partial\eta + c.c.)\right)(x)$$

coadjoint repres. group

$$\mathcal{J}'(x') = e^{-3\Xi_R(x')}\, e^{i\,\Xi_I(x')}\left(\mathcal{J} + \tfrac{1}{2}\partial\bar\partial\mathcal{P} + \tfrac{3}{2}\bar\partial\partial\mathcal{P}\right)(x)$$

$$\mathcal{P}'(x') = e^{-3\Xi_R(x')}\, \mathcal{P}(x)$$

in terms of conf. fields

$$\tilde\eta'(\xi') = (c\xi+d)^{-2}\,\tilde\eta(\xi)$$

$$\tilde\beta'(x') = (c\xi+d)^{-1}(\bar c\bar\xi+\bar d)^{-1}\left(\tilde\beta - \tilde\eta\,\partial\tilde\alpha - \tfrac{1}{2}\tilde\alpha\,\partial\tilde\eta + c.c.\right)(x)$$

$$\tilde{\mathcal{J}}'(x') = (c\xi+d)^{2}(\bar c\bar\xi+\bar d)^{4}\left(\tilde{\mathcal{J}}(x) + \left(\tfrac{1}{2}\tilde\partial\partial\tilde{\mathcal{P}} + \tfrac{3}{2}\partial\tilde\partial\tilde{\mathcal{P}}\right)\right)(x)$$

$$\tilde{\mathcal{P}}'(x') = (c\xi+d)^{3}(\bar c\bar\xi+\bar d)^{3}\,\tilde{\mathcal{P}}(x)$$

Expansions : spin weighted spherical harmonics : ${}_sZ_{jm}$ un normalized ${}_sY_{jm}$ normalized

Gelfand, Minlos, Shapiro (1958) ; Wu & Yang, Nucl. Phys. B (1976)
Newman, Penrose, JMP (1966) ; Thorne, Rev. Mod. Phys (1980)

conformal Killing eq. on S^2 $\quad \bar{\eth} Y^{[-1,1]} = 0 = \eth \bar{Y}^{[1,-1]}$

$$Y_m = -R\sqrt{2}\ {}_{-1}Z_{1,m} \qquad m = -1, 0, 1 \qquad Y = \sum_{m=-1}^{1} y_m Y_m$$

$$T_{jm} = {}_0Z_{j,m} \qquad T = \sum_{j,\,|m|\leq j} t_{jm} T_{j,m}, \qquad \bar{t}_{jm} = (-1)^m t_{j,-m}$$

dual basis $\quad Y_*^m = \dfrac{-6}{R\sqrt{2}\,(1+m)!\,(1-m)!}\ {}_{-1}Z_{1,m} \qquad J = \sum_{m=-1}^{1} j_m Y_*^m$

$$T_*^{j,m} = \frac{(2j+1)!\,(2j)!}{j!\,j!\,(j+m)!\,(j-m)!}\ {}_0Z_{j,m} \qquad P = \sum_{j,\,|m|\leq j} p_{j,m} T_*^{j,m}, \qquad \bar{p}_{jm} = (-1)^m p_{j,-m}$$

NB: conformal fields : $\tilde{Y}_m = Y_m P_S = \zeta^{1-m} \Rightarrow [\tilde{Y}_m, \tilde{Y}_n] = (m-n)\, \tilde{Y}_{m+n}$

$\longrightarrow$ all other structure constants can be worked out explicitly (ugly)

Remark (i) Penrose & Rindler Vol I, section 4.15

$w \geq |s|$ $\qquad \eth^{w+s+1} \eta^{s,w}$ $\qquad \bar\eth^{w+s+1} \eta^{s,w}$

$\qquad [w+1, s-1] \qquad [-w-1, -s-1]$

definite boost weight

In the study of spin-weighted spherical harmonics it is useful to contemplate the following array:

j													
0							1						
$\frac{1}{2}$						2		2					
1		$\overleftarrow{\eth'}$			3		3		3		$\overrightarrow{\eth}$		
$\frac{3}{2}$				4		4		4		4			
2			5		5		5		5		5		
$\frac{5}{2}$		6		6		6		6		6		6	
$\vdots$	$\cdot^{\cdot^{\cdot}}$												$\ddots$
$s = \ldots$		$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$\ldots$

(4.15.60)

The numbers in this triangular array (which extends indefinitely downwards) represent the complex *dimensions* of the various spaces of spin-weighted spherical harmonics, as discussed in (4.15.43) *et seq.* Each of these spaces is characterized by its values of s and j, as shown. The dimension *zero* is assigned wherever a blank space appears in the array. The operator $\eth$ carries us a step of one s-unit to the right and $\eth'$ one s-unit to the left. (From our earlier discussion, the j-value is not affected by $\eth$ or $\eth'$.) Whenever such a step carries us off the array, the result of the operator $\eth$ or $\eth'$ is zero. Note that the dimension remains constant whenever it does not drop to, or increase from, zero.

$\bar\eth \eta = 0 \Leftrightarrow \eth^3 \eta = 0$ $\qquad$ weight

$\eth \bar\eta = 0 \Leftrightarrow \bar\eth^3 \bar\eta = 0$ $\qquad$ same solutions

dual situation $\quad w \leq -|s| - 2$

$\eth^{s-w-1} \kappa^{w+1, s-1} \qquad \bar\eth^{-s-w-1} \kappa^{-w-1, -s-1}$

$[s, w] \qquad [s, w]$ $\qquad$ definite boost weight

$[2, -2] \qquad [-2, 0]$

$\mathcal{J} \sim \mathcal{J} + \bar\eth \bar{\mathcal{I}} \quad \Leftrightarrow \quad \bar{\tilde{\mathcal{J}}} \sim \bar{\tilde{\mathcal{J}}} + \eth^3 \mathcal{M}$

same equivalence classes

$j \leq w$: finite dim. repres of Lorentz, "heads"

$j > w$: ∞ dim , "tails"

Remark (ii) reduction to Poincaré

$\eth^2 \mathcal{J} = 0 = \bar\eth^2 \mathcal{J} \qquad \mathcal{P} \sim \mathcal{P} + \eth^2 \mathcal{N} + \bar\eth^2 \bar{\mathcal{N}}$

$\qquad [-2, -1] \qquad [2, -1]$

Realization on punctured plane

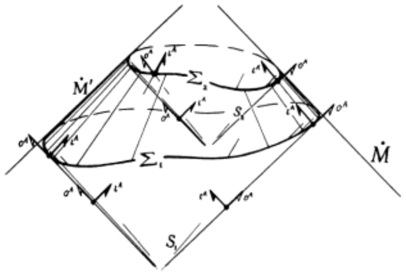
- Weyl trsf $e^{-\Phi(\xi,\bar\xi)} = \frac{\sqrt{2}}{1+\xi\bar\xi}$ $\quad \xi = R^{-1} z \quad ds^2 = -2\, dz\, d\bar z$

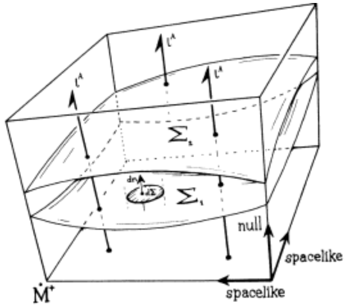
- 2-punctures: remove points at origin & infinity $\mathbb{C}_0$

- on the level of the algebra, look at the algebra of all infinitesimal local conformal trsf.

Not the Lie algebra of globally well-defined trsf.

For asymptotically flat spaces, $\dot M$ is in fact a null hypersurface [7]. The structure of $\dot M$ is essentially the same as for Minkowski space (Figure 4). We shall omit the three points I^-, I^0, I^+ here. Then $\dot M$ consists of two portions, each of which is topologically a "cylinder" $S^2 \times E^1$. We are concerned, here, only with the future portion $\dot M^+$, and by judicious choice of conformal factor Ω, we can ensure that the geometry of $\dot M^+$ is as simple as possible. In fact, by taking one generator of $\dot M^+$ "back to infinity" we can open out the cylinder into a space with Euclidean three-space topology. Furthermore, it turns out that we can also make this three-space metrically *flat* (Figure 6). This will simplify matters considerably.





Penrose 1967 AMS

$P=1 \Rightarrow$ weighted scalars = conformal fields

gravity : sphere → plane

CFT : plane → sphere

$$e^{\Phi(x')} = \frac{\partial z'}{\partial z} \qquad e^{\Phi_R(x')} = \left(\frac{\partial z'}{\partial z}\,\frac{\partial \bar z'}{\partial \bar z}\right)^{1/2}, \qquad e^{i\Phi_I(x')} = \left(\frac{\partial z'/\partial z}{\partial \bar z'/\partial \bar z}\right)^{1/2}$$

Coulomb gas ?

Expansions

$$\phi_{h,\bar h}(z,\bar z) = \sum_{k,l} a_{k,l}\ \tilde{Z}_{h,\bar h\, k,l}\,, \qquad \tilde{Z}_{h,\bar h\, k,l} = z^{-h-k}\ \bar z^{-\bar h - l}$$

$h, \bar h \in \mathbb{N} \Rightarrow k, l \in \mathbb{Z}$

$h, \bar h \in \frac{\mathbb{N}}{2} \Rightarrow k, l \in \frac{1}{2} + \mathbb{Z}$

(NS)

Pairing $\quad \widetilde{\langle \psi_{-\bar h+1, -h+1}, \phi_{h,\bar h} \rangle} = \mathrm{Res}_z\ \mathrm{Res}_{\bar z} \left[\overline{\psi_{-\bar h+1, -h+1}}\ \phi_{h,\bar h} \right]$

assumptions ✓

$$\mathrm{Res}_z (\partial \phi) = 0 = \mathrm{Res}_{\bar z} (\bar\partial \phi)$$

adjoint repr. group $\quad \tilde{Y}'(z') = \left(\frac{\partial z}{\partial z'}\right)^{-1} \tilde{Y}(z)$

$$\tilde{\beta}'(x') = \left(\frac{\partial z}{\partial z'}\right)^{-1/2} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{-1/2} \left(\tilde{\beta} - (\tilde{Y}\, \partial \tilde{\alpha} - \tfrac{1}{2} \tilde{\alpha}\, \partial \tilde{Y} + \text{c.c.})\right)(x)$$

coadjoint repr group $\quad J'(x') = \left(\frac{\partial z}{\partial z'}\right)^{1} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{2} \left(J + \tfrac{1}{2} \tilde{\mathcal{F}}\, \partial \tilde{\mathcal{F}} + \tfrac{3}{2} \partial \tilde{\partial} \tilde{\mathcal{F}} \right)(x)$

$$\tilde{\mathcal{F}}'(x) = \left(\frac{\partial z}{\partial z'}\right)^{3/2} \left(\frac{\partial \bar z}{\partial \bar z'}\right)^{3/2} \tilde{\mathcal{F}}(x)$$

to be used for conformal mapping.

Expansions

$$\langle {}_{-\bar h+1,-h+1}\tilde Z_{k',l'},\ {}_{h,\bar h}\tilde Z_{k,l} \rangle = \delta^0_{l'+k}\,\delta^0_{k'+l}$$

$$\tilde Y_m = z^{1-m}, \qquad \tilde J_{k,l} = z^{1/2-k}\,\bar z^{1/2-l} \qquad m,\ \tfrac{1}{2}+k,\ \tfrac{1}{2}+l \in \mathbb{Z}$$

$$\tilde Y^m_* = z^{-1}\,\bar z^{-2+m} \qquad \tilde J^{k,l}_* = z^{-3/2+l}\,\bar z^{-3/2+k}$$

$$\tilde Y_m \cdot {}_{h,\bar h}\tilde Z_{k,l} = -(hm+k)\ {}_{h,\bar h}\tilde Z_{k+m,l}, \qquad \tilde{\bar Y}_m \cdot {}_{h,\bar h}\tilde Z_{k,l} = -(\bar h m+l)\ {}_{h,\bar h}\tilde Z_{k,l+m}$$

Structure constants

$$[\tilde Y_m, \tilde Y_n] = (m-n)\,\tilde Y_{m+n} \qquad [\tilde Y_m, \tilde J_{k,l}] = (\tfrac{1}{2}m-k)\,\tilde J_{k+m,l}$$

$$[\tilde{\bar Y}_m, \tilde J_{k,l}] = (\tfrac{1}{2}m-l)\,\tilde J_{k,l+m}$$

$$[\tilde Y_m, \tilde{\bar Y}_n] = 0 = [\tilde J_{k,l}, \tilde J_{r,s}]$$

coadjoint repr. algebra

$$\mathrm{ad}^*_{\tilde{y}_m} \tilde{y}^n_* = (-2m+n)\, \tilde{y}^{n-m}_* \,, \qquad \mathrm{ad}^*_{\tilde{y}_m} \tilde{J}^{k,l}_* = \left(-\tfrac{3}{2}m+k\right) \tilde{J}^{k-m,l}_* \,,$$

$$\mathrm{ad}^*_{\tilde{J}_{k,l}} \tilde{J}^{r,s}_* = \left(\frac{r-3k}{2}\right) \delta^s_l \, y^{r-k}_* + \left(\frac{s-3l}{2}\right) \delta^r_k \, \tilde{y}^{s-l}_*$$

Realization on cylinder

$$z = e^{-i\frac{2\pi}{L_1}w} \,, \quad w = w_1 + i w_2 \,, \quad w_1 \sim w_1 + L_1 \,, \quad \phi^C_{h,\bar{h}}(w,\bar{w}) = \left(-i\frac{2\pi}{L_1} z\right)^h \left(i\frac{2\pi}{L_1}\bar{z}\right)^{\bar{h}} \phi_{h,\bar{h}}(z,\bar{z})$$

use formulas for the group to map generators $\quad y^C_m = i\left(\frac{2\pi}{L_1}\right)^{-1} e^{i\frac{2\pi}{L_1}mw} \ldots$

same structure constants, obtained from ad^* still provide a representation

but pairing issues ....

(Torus: $\quad w^T_2 \sim w^T_2 + L_2 \quad e^{i\frac{2\pi}{L_1}(w_1+iw_2)} \to e^{i\frac{2\pi}{L_1}w^T_1}\, e^{i\frac{2\pi}{L_2}w^T_2}$

$$w_2 = +i\frac{L_2}{L_1} w^T_2$$)

Identification in non-radiative asymptotically flat spacetimes at $\mathcal{J}^+$

Back to S^2 & GR: BMS metric $\Longleftrightarrow$ $\boxed{\text{NP first order}}$ (similar analysis at $\mathcal{J}^-$)

Solution space, free data at $\mathcal{J}^+$: $\boxed{\Psi_2^0 + \bar{\Psi}_2^0, \quad \Psi_1^0, \quad \sigma^0}$ undetermined u-dependence

$\dot{\bar{\sigma}}^0$ news

evolution equations. $\partial_u \Psi_2^0 = \eth \Psi_3^0 + \sigma^0 \Psi_4^0$, $\quad \partial_u \Psi_1^0 = \eth \Psi_2^0 + 2\sigma^0 \Psi_3^0$

constraints $\quad \Psi_2^0 - \bar{\Psi}_2^0 = \bar{\eth}^2 \sigma^0 - \eth^2 \bar{\sigma}^0 + \dot{\sigma}^0 \bar{\sigma}^0 - \sigma^0 \dot{\bar{\sigma}}^0$

$\Psi_3^0 = -\eth \dot{\bar{\sigma}}^0$, $\quad \Psi_4^0 = -\ddot{\bar{\sigma}}^0$

additional data to construct solutions

$$\Psi_0 = \sum_{n \geq 0} \Psi_0^n(\zeta, \bar{\zeta}, u_0)\, r^{-5-n}$$

Transformation of (relevant) free data at $\mathscr{I}^+$ $\qquad s=(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})$, $\quad f=\mathcal{T}+\frac{1}{2}u(\eth\mathcal{Y}+\bar{\eth}\bar{\mathcal{Y}})$

$$\delta_s\sigma^0=[f\partial_u+\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+\tfrac{3}{2}\eth\mathcal{Y}-\tfrac{1}{2}\bar{\eth}\bar{\mathcal{Y}}]\sigma^0-\eth^2 f$$

$$\delta_s\dot{\sigma}^0=[f\partial_u+\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+2\eth\mathcal{Y}\qquad]\dot{\sigma}^0-\tfrac{1}{2}\eth^2(\underline{\eth\mathcal{Y}}+\bar{\eth}\bar{\mathcal{Y}})$$

$\begin{cases}\text{EM tensor}\\ \text{Schwarzian derivative}\end{cases}$

$$\delta_s\psi^0_2=[\;''\quad''\quad''+\tfrac{3}{2}\eth\mathcal{Y}+\tfrac{3}{2}\bar{\eth}\bar{\mathcal{Y}}]\psi^0_2+2\eth f\,\psi^0_3$$

$$\delta_s\psi^0_1=[\;''\quad''\quad''+2\eth\mathcal{Y}+\bar{\eth}\bar{\mathcal{Y}}]\psi^0_1+3\eth f\,\psi^0_2$$

broken current algebra

$$J_s=\frac{i}{R^2}\left[(P_S\bar{P}_S)^{-1}J^u_s\,d\xi\wedge d\bar{\xi}+P_S^{-1}J^{\bar{\xi}}_s\,du\wedge d\xi-\bar{P}_S J^{\xi}_s\,du\wedge d\bar{\xi}\right]$$

$$\delta_{s_1}J_{s_2}+\boxed{\Theta_{s_2}(\delta_{s_1}X)}\approx-J_{[s_1,s_2]}+d\,L_{s_1,s_2}$$

non-conservation $\qquad d\,J_s+\Theta_s(\delta_{(0,0,1)}X)\approx 0$

$s_1:(\mathcal{Y}_1,\bar{\mathcal{Y}}_1,\mathcal{T}_1)=(0,0,1)$ $\qquad \Theta_s(\delta X)\sim\dot{\sigma}^0,\dot{\bar{\sigma}}^0$ vanishes in the absence of news time components

$$J^u_s = -\frac{1}{8\pi G}\Big\{\big[\overbrace{\Psi^0_2+\bar\Psi^0_2}^{BH}+\overbrace{\sigma^0\dot{\bar\sigma}^0+\bar\sigma^0\dot\sigma^0}^{\text{gravitons}}\big]f+\big[\Psi^0_1+\sigma^0\eth\bar\sigma^0+\tfrac{1}{2}\eth(\sigma^0\bar\sigma^0)\big]\mathcal{Y}+\big[\bar\Psi^0_1+\bar\sigma^0\bar\eth\sigma^0+\tfrac{1}{2}\bar\eth(\sigma^0\bar\sigma^0)\big]\bar{\mathcal{Y}}\Big\}$$

$$\Theta^u_s(\delta X) = \frac{1}{8\pi G}\big[\dot{\bar\sigma}^0\delta\sigma^0+\dot\sigma^0\delta\bar\sigma^0\big]f$$

charges $\quad Q_s = \int_{S^2,\,u=cte} \frac{i}{R^2}\frac{d\zeta d\bar\zeta}{P_S\bar P_S} J^u_s \,. \qquad \Theta_s[\delta X] = \int_{S^2,\,u=cte}\frac{i}{R^2}\frac{d\zeta d\bar\zeta}{P_S\bar P_S}\Theta^u_s[\delta X]$

algebra $\quad \delta_{s_1}Q_{s_2} + \Theta_{s_2}[\delta_{s_1}X] = -Q_{[s_1,s_2]}$

(non-)conservation of BMS_4 charges

G.B. & C. Troessaert JHEP (2011)

JHEP (2013)

$$\frac{d}{du}Q_s = -\underbrace{\int_{S^2,\,u=cte}\frac{i}{R^2\,8\pi G}\frac{d\zeta d\bar\zeta}{P_S\bar P_S}\big[\dot{\bar\sigma}^0\delta_s\sigma^0+\dot\sigma^0\delta_s\bar\sigma^0\big]}_{\text{fluxes}\quad\text{generalizes mass loss}}$$

$\boxed{\text{non-radiative space-times (no news)}}$ $\sigma^0 = \sigma^0(\xi, \bar{\xi}, \cancel{u})$ $\left(\Rightarrow \dot{\sigma}^0 = 0 = \psi_3^0 = \psi_4^0, \quad \mathcal{O}_s[\delta\chi] = 0 \right)$

compare "abstract" construction of $\mathfrak{bms}_4^*$

identification at $u=0$ $\qquad \mathcal{P} = -\frac{1}{2G}(\psi_2^0 + \bar{\psi}_2^0) \qquad \bar{\mathcal{J}} = -\frac{1}{2G}\Big(\underbrace{\psi_1^0 + \sigma^0 \eth \bar{\sigma}^0 + \frac{1}{2}\eth(\sigma^0\bar{\sigma}^0)}_{\psi^0_{1\bar{\mathcal{J}}}}\Big)$

super-momentum = Bondi mass aspect

~~super~~- angular momentum = Bondi angular momentum aspect

(pre) momentum map: $\mathfrak{F}$. algebra of non-radiative free data

$\mathfrak{bms}_4$ representation δ_s, $[\delta_{s_1}, \delta_{s_2}] = \delta_{[s_1, s_2]}$

$$\mu : \mathfrak{F} \longrightarrow \mathfrak{bms}_4^*$$

$$\mu\left(-\frac{1}{2G}(\psi_2^0 + \bar{\psi}_2^0)\right) = \mathcal{P}, \qquad \mu\left(-\frac{1}{2G}\psi^0_{1\bar{\mathcal{J}}}\right) = [\bar{\mathcal{J}}], \qquad \mu \circ \delta_s = \mathrm{ad}_s^* \circ \mu$$

transformation laws at $u=0$ $\quad \delta_s(\psi_2^0+\bar\psi_2^0) = (Y\eth + \bar Y\bar\eth + \frac32 \eth Y + \frac32 \bar\eth \bar Y)(\psi_2^0+\bar\psi_2^0)$ ✓

$$\delta_s \psi_1^0 = [Y\eth + \bar Y\bar\eth + 2\eth Y + \bar\eth\bar Y]\psi_1^0 + \frac12 T\eth(\psi_2^0+\bar\psi_2^0+\bar\eth^2\cancel{\sigma^0}-\eth^2\cancel{\bar\sigma^0})$$
$$+\frac32 \eth T(\psi_2^0+\bar\psi_2^0+\bar\eth^2\cancel{\sigma^0}-\eth^2\cancel{\bar\sigma^0})$$

$$\delta_s \psi^0_{1\tilde{\jmath}} = [Y\eth+\bar Y\bar\eth+2\eth Y+\bar\eth\bar Y]\psi^0_{1\tilde{\jmath}} + \frac12 T\eth(\psi_2^0+\bar\psi_2^0) + \frac32 \eth T(\psi_2^0+\bar\psi_2^0)$$
$$+\frac12\bar\eth\left(T\bar\eth\eth\sigma^0 - \bar\eth T\eth\sigma^0 + 3\eth T\bar\eth\sigma^0 - 3\bar\eth\eth T\sigma^0 - \frac32 T\sigma^0\right) \boxed{-\frac12\eth^3(T\bar\sigma^0)}$$

$\boxed{\text{trivial!}}$

Remark: electric case $\bar\eth^2\sigma^0_e = \eth^2\bar\sigma^0_e \iff \sigma^0_e = \eth^2\chi_e$

$$\delta_s\chi_e = [Y\eth+\bar Y\bar\eth-\frac12\eth Y-\frac12\bar\eth\bar Y] - T$$

Newman Penrose JMP 1966

Strominger et al. 2013-

Compère et al. 2016

simplified pre-momentum map $\mu' : \mathfrak{F}_e \longrightarrow \mathfrak{bms}_4^*$ (not physically relevant!)

$$\mu'[-\frac{1}{2G}(\psi_2^0+\bar\psi_2^0)] = T, \quad \mu'[-\frac{1}{2G}\psi_1^0] = [Y], \quad \mu'\circ\delta_s = ad^*_s\circ\mu'$$

Perspectives

1) classify coadjoint orbits of BMS_4, revisit UIRREPS McCarthy 1972-93

2) coadjoint repres. of generalized BMS_4 Campiglia & Laddha Phys. Rev. 2014

$\mathrm{Diff}(S^2) \ltimes C^\infty(S^2)$ on S^2 drop $\eth \bar{\mathcal{Y}} = 0 = \bar{\eth} \mathcal{Y}$ $\quad \eth^3 \mathcal{Y} = 0 = \bar{\eth}^3 \bar{\mathcal{Y}}$

and also equivalence relations

$$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar{\eth} \mathcal{L} \;, \quad \bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \eth^3 \mathcal{M}$$

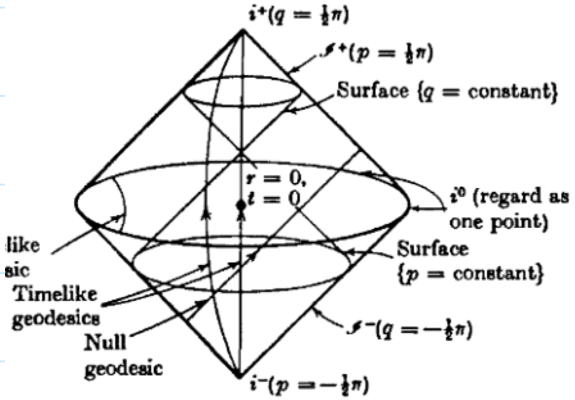
simply expand everything in spin-weighted spherical harmonics

4) Complete pre-momentum map to bona fide one connection to spatial infinity   Henneaux & Troessaert JHEP 2018

Torre CQG 1986

Oliveri & Speziale 2019

Wieland 2020



5) Study interactions of this group theory sector with radiative Dof at $\mathcal{J}^+$

Ashtekar & Streubel Proc. Roy. Soc. 1981

Ashtekar (1984)