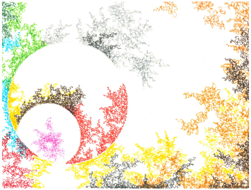
Pseudospectrum and black hole QNM instability: ultraviolet and infrared universality conjectures

José Luis Jaramillo

Institut de Mathématiques de Bourgogne (IMB)
Université de Bourgogne Franche-Comté
Jose-Luis.Jaramillo@u-bourgogne.fr



Journées Relativistes de Tours
Tours, 31 May - 2 June 2023

Scheme

1. The Problem in a nutshell: (asymptotically flat) BH QNM instability
2. Non-normal operators: spectral instability and Pseudospectrum
3. Hyperboloidal approach to QNMs: the Pöschl-Teller toy model
4. Black Hole QNMs: ultraviolet (in)stability, infrared instability
5. Discussion, Conclusions and Perspectives: universality conjectures

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BH quasi-normal modes as a spacetime probe

Black hole quasi-normal modes (QNMs): a cornerstone in Gravitation

Resonant response of black holes under linear perturbations: complex frequencies providing an invariant probe into the background spacetime geometry.

i) Relativistic astrophysics and gravitational wave physics.
ii) Gravity and the quantum: semiclassical gravity, AdS/CFT-fluid/gravity dualities, analogue gravity...
iii) Mathematical relativity.
iv) ... (Interdisciplinary physics)

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Are QNMs "structurally stable" under small perturbations?

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A further (bold!) question [JLJ, R.P. Macedo, L. Al Sheikh 21, ...]: :

Can we measure the '(effective) regularity' of spacetime
from the perturbation of quasi-normal mode (QNM) overtones?

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Perturbation theory on a Schwarzschild Black Hole: spherically symmetric case

Scalar, electromagnetic and gravitational perturbations reduced to (Minkowski) 1+1 wave equation for $\phi_{\ell m}(t, r_*)$ with a potential V_ℓ [Regge-Wheeler 57, Zerilli 70]:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\ , \ r^* \in]-\infty, \infty[$$

Schwarzschild quasi-normal modes

Convention for a "mode": $\phi_{\ell m}(t, r_*) \sim e^{i\omega t}\hat{\phi}_{\ell m}(r_*)$.
"Spectral" problem with **"outgoing boundary" conditions**:

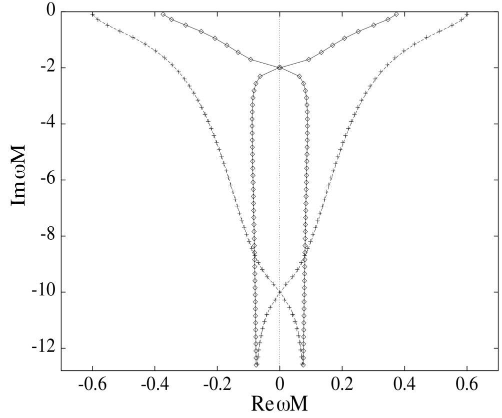
$$\left(-\frac{\partial^2}{\partial r_*^2} + V_\ell\right)\hat{\phi}_{\ell m} = \omega^2\hat{\phi}_{\ell m} \quad , \quad r^* \in]-\infty, \infty[$$

$$\hat{\phi}_{\ell m} \sim e^{-i\omega r_*} \ , \ (r_* \to \infty) \quad , \quad \hat{\phi}_{\ell m} \sim e^{i\omega r_*} \ , \ (r_* \to -\infty)$$

Time evolution stability: $\mathrm{Im}(\omega) > 0$. Exponential divergence of $\hat{\phi}_{\ell m}$ at $\pm\infty$.

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Schwarzschild gravitational QNMs



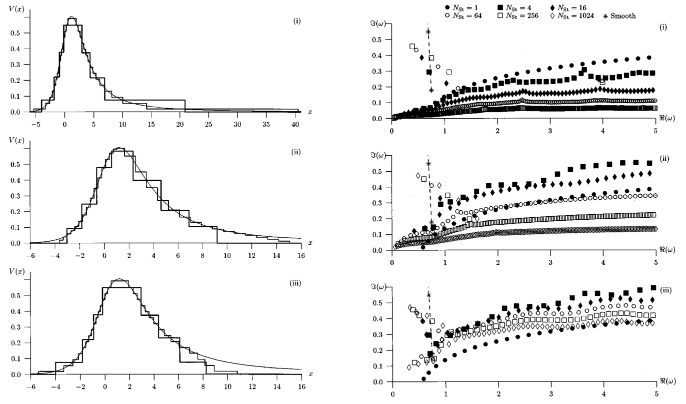
Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses)

[e.g. Kokkotas & Schmidt 99, Berti et al. 06, Konoplya & Zhidenko 11]

QNM frequencies ω_n are invariant probes into the background geometry

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Nollert's work on stair-case discretizations of Schwarzschild (revisited in [Daghigh, Green & Morey 20])



- Instability of the slowest decaying QNM (but ringdown "stability").
- Instabilities of "highly damped QNMs".

Posing the problem

Consider the operator on functions (defined on "appropriate" functional spaces) with non-compact-domain and with $V > 0$ with appropriate decay at infinity:

$$P_V = -\Delta + V$$

QNMs in the theory of Scattering Resonances

[Lax & Phillips, Vainberg; Sjöstrand, Zworski, Petkov, Iantchenko, ...many others; e.g. Dyatlov & Zworski 20]

Resolvent $R_V(\lambda) = (P_V - \lambda)^{-1}$ analytic $\mathrm{Im}(\lambda) > 0$. Scattering resonances: poles of the meromorphic extension of (truncated) $R_V(\lambda)$ to $\mathrm{Im}(\lambda) < 0$.

QNMs as a "proper" eigenvalue problem: **non-selfadjoint operators**

- "Complex scaling" [Simon 78, Reed & Simon 78, Sjöstrand...]: not the approach followed here.
- Hyperboloidal approach [Friedman & Schutz 75, Schmidt 93, Bizon, Zenginoglu 11, Vasy 13, Warnick 15, Ansorg & P.-Macedo 16, Gajic & Warnick 19, Bizon et al. 20, Galkowski & Zworski 20, ...] Problem in terms of "eingenvalue problem" of **non-selfadjoint operator** L:

$$\boxed{L\, u_{\ell m} = \omega\, u_{\ell m}} \quad , \quad u_{\ell m} \in H \text{ (Hilbert space)}$$

  - **Geometric boundary conditions**: Null infinity reached by hyperboloidal slices.
  - **Regularity conditions on** $u_{\ell m}$: choice of appropriate H, then $\omega \in \sigma_p(L)$.

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Spectral Theorem. Normal and 'non-normal' operators

Normal operators: Spectral Theorem

- **Normality**: denoting the adjoint matrix by $L^\dagger$, then L is normal iff
$$\boxed{[L, L^\dagger] = LL^\dagger - L^\dagger L = 0}$$
Matrix examples: symmetric, hermitian, orthogonal, unitary...
- **Spectral Theorem** ("moral statement"):
L is normal iff is unitarily diagonalisable.

Note: this depends on the adjoint $L^\dagger$, then on the Hilbert space (scalar product).

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- Completeness more difficult to study.
- Eigenvectors not necessarily orthogonal.
- **Spectral instabilities**.

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Methodology here: "exploration" stage

Numerical spectral methods: **Chebyshev polynomials** truncations.

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Example of spectral instability

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$$

acting on functions in $L^2([0,1])$, with homogeneous Dirichlet conditions (Chebyshev finite-dimensional matrix approximates).

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$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\mathrm{Random}} \quad , \quad a, b, c \in \mathbb{R}, \ ||E_{\mathrm{Random}}|| = 1$$

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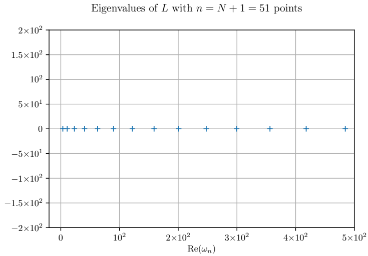

$$a = -1,\ b = 0,\ c = 1,\ \epsilon = 0$$

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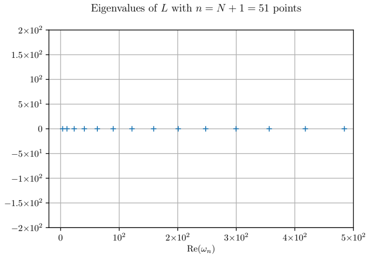

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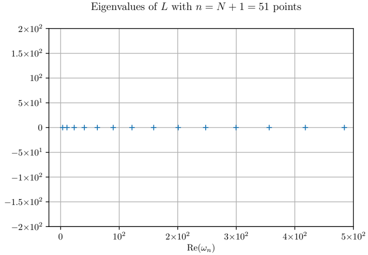

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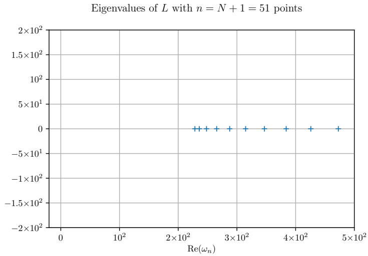

$a = -1, \; b = 30, \; c = 1, \; \epsilon = 0$

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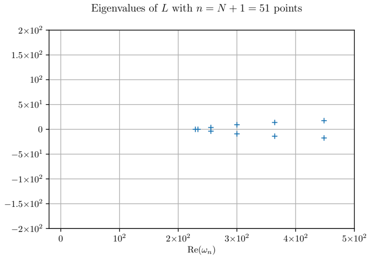

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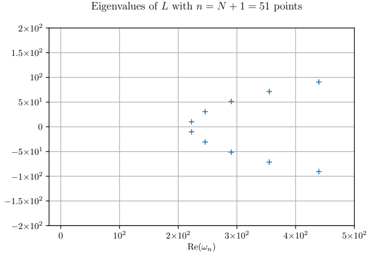

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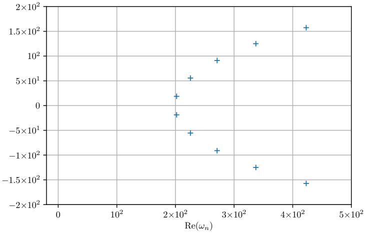
Eigenvalues of L with $n = N + 1 = 51$ points



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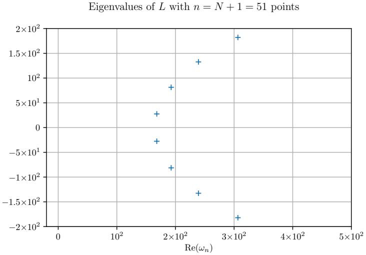

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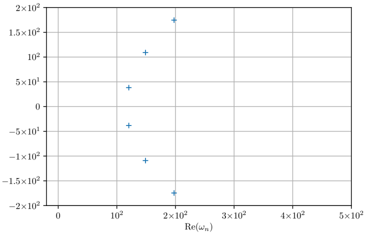

$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-2}$

Spectral (in)stability: eigenvalue condition number

Left- and right-eigenvectors, respectively u_i and v_i, of A

$$A^\dagger u_i = \bar{\lambda}_i u_i \ \ (\Leftrightarrow u_i^\dagger A = \lambda_i u_i^\dagger) \ \ , \ \ A v_i = \lambda_i v_i \ \ , \ \ i \in \{1, \ldots, n\} \ ,$$

Perturbation theory of eigenvalues [cf. Kato 80, ...; e.g. Trefethen, Embree 05]:

$$A(\epsilon) \ = \ A + \epsilon \delta A \ \ , \qquad ||\delta A|| = 1 \ .$$

$$|\lambda_i(\epsilon) - \lambda_i| = \epsilon \frac{|\langle u_i, \delta A \, v_i(\epsilon)\rangle|}{|\langle u_i, v_i\rangle|} \leq \epsilon \frac{||u_i|| \ ||\delta A \, v_i||}{|\langle u_i, v_i\rangle|} + O(\epsilon^2) \leq \epsilon \frac{||u_i|| \ ||v_i||}{|\langle u_i, v_i\rangle|} + O(\epsilon^2).$$

Eigenvalue condition number: $\kappa(\lambda_i)$

$$\boxed{\kappa(\lambda_i) = \frac{||u_i|| ||v_i||}{|\langle u_i, v_i\rangle|}}$$

Spectral (in)stability and Pseudospectrum

Pseudospectrum

Given $\epsilon > 0$, the ϵ-pseudospectrum $\sigma_\epsilon(L)$ of L is defined as [e.g Trefethen & Embree 05]:

$$
\begin{aligned}
\sigma_\epsilon(L) &= \boxed{\{\lambda \in \mathbb{C}, \text{ such that } \lambda \in \sigma(L + \delta L) \text{ for some } \delta L \text{ with } ||\delta L|| < \epsilon\}} \\
&= \{\lambda \in \mathbb{C}, \text{ such that } ||Lv - \lambda v|| < \epsilon \text{ for some } v \text{ with } ||v|| = 1\} \\
&= \{\lambda \in \mathbb{C}, \text{such that } ||(\lambda I - L)^{-1}|| > \epsilon^{-1}\}
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Normal case: bounds on the norm of the resolvent $R_L(\lambda) = (\lambda I - L)^{-1}$

Given $\lambda \in \mathbb{C}$ and $\sigma(L)$ the spectrum of L, it holds

$$||(\lambda I - L)^{-1}||_2 = \frac{1}{\text{dist}(\lambda, \sigma(L))}$$

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Non-normal case: bad control on the resolvent $R_L(\lambda)$. **Pseudospectrum**

The norm of the resolvent can become very large far from the spectrum:

$$||(\lambda I - L)^{-1}||_2 \leq \frac{\kappa}{\mathrm{dist}(\lambda, \sigma(L))}$$

where κ is a "condition number" assessing the lack of proportionality of 'left' and 'right' eigenvectors of L, and can become very large in the non-normal case.

Bauer-Fike theorem. Random perturbations

Pseudospectrum and condition number: Bauer-Fike theorem [e.g Trefethen & Embree 05]

Defining "tubular neighbourhood" of radius ϵ around $\sigma(A)$

$$\Delta_\epsilon(A) = \{\lambda \in \mathbb{C} : \mathrm{dist}\,(\lambda, \sigma(A)) < \epsilon\} \ ,$$

it holds: $\Delta_\epsilon(A) \subseteq \sigma_\epsilon(A)$. For normal operators: $\sigma_\epsilon(A) = \Delta_\epsilon(A)$.
Non-normal case, $\kappa(\lambda_i) \neq 1$, it holds (for small ϵ):

$$\sigma_\epsilon(A) \subseteq \bigcup_{\lambda_i \in \sigma(A)} \Delta_{\epsilon\kappa(\lambda_i) + O(\epsilon^2)}(\{\lambda_i\}) \ ,$$

Therefore $\sigma_\epsilon(A)$ larger tubular neighbourhood of radius $\sim \epsilon\kappa(\lambda_i)$.

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Random perturbations and Pseudospectrum

Random perturbations ΔL with $||\delta L|| < \epsilon$ "push" eigenvalues into $\sigma_\epsilon(A)$, providing an insightful and systematic manner of exploring $\sigma_\epsilon(L)$.

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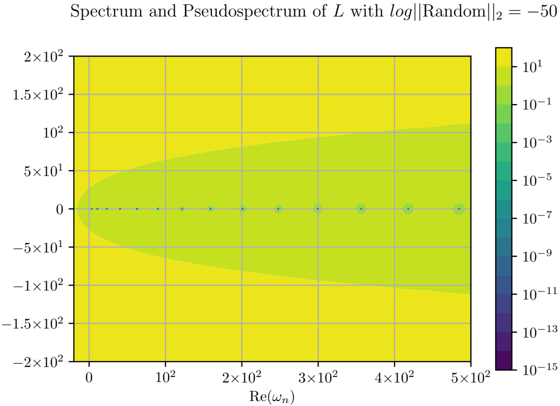
The 'role' of random perturbations [Sjöstrand 19; Hager 05, Montrieux, Nonnenmacher, Vogel,...]

Random perturbations improve the analytical behaviour of $R_L(\lambda)$!!!

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\rm Random}$



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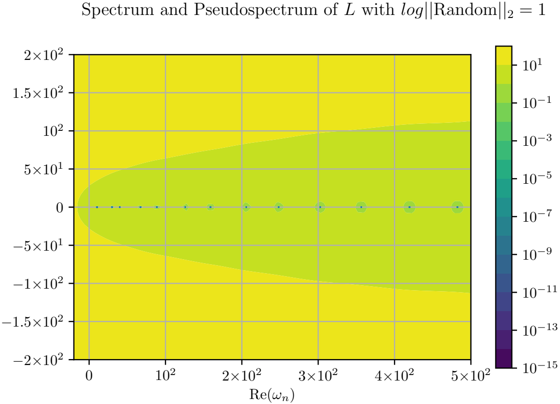

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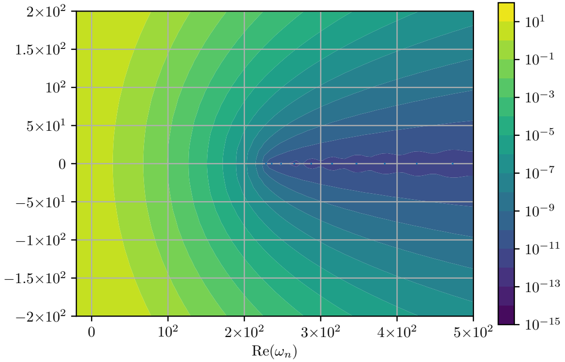
Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -15$

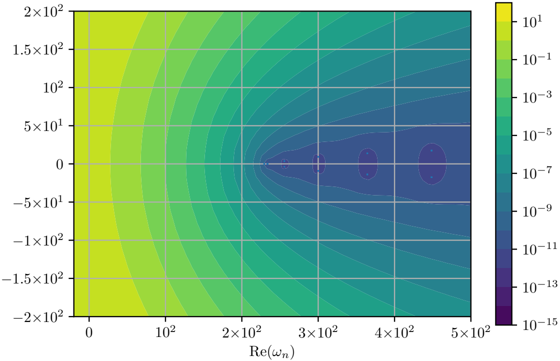


$a = -1, \ b = 30, \ c = 1, \ \epsilon = 0$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$

Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -10$



$a = -1,\ b = 30,\ c = 1,\ \epsilon = 10^{-10}$

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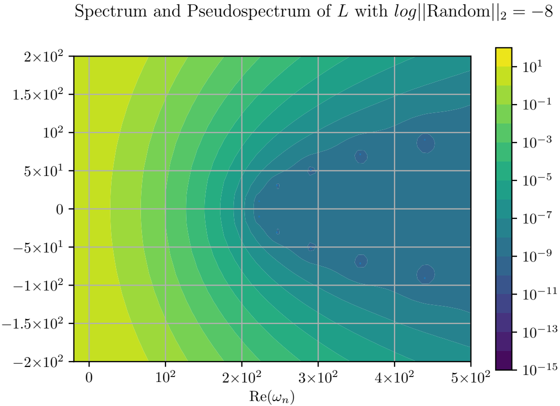
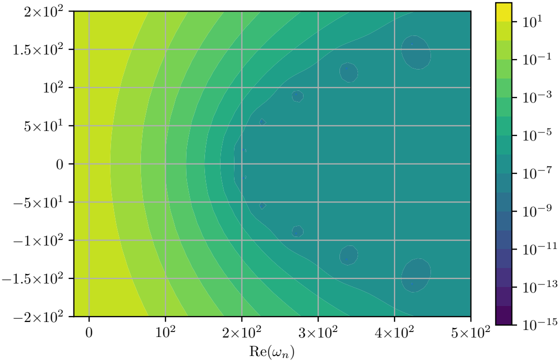


$a = -1,\ b = 30,\ c = 1,\ \epsilon = 10^{-8}$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -6$

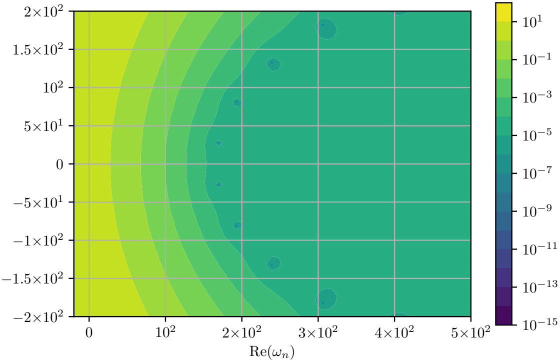


$a = -1, \ b = 30, \ c = 1, \ \epsilon = 10^{-6}$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\dfrac{d^2}{dx^2} + b\dfrac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$

Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -4$

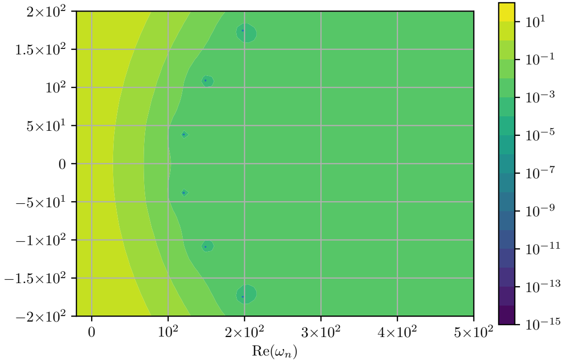


$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-4}$

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -2$



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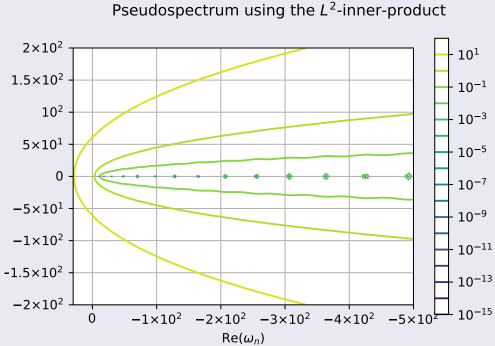
The relevance of the scalar product: assessing large/small

The illustrative operator: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$

- Non-selfadjoint in standard $L^2([0,1])$ for $b \neq 0$.
- Formally normal!
- Non-normal: domain of $L^\dagger L$ and $LL^\dagger$ different.
- But actually self-adjoint...

Cast in Sturm-Liouville form: selfadjoint for appropriate scalar product $\langle \cdot, \cdot \rangle_w$!!!



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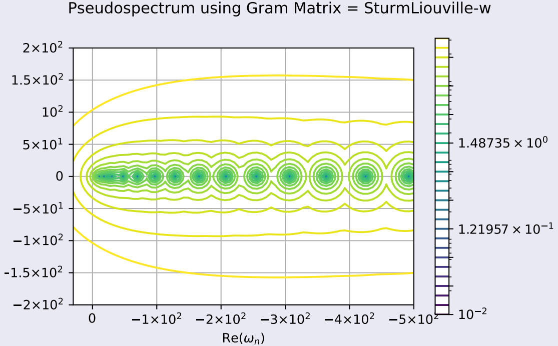
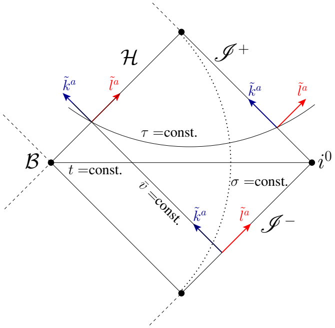


Scheme

Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to QNMs

- **Spectral problem**: homogeneous wave equation with purely outgoing boundary conditions.
- Outgoing BCs naturally imposed at $\mathscr{I}^+$.
- Outgoing BCs actually "incorporated" at $\mathscr{I}^+$:
  - Geometrically: null cones outgoing.
  - Analytically: BCs encoded into a singular operator, **"BCs as regularity conditions"**.
- **Eigenfunctions** do not diverge when $x \to \infty$: actually **integrable**. Key to Hilbert space.



Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

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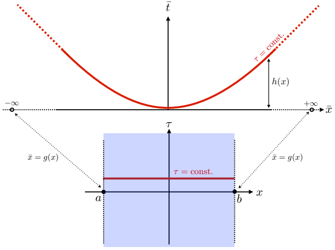

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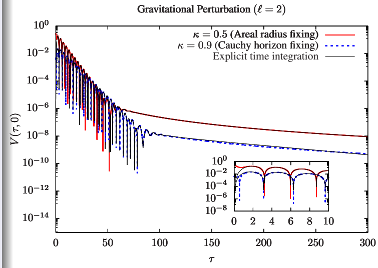

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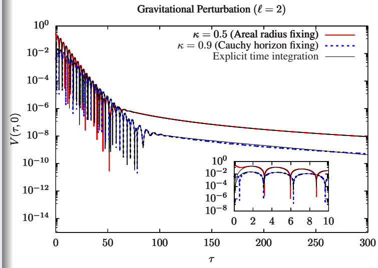

1. Wave problem in spherically symmetric asymptotically flat case

As starting point, consider the problem for a $\phi_{\ell m}$ mode in tortoise coordinates:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\ , \ r^* \in]-\infty, \infty[$$

Compactification along hyperboloidal slices

2. Choice of hyperboloidal foliation and compactification

Make the change to Bizoń-Mach variables [Bizoń & Mach 17]:

$$\left\{ \begin{array}{lcl} \tau &=& t - \ln\left(\cosh r_*\right) \\ x &=& \tanh r_* \end{array} \right. \quad , \quad \tau \in]-\infty, \infty[\ , \ x \in]-1, 1[$$

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3. We add the boundaries $y = \pm 1$.

3. Wave equation in hyperboloidal coordinates: no boundary conditions allowed

For $x = \pm 1$, $V_\ell = 0$. In the interior, $x \in]-1, 1[$:

$$\left(\partial_\tau^2 + 2x\partial_\tau\partial_x + \partial_\tau + 2x\partial_x - (1-x^2)\partial_x^2 + \tilde{V}_\ell\right)\phi_{\ell m} = 0 \ ,$$

with $\tilde{V}_\ell = \dfrac{V_\ell}{(1-x^2)}$.

Wave equation: reduction to first order system

4. Evolution equation in first order form

Introducing the auxiliary field

$$\psi_{\ell m} = \partial_\tau \phi_{\ell m} \ ,$$

we can write the wave equation in first-order form:

$$\partial_\tau \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline (1-x^2)\partial_x^2 - 2x\partial_x - \tilde{V}_{\ell m} & -(2x\partial_y + 1) \end{array} \right) \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} \ .$$

Spectral problem: first order formulation

5. Eigenvalue problem for a non-selfadjoint operator, no BCs

Our problem: study

$$L \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \omega \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} \quad , \quad L = \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array} \right) ,$$

where

$$L_1 = (1 - x^2)\partial_x^2 - 2x\partial_x - \tilde{V}_{\ell m} \ \ , \ \ L_2 = -(2x\partial_x + 1) \ .$$

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QNM problem as a “proper” eigenvalue problem. But...**Hilbert space?**

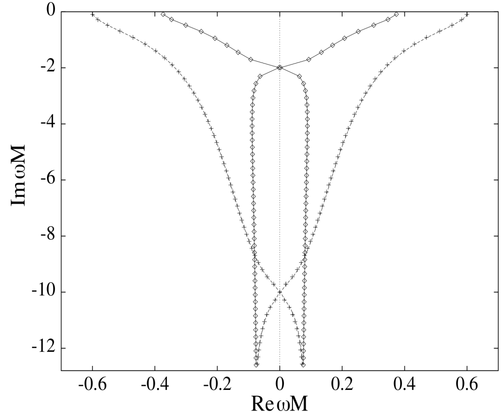
Spectral problem in a Hilbert space with **“Energy” scalar product** ($\tilde{V} > 0$):

$$\left\langle \begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix} \right\rangle_E = \int_{\Sigma_\tau}\left(\bar{\psi}_1\psi_2 + (1-x^2)\partial_x\bar{\phi}_1\partial_x\phi_2 + \tilde{V}\bar{\phi}_1\phi_2\right) d\Sigma_t$$

Scheme

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

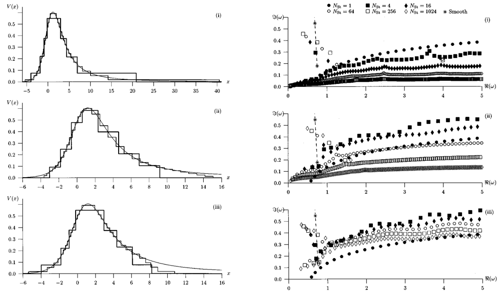
Schwarzschild gravitational QNMs



Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses) [e.g. Kokkotas & Schmidt 99]

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Nollert's work on stair-case discretizations of Schwarzschild (revisited in [Daghigh, Green & Morey 20])



- Instability of the slowest decaying QNM (but ringdowm "stability").
- Instabilities of "highly damped QNMs".
- Various interests in BH QNM perturbations:
  - i) "Dirty" astrophysical black holes [Leung et al. 97; Barausse, Cardoso & Pani 14;...]
  - ii) Quantum (highly damped QNMs/high frequency instability) [Hod 98; Maggiore 08; Babb, Daghigh & Kunstatter 11; Ciric, Konjik & Samsarov 19, Olmedo; ...].

Test-bed study: Pöschl-Teller potential

Pöschl-Teller potential [JLJ, Macedo & Al Sheikh 21] (toy-model in [Bizoń, Chmaj & Mach 20])

$$V = V_o \,\mathrm{sech}^2(r_*) = V_o(1-y^2) \quad \Longrightarrow \quad \boxed{\tilde{V} = V_o}$$

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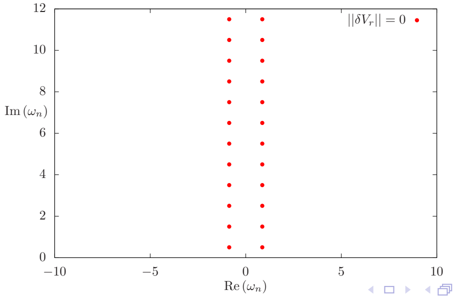
Pölsch-Teller QNM Perturbed-Spectra: Random Potential



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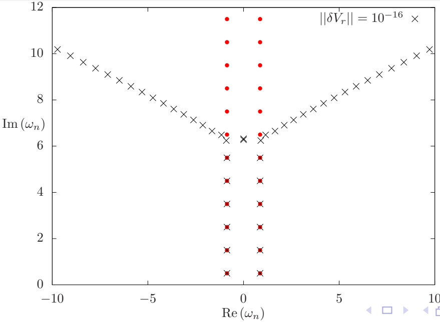

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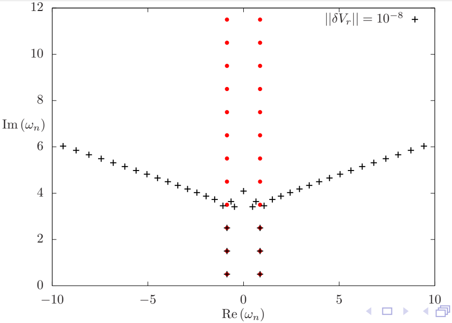

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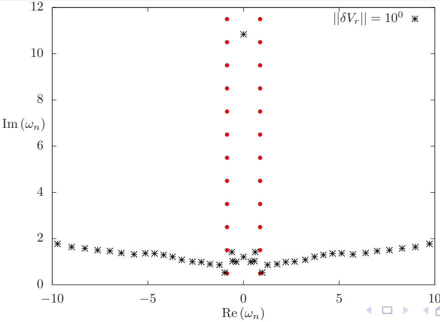

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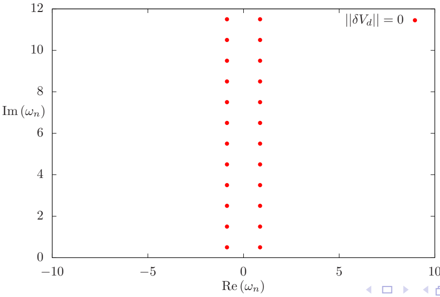
Pölsch-Teller QNM Perturbed-Spectra: Deterministic Potential



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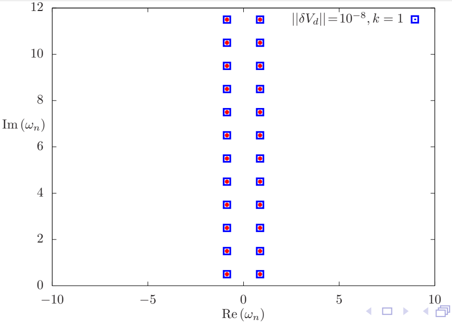

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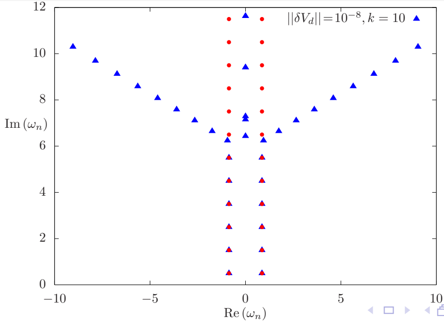

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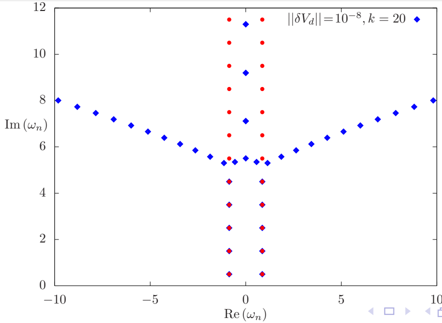

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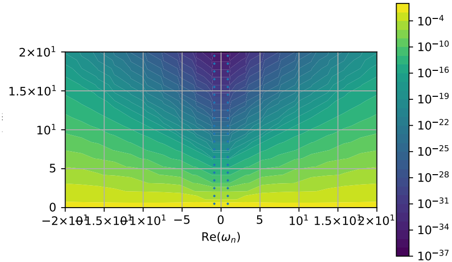
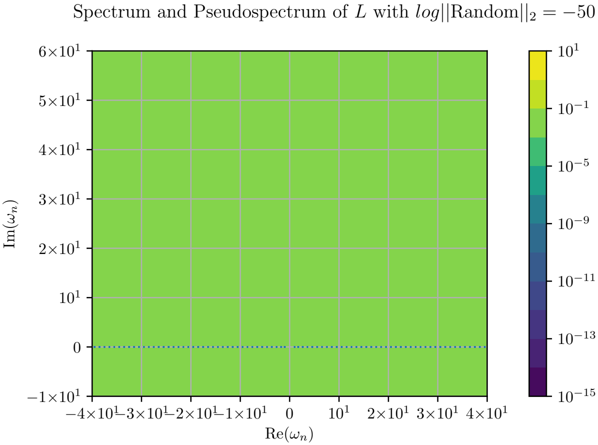
Poschel-teller Spectrum and Pseudospectrum of L



Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

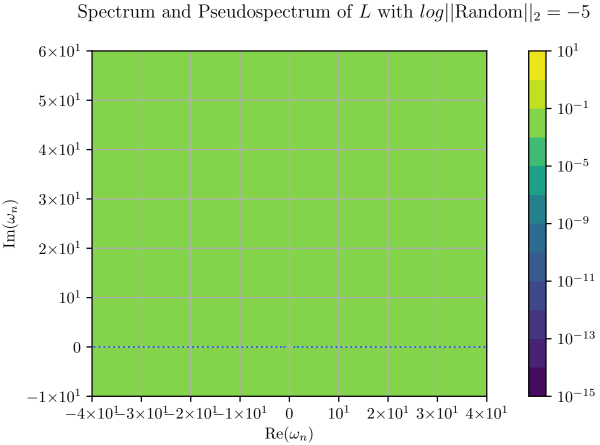


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 0}$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

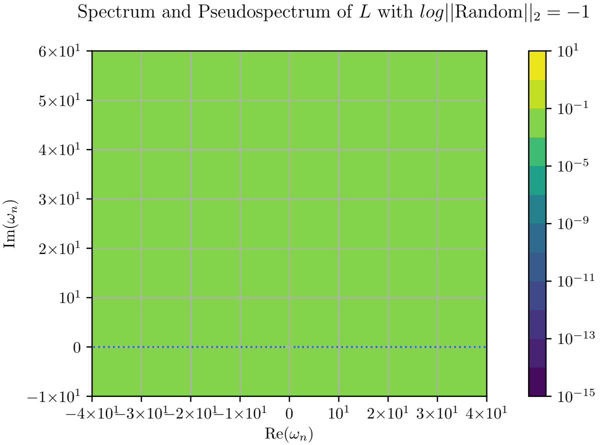


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-5}$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

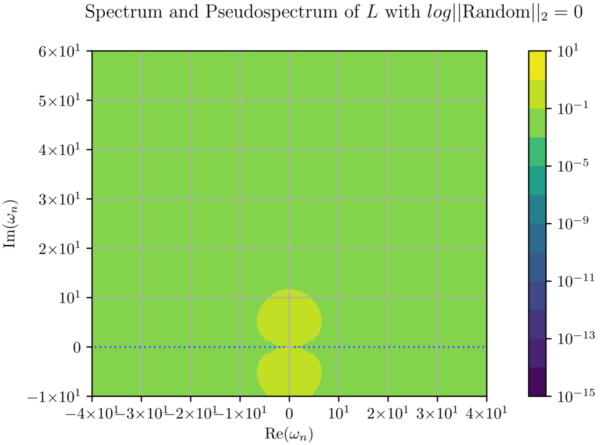


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-1}$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

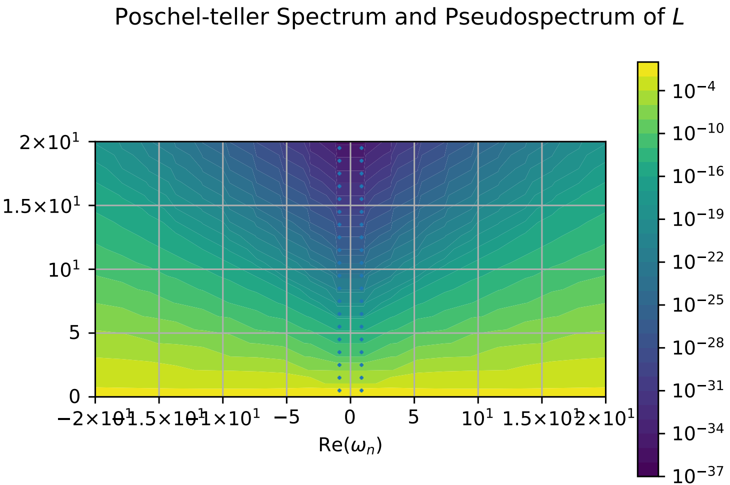


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^0$$

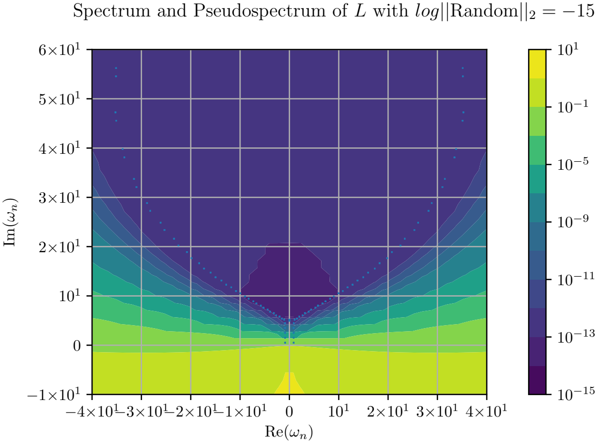
Test-bed study: Pöschl-Teller potential

QNM problem: $\hat{L}_2 \neq 0$.



Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

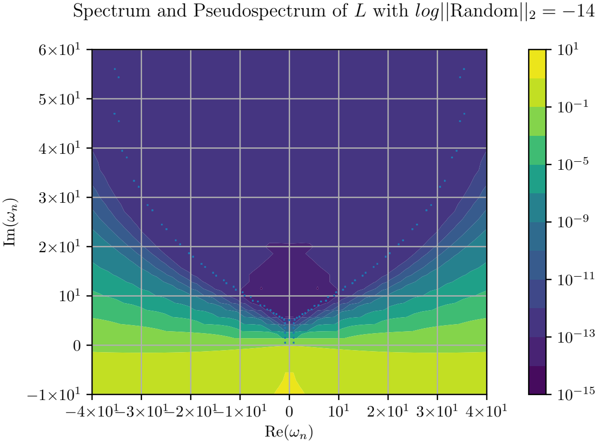


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-15}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

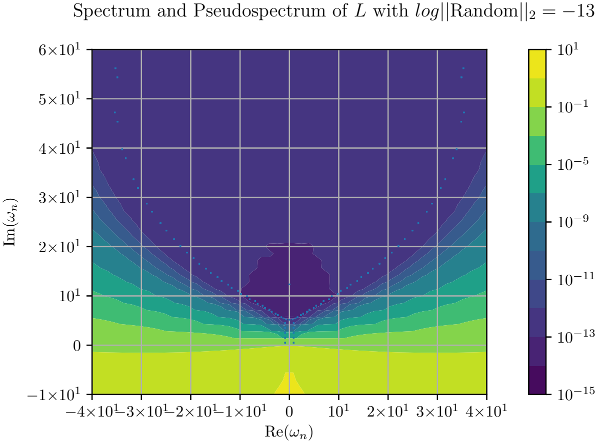


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-14}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

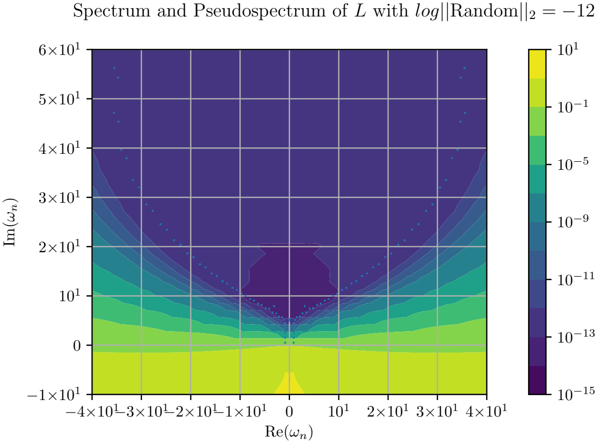


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-13}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



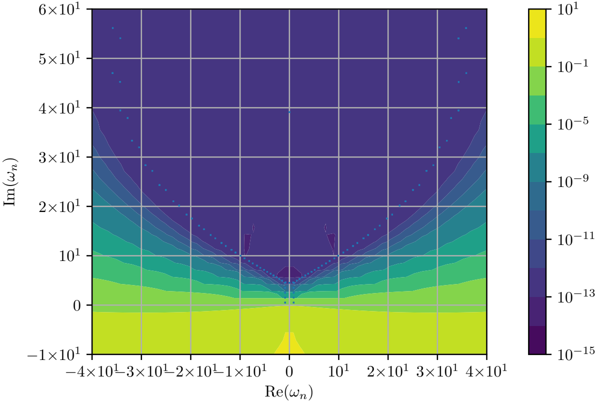
$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-12}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -11$

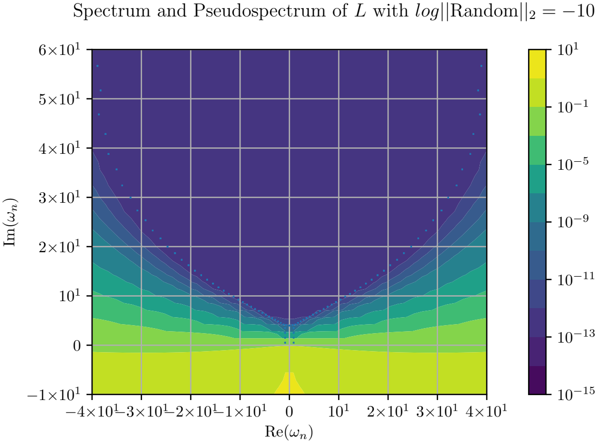


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-11}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



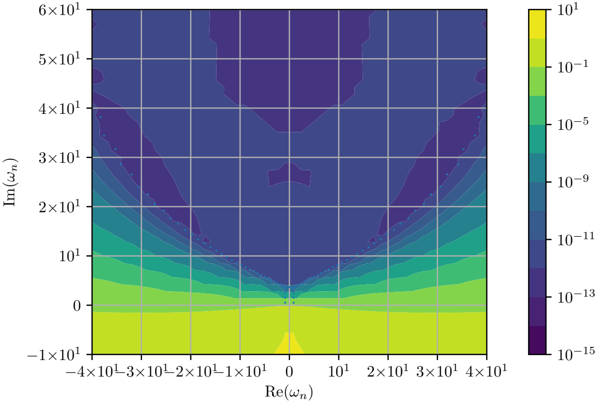
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-10}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -9$

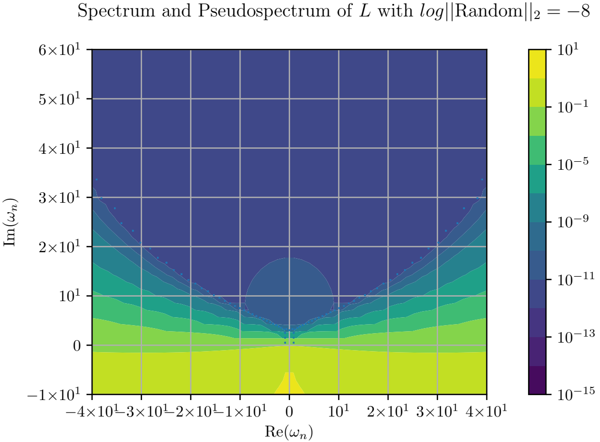


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-9}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



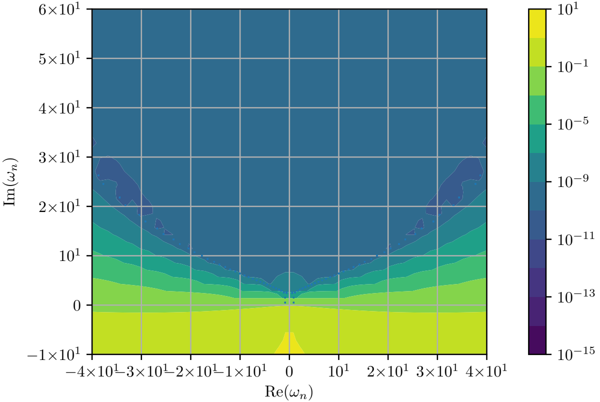
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-8}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -7$

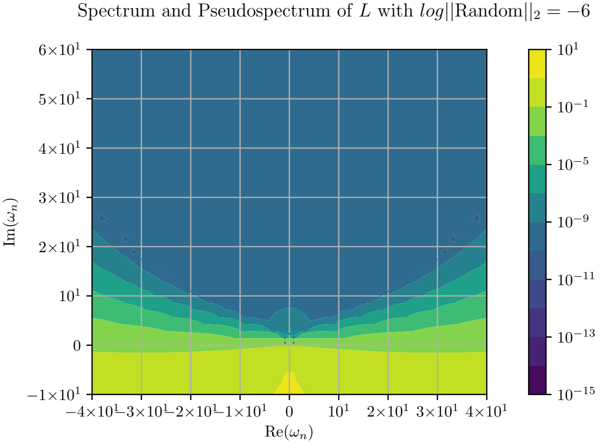


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-7}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

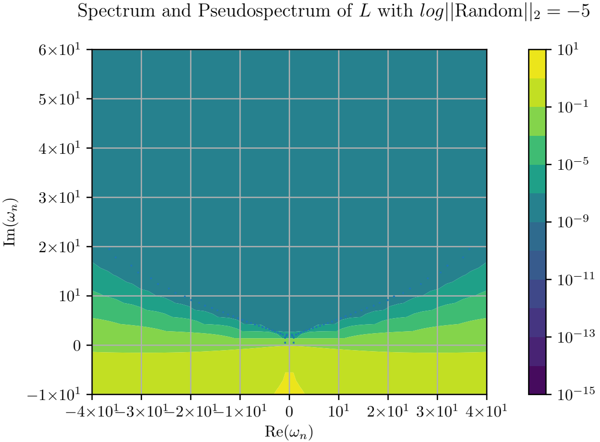


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-6}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

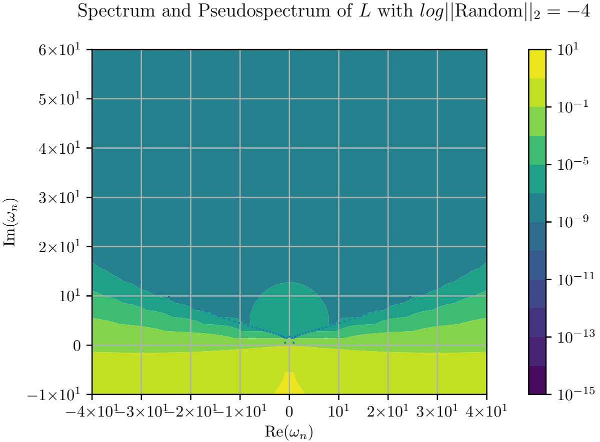


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-5}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

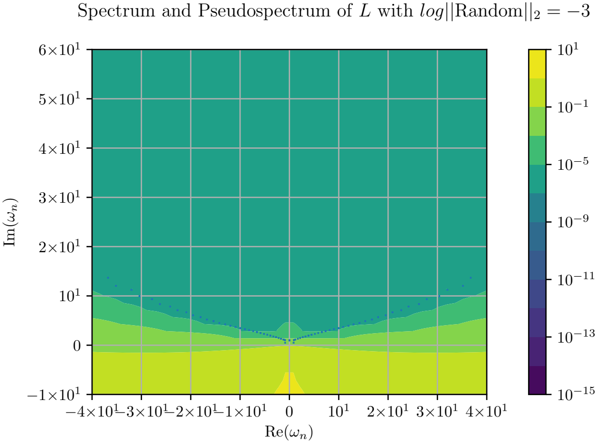


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-4}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

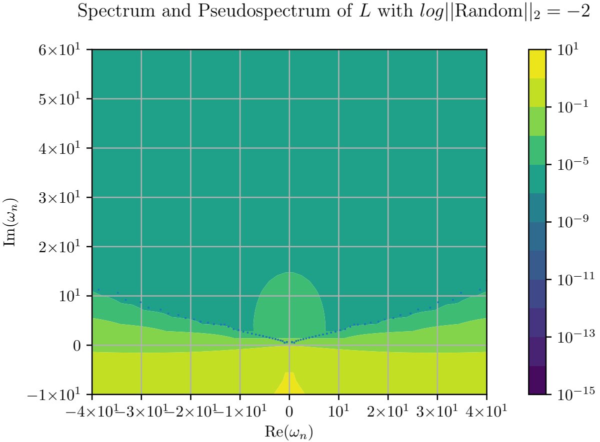


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-3}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

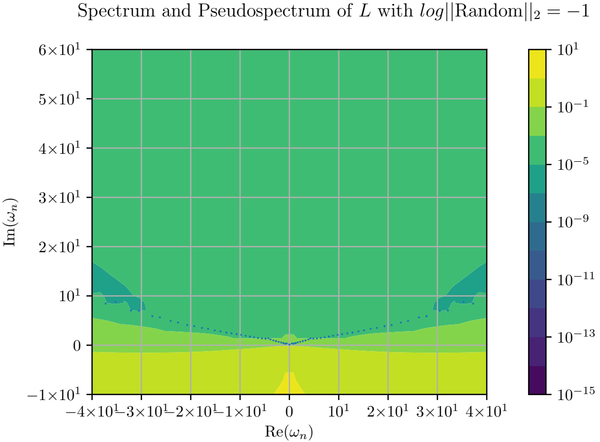


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-2}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

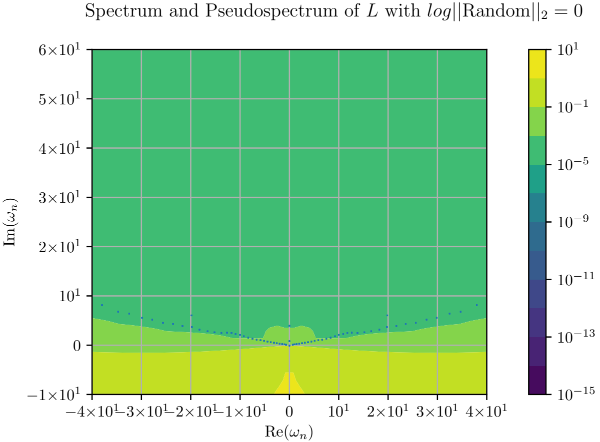


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-1}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^0$$

Test-bed study: Pöschl-Teller potential

From this we learn:

- **ALL overtones QNMs, unstable under high frequency perturbations**: instability grows as damping grows [JLJ, Macedo & Al Sheikh 21].
- Perturbations make **QNMs to "migrate" towards Pseudospectrum contour lines** ("extended pattern", cf. Bauer-Fike theorem).
- **Slowest damped QNM, stable under high frequency perturbations**:
  - Directly from the Pseudospectrum.
  - From the size of the needed perturbations.
- It can be repeated with deterministic **high frequency** k perturbations.
- For low frequency perturbations: much milder effect.

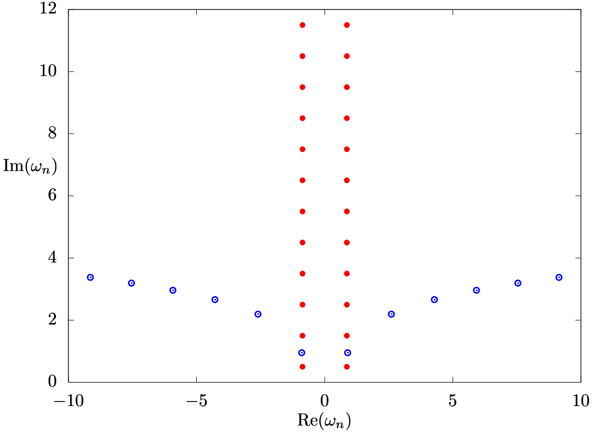
$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon \cos(2\pi k y) \quad , \quad k \gg 1$$

- It can also be seen:
  Slowest damped QNM unstable under "infrared perturbations" (cutting V at larges distances): **Nollert's instability of the fundamental QNM**.

Test-bed study: Pöschl-Teller potential

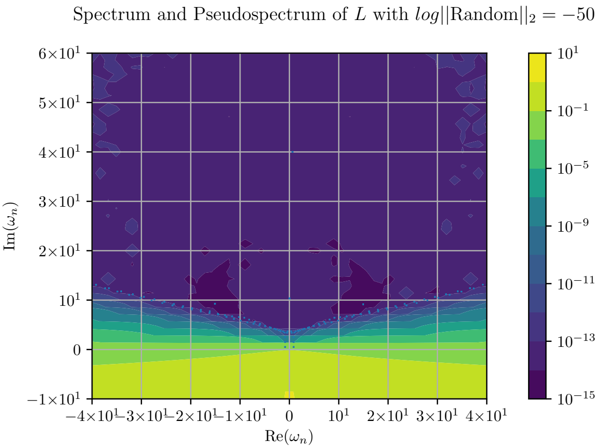
Infrared effect on fundamental QNM

Comparison between QNMs for Pöschl-Teller potential (red circles) and QNMs of "cut-Pöschl-Teller potential", set to zero at large distances (blue circles).



Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

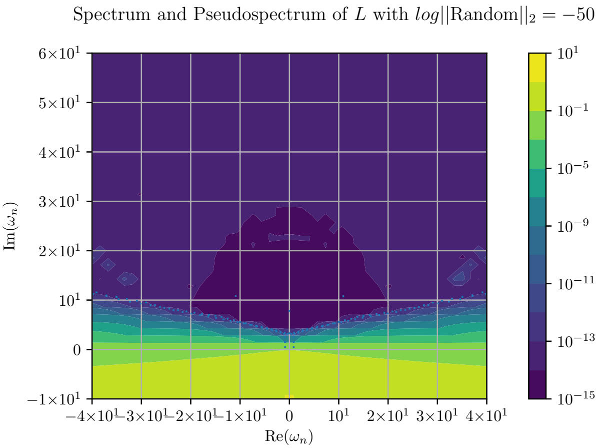


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$N = 120$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

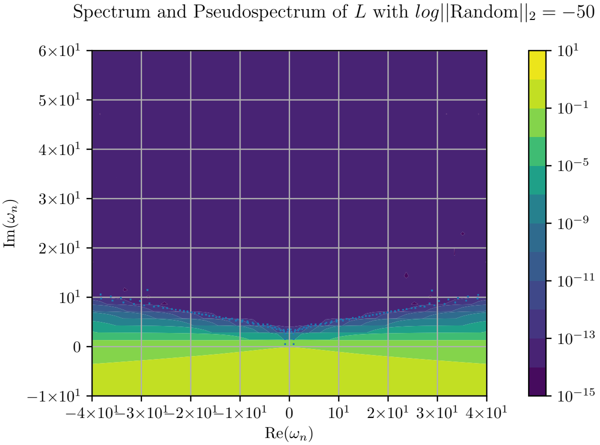


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$N = 140$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

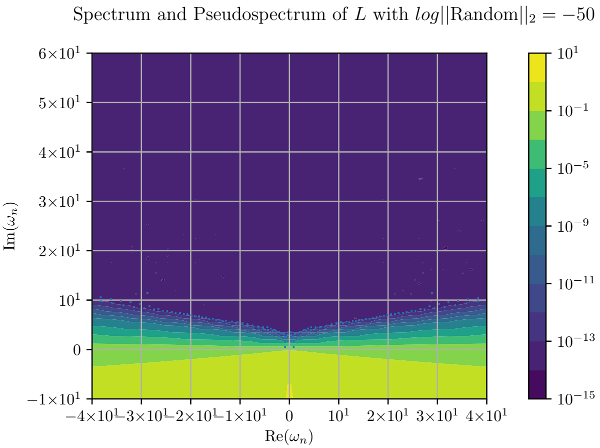


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$N = 160$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

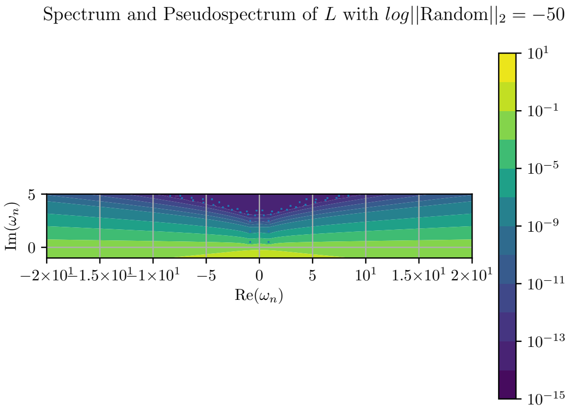


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$N = 160$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

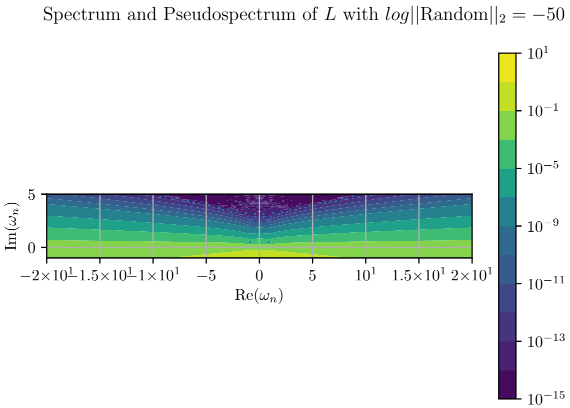


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{N = 160}$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

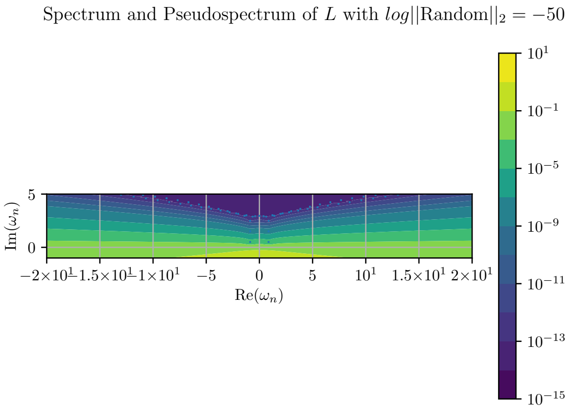


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{N = 180}$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.



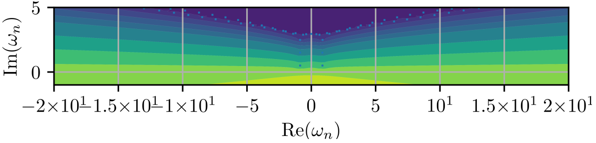
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

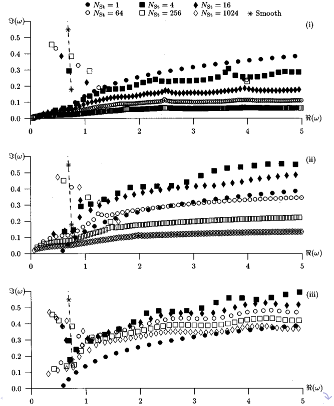
$$\boxed{N = 200}$$

Test-bed study: Pöschl-Teller potential

From this we learn:

- Pseudospectrum contour lines seem to converge to QNM curves observed by Nollert [cf. Nollert 96].
- Similarity to Neutron-Star "w-modes".





Test-bed study: Pöschl-Teller potential

From this we learn:

- Pseudospectrum contour lines seem to converge to QNM curves observed by Nollert [cf. Nollert 96].
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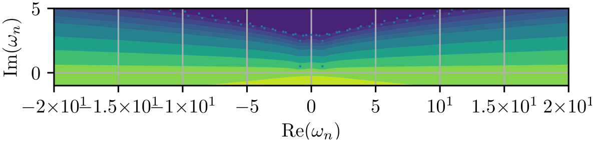
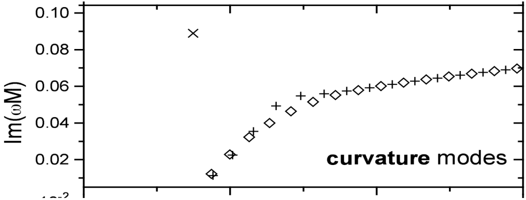
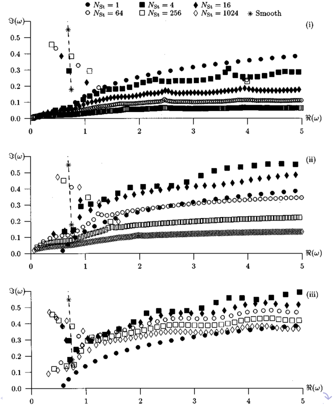


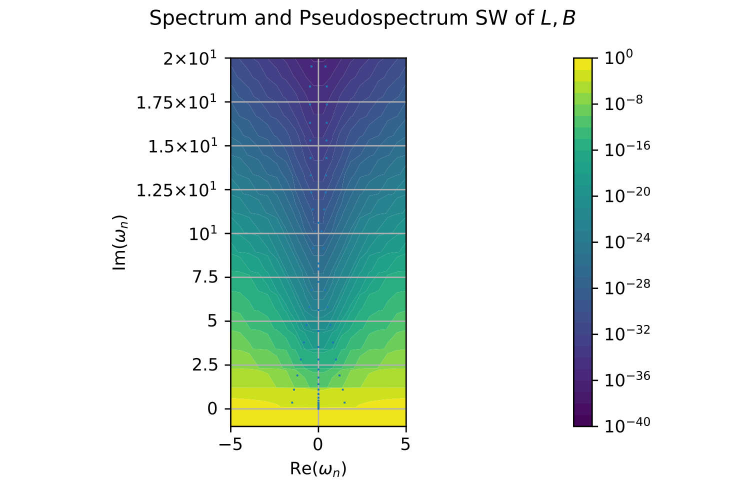


Curvature/w-modes in neutron stars [Kokkotas & Schmidt 99]



Schwarzschild QNMs

Schwarzschild Pseudospectrum: same qualitative behaviour (note: branch cut)



Highly damped QNMs unstable, slowest decaying QNMs stable.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Starting point: (scalar) wave equation in "tortoise" coordinates

On a stationary spatime (with timelike Killing ∂_t):

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0\ ,$$

Dimensionless coordinates: $\bar{t} = t/\lambda$ and $\bar{x} = r_*/\lambda$ (and $\bar{V}_\ell = \lambda^2 V_\ell$),

Conformal hyporboloidal approach

$$\left\{\begin{array}{lcl} \bar{t} & = & \tau - h(x) \\ \bar{x} & = & f(x) \end{array}\right. .$$

- $h(x)$: implements the hyperboloidal slicing, i.e. $\tau = \mathrm{const.}$ is a horizon-penetrating hyperboloidal slice Σ_τ intersecting future $\mathscr{I}^+$.
- $f(x)$: spatial compactification between $\bar{x} \in [-\infty, \infty]$ to $[a, b]$.
- Timelike Killing: $\lambda\partial_t = \partial_{\bar{t}} = \partial_\tau$.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

First-order reduction: $\psi_{\ell m} = \partial_\tau \phi_{\ell m}$

$$\boxed{\partial_\tau u_{\ell m} = i L u_{\ell m}} \quad , \quad \text{with} \quad u_{\ell m} = \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$L_1 = \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)}$$

$$L_2 = \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right)$$

with $w(x) = \dfrac{f'^2 - h'^2}{|f'|} > 0$, $p(x) = \dfrac{1}{|f'|}$, $q(x) = |f'|\, V_\ell$, $\gamma(x) = \dfrac{h'}{|f'|}$.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Spectral problem

Taking Fourier transform, dropping (ℓ, m) (convention $u(\tau, x) \sim u(x)e^{i\omega\tau}$):

$$\boxed{L\, u_n = \omega_n\; u_n} \; .$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$L_1 = \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)}$$

$$L_2 = \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right)$$

Conformal hyperboloidal approach: **No boundary conditions**

It holds $p(a) = p(b) = 0$, L_1 is “singular”: **BCs “in-built” in** L.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Scalar product

Natural scalar product (where $\tilde{V}_\ell := q(x) > 0$):

$$\langle u_1, u_2 \rangle_{_E} = \frac{1}{2}\int_a^b \Big(w(x)\bar{\psi}_1\psi_2 + p(x)\partial_x\bar{\phi}_1\partial_x\phi_2 + \tilde{V}_\ell\bar{\phi}_1\phi_2 \Big) dx \ ,$$

associated with the "total energy" of ϕ on Σ_t, defining the **"energy norm"**

$$||u||^2_{_E} = \langle u, u \rangle_{_E} = \int_{\Sigma_\tau} T_{ab}(\phi, \partial_\tau\phi) t^a n^b d\Sigma_\tau \ ,$$

Spectral problem of a non-selfadjoint operator

- Full operator L: not selfadjoint.
- L_2: dissipative term encoding the energy leaking at $\mathscr{I}^+$.
- L selfadjoint in the non-dissipative $L_2 = 0$ case.

Non-normal operators spectral tools: "energy norm" for Pseudospectrum.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Adjoint operator

$$L^\dagger = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 + L_2^\partial \end{array}\right)$$

where

$$L_2^\partial = 2\frac{\gamma}{w}\Big(\delta(x-a) - \delta(x-b)\Big)$$

Loss of "self-adjointness" happens at the boundaries (as expected)

Loss of "self-adjointness": an "infrared" phenomenon

"Ultraviolet" QNM overtone instability connected with a essentially asymptotic ("infrared") structure.

Later discussion: "missing degrees of freedom and asymptotic symmetries".

Application to Pöschl-Teller

Conformal compactification

$$\left\{ \begin{array}{lcl} \bar{t} & = & \tau - \frac{1}{2}\ln(1-x^2) \\ \bar{x} & = & \mathrm{arctanh}(x) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{lcl} \tau & = & \bar{t} - \ln\left(\cosh \bar{x}\right) \\ x & = & \tanh \bar{x} \end{array} \right.$$

mapping $[-\infty, \infty]$ to $[a, b] = [-1, 1]$.

Spectral problem

Operators in L, with potential $V(x) = V_o \mathrm{sech}^2(\bar{x})$ (with $V_o = 1$):

$$\begin{array}{lcl} L_1 & = & \partial_x\left((1-x^2)\partial_x\right) - 1 \\ L_2 & = & -\left(2x\partial_x + 1\right) \ . \end{array}$$

where:

$$w(x) = 1 \ \ , \ \ p(x) = (1-x^2) \ \ , \ \ q(x) = \frac{V}{1-x^2} =: \tilde{V}(x) \ \ , \ \ \gamma(x) = -x.$$

Application to Schwarzschild

Schwarzschild potential

Axial (Regge-Wheeler) case (also for polar (Zerilli) parity):

$$V_\ell^s = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right) \ ,$$

where

- $r_* = r + 2M\ln(r/2M - 1)$
- $s = 0, 1, 2$, respectively, to the scalar, electromagnetic and gravitational cases.

Numerical Chebyshev methods: analyticity of $V(x)$

Bizoń-Mach coordinates used in Pöschl-Teller not well adapted now: the potential is non-analytic in x, spoiling the accuracy of Chebyshev's methods.
Same problem for the polar (Zerilli) case.

Solution

We resort rather to the 'minimal gauge' hyperboloidal slicing [Ansorg, Macedo 16; Macedo 18] guaranteeing the analyticity of the Schwarzschild potential.

Application to Schwarzschild

Conformal compactification: "minimal gauge" [Ansorg, Macedo 16; Macedo 18]

$$\begin{cases} \bar{t} &= \tau - \frac{1}{2}\left(\ln\sigma + \ln(1-\sigma) - \frac{1}{\sigma}\right) \\ \bar{x} &= \frac{1}{2}\left(\frac{1}{\sigma} + \ln(1-\sigma) - \ln\sigma\right) \end{cases} ,$$

mapping $[-\infty, \infty]$ to $[a, b] = [0, 1]$ (we use σ, rather than x).

Spectral problem

Operators in L:

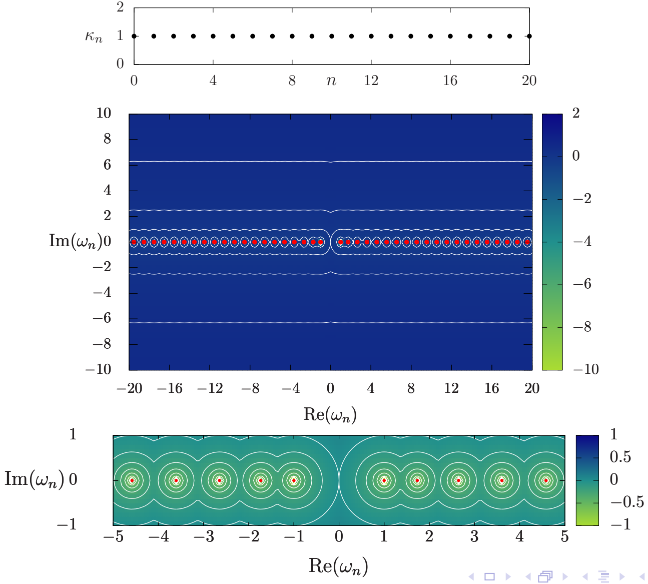
$$\begin{aligned} L_1 &= \frac{1}{1+\sigma}\left[\partial_\sigma\left(\sigma^2(1-\sigma)\partial_\sigma\right) - \left(\ell(\ell+1) + (1-s^2)\sigma\right)\right] \\ L_2 &= \frac{1}{1+\sigma}\left((1-2\sigma^2)\partial_\sigma - 2\sigma\right) . \end{aligned}$$

where:

$$w(\sigma) = 1+\sigma \ , \ p(\sigma) = \sigma^2(1-\sigma) \ , \ q(\sigma) = \frac{V}{\sigma^2(1-\sigma)} =: \tilde{V}(\sigma) \ , \ \gamma(\sigma) = 1 - 2\sigma^2 .$$

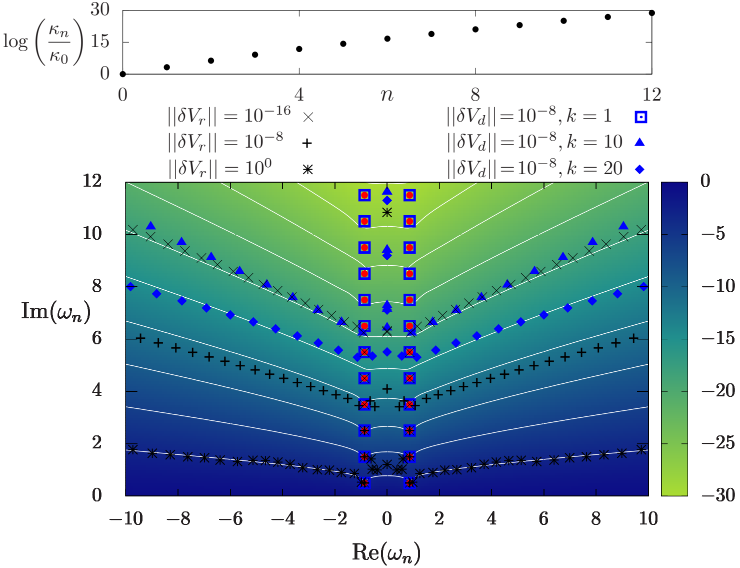
Trying to understand: the "current picture"... in pictures

Pöschl-Teller, $L_2 = 0$:



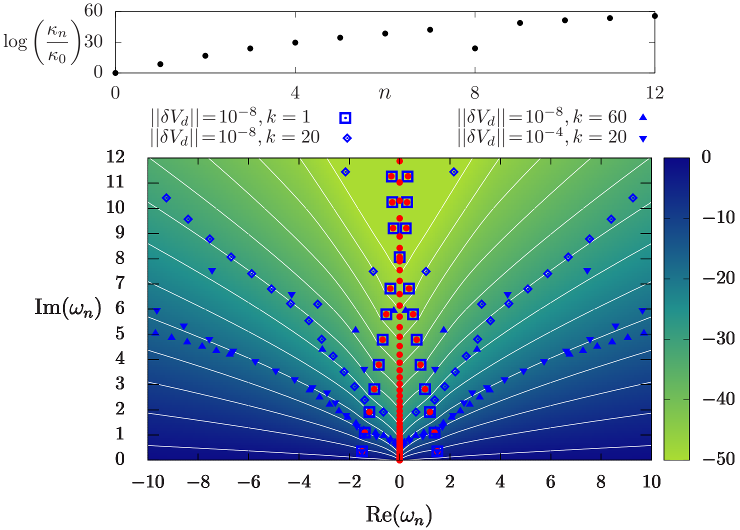
Trying to understand: the "current picture"... in pictures

Pöschl-Teller, $L_2 \neq 0$:



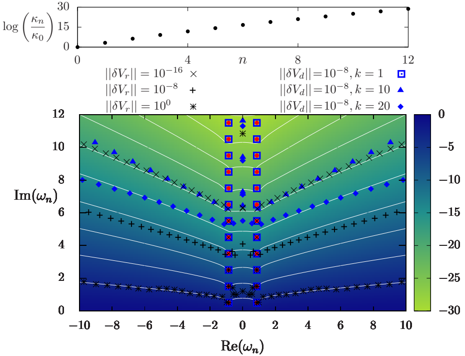
Trying to understand: the "current picture"... in pictures

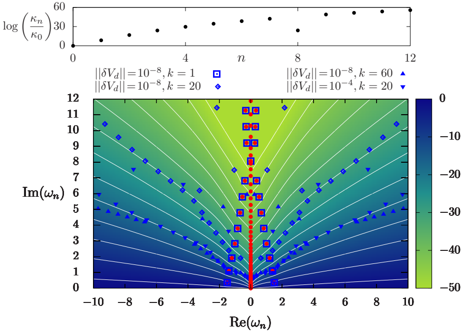
Schwarzschild ($s = 2$, $\ell = 2$):



Trying to understand: the "current picture"... in pictures

Comparison Pöschl-Teller versus Schwarzschild:





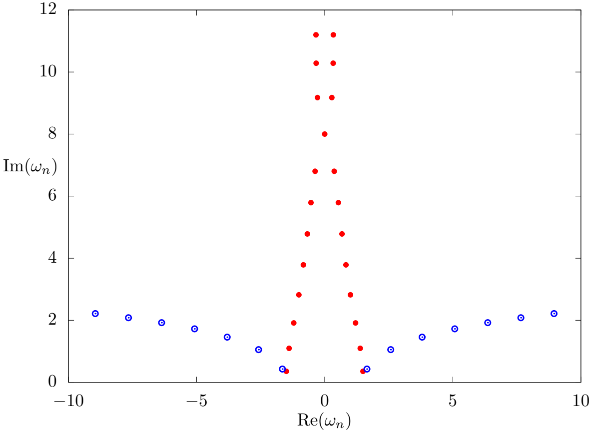
Remarks:

- High frequency perturbations: random δV_r, deterministic $\delta V_d \sim \cos(2\pi k x)$.
- **Fundamental QNM: ultraviolet stable, infrared unstable**.
- **Perturbed QNMs "migrate" towards $\epsilon-$contour lines of Pseudospectra.**
- **'Universality' phenomenon?**

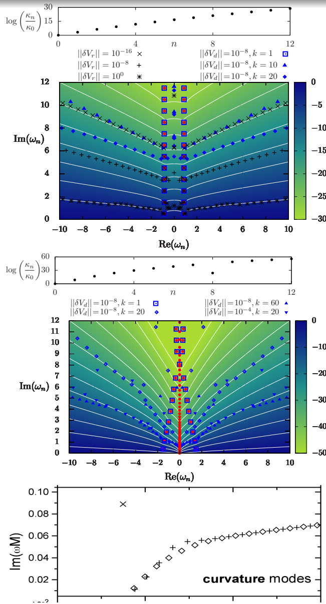
Trying to understand: the "current picture"... in pictures

Infrared effect on fundamental QNM: Schwarzschild case

Comparison between QNMs for Schwarzschild potential (red circles) and QNMs of "cut-Schwarzschild", set to zero at large distances (blue circles).



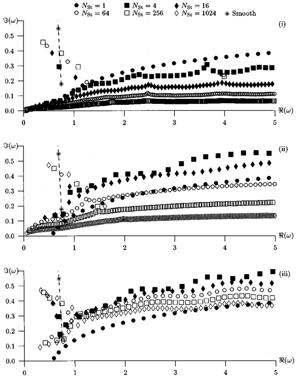
Trying to understand: the "current picture"... in pictures



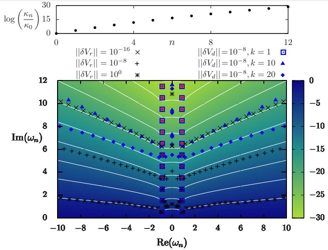
Black Hole and Neutron Star QNMs

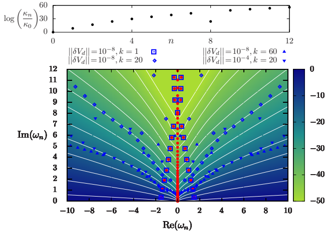
Comparison with:

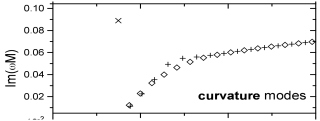
- Nollert's high-frequency Schwarzschild perturbations.
- Nollert's remark on Neutron Stars (w-modes) curvature QNMs.



Trying to understand: the "current picture"... in pictures



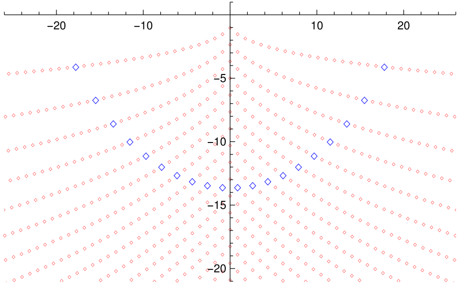




"Duality" QNMs (long-range potentials) and Regge poles (compact obstacles)?

QNM of an spherical obstacle [Stefanov 06]:

- Red-diamonds: fixed "n", running angular ℓ.
- Blue-diamonds: fixed ℓ (here $\ell = 20$), running n.



Scheme

Connecting with Lax-Phillips: Keldysh expansion

Hilbert space: Keldysh's expansion of the resolvent [Keldysh 51, 71]

Given v_j and w_j (right- . left and right eigenvectors of L), with $\langle w_n, v_n\rangle = -1$, the resolvent of L can be expanded in a domain Ω around the poles as

$$(L-\lambda)^{-1} = \sum_{\lambda_j \in \Omega} \frac{|v_j\rangle\langle w_j|}{\lambda - \lambda_j} + H(\lambda) \quad , \quad \lambda \in \Omega \setminus \sigma(L)$$

QNM resonant expansions, "1st-order time reduction" [Al Sheikh, JLJ, Gasperin 21, 22]

The field u satisfying $\partial_\tau u = iLu$, with $u(\tau=0,x) = u_0$, can be written in an "asymptotic expansion" as

$$u(\tau,x) \quad = \quad \sum_{j=1}^{N} e^{i\omega_j\tau}\kappa_j \langle \hat{w}_j | u_0\rangle_{E} \hat{v}_j + E_N(\tau;S)$$

$$\text{with} \qquad ||E_N(\tau;S)||_{E} \leq C_N(a,L)e^{-a\tau}||S||_{E}$$

Note the presence of the condition number κ_j.

Connecting with Lax-Phillips: Keldysh expansion

Hilbert space: Keldysh's expansion of the resolvent [Keldysh 51, 71]

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QNM resonant expansions, "scattered field" [Al Sheikh, JLJ, Gasperin 21, 22]

In terms of the scattered field satisfying the 2nd order equation

$$\phi(\tau,x) \sim \sum_n e^{i\omega_j\tau} a_n \hat{\phi}^R_j(x) \quad , \quad \phi(t,x) \sim \sum_n e^{i\omega_j t} a_n e^{i\omega_j h(x)} \hat{\phi}^R_j(x)$$

with $\quad a_j = \frac{\kappa_j}{2}\left(\langle \hat{\phi}^L_j, \varphi_0\rangle_{H^1_{(V,p)}} - i\omega_j \langle \hat{\phi}^L_j, \varphi_1\rangle_{(2,w)}\right)$

that recovers the expression in terms of normal modes in the selfadjoint (normal) case: $\kappa_j = 1$, $\hat{\phi}^R_j(x) = \hat{\phi}^L_j(x)$.

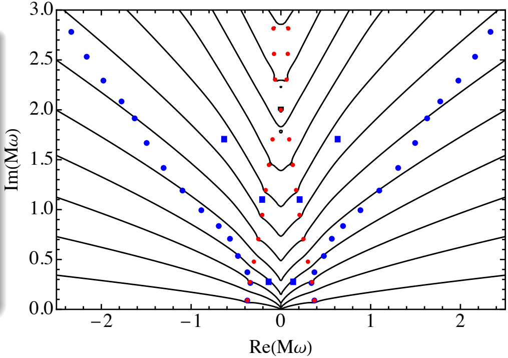
Are the perturbed QNMs in the ringdown signal? Yes

Extracting QNM: Mode Filtering

[JLJ, Macedo, Al Sheikh 21]

$$\Phi^N_{\mathrm{spec}}(t) := \sum_{n=0}^{N} \mathcal{A}_n e^{i\omega_n t}$$

$$\mathcal{F}^N(t) = \Phi_{\mathrm{evol}}(t) - \Phi^N_{\mathrm{spec}}(t)$$



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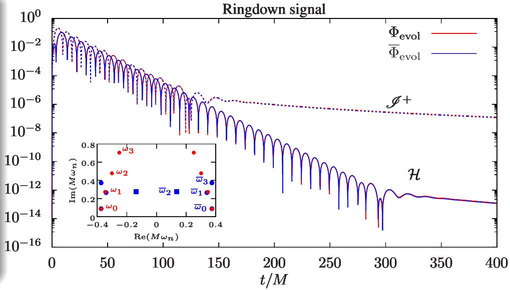

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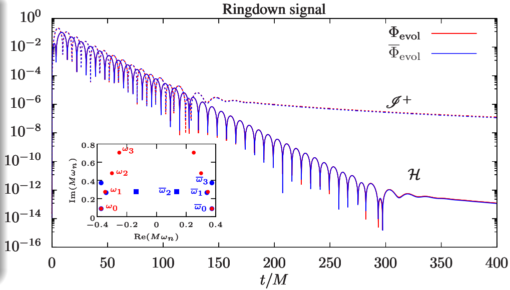
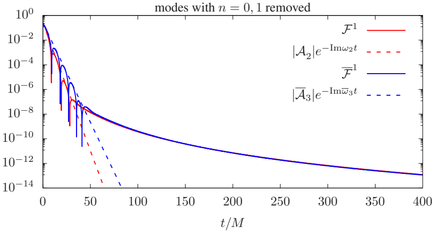
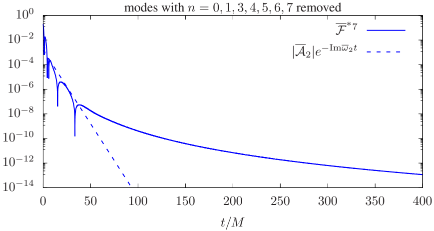


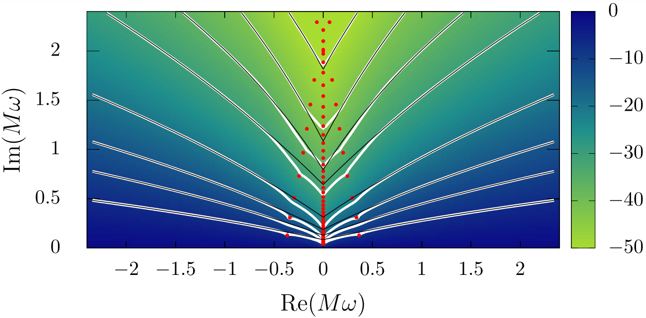




Emerging picture: "Universality" of compact-object QNMs

Logarithm QNM-free regions: Pseudospectrum contour lines [JLJ, Macedo, Al Sheikh 21]

- QNM overtones: ultraviolet (high-frequency) unstable.
- Fundamental QNM: ultraviolet (high-frequency) stable, infrared unstable. (**Warning**: "Flea on the Elephant" effect for "trapping potentials" [Jona-Lasinio et al. 81, 84; Helffer & Sjöstrand 83, 85; Simon 85; Cheung et al. 21, Konoplya & Zhidenko 22...])
- Logarithmic $|\mathrm{Re}(\omega)| \gg 1$ asymptotics: $\mathrm{Im}(\omega) \sim C_1 + C_2 \ln(|\mathrm{Re}(\omega)| + C_3)$



Emerging picture: "Universality" of compact-object QNMs

C^p regularity: Regge QNM branches, logarithmic asymptotics

- Compact support potential: **theorem** by Zworski (Weyl law) [Zworski 87] (Note: this **fully** explains Nollert's results [Nollert 96]
- Non-compact support, $C^{p<\infty}$ potential: WKB Berry's treatment [Berry 82].
- Related results by Nollert-Price [Nollert & Price 99]
- Polytropic neutron stars: w-modes [Zhang, Wu & Leung 11]

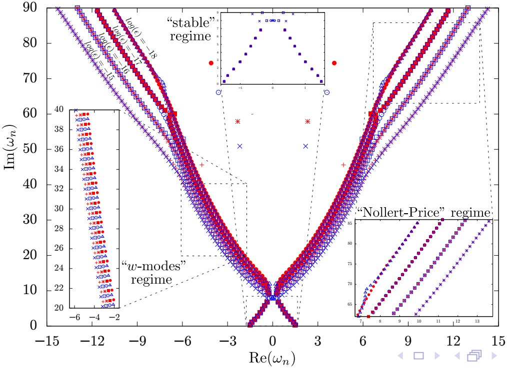
$$
\begin{aligned}
\mathrm{Re}(\omega_n) &\sim \pm\left(\frac{\pi}{L}n + \frac{\pi\gamma_p}{2L}\right) \\
\mathrm{Im}(\omega_n) &\sim \frac{1}{L}\left[\gamma \ln\left(\left(\frac{\pi}{L}n + \frac{\pi\gamma_p}{2L}\right) + \frac{\pi\gamma_\Delta}{2L}\right) - \ln S\right]
\end{aligned}
$$

- **"Length scale"** L, **"Small structure" coefficient** γ, **"Strength coefficient"** S.
- *"Real shift"* γ_Δ.
- *"Parity term"* γ_p.

Emerging picture: "Universality" of compact-object QNMs

Regular perturbations: "Nollert-Price-like" branches and "inner QNMs"

- Always above the pseudospectra log-lines: **BH QNM Isospectrality Loss**.
- "Nollert-Price-like": different branches, transitions at critical values.
- "Inner modes": appearing at the critical values. Weyl law.
- Nollert-Price branches towards logarithm pseudospectra as perturbation frequency increases: **Do they get to the logarithmic pseudospectra lines?**



From Burnett conjecture to a Regge QNM conjecture

Burnett's conjecture: high-frequency limit of spacetimes [Burnett 89; Huneau, Luk 18, 19]

The limit of high-frequency oscillations of a vacuum spacetime is effectively described by an effective matter spacetime described by massless Einstein-Vlasov.

Luk & Rodnianski: "null shells" as low-regularity problems in Einstein equations

- Spikes are more efficient in triggering instabilities [Gasperin & JLJ 21]:

$$|\delta g_{ab}| \, ||\delta L||_{_E} \sim \max_{\mathrm{supp}(\delta g)} |\partial \delta g_{ab}|^2 \sim \max_{\mathrm{supp}(\delta g)} \rho_E(\delta g; x)$$

- Luk & Rodnianski's treatment [Luk & Rodnianski 20]: Allowing for "concentrations", the infinite high-frequency limit is a **low regularity limit**

$$g_n \to g_\infty \ \text{ in } C^0 \cap H^1 \quad , \quad \partial g_n \to \partial g_\infty \ \text{ in } L^2$$

Regge QNM branches conjecture: a low-regularity problem [JLJ, Macedo, AL Sheikh 21, Gasperin & JLJ 21]

In the limit of infinite frequency, generic ultraviolet perturbations push BH QNMs in Nollert-Price branches to asymptotically logarithmic branches along the QNM-free region boundary, exactly following the Regge QNM asymptotic pattern.

QNMs and norms: "Definition" versus "Stability" problems

Definition of QNMs: we can (need to) choose the norm to control QNMs

- Ansorg & Macedo [Ansorg & Macedo 16]: if C^∞-regularity, then every point in the upper complex plane is an eigenvalue. Need of "more control".
- Warnick & Gajic [Warnick & Gajic 19]: Fundamental contribution by identifying in **Gevrey-2** regularity classes in (**asymptotically flat**) hyperboloidal framework (complemented in [Galkowski & Zworski 20] in a complex scaling setting).
  Hilbert spaces of $(\sigma, 2)$-Gevrey functions on $[0, \rho_0]$: f, $g \in C^\infty((0, \rho_0); \mathbb{C})$

$$\langle f, g\rangle_{G^2_{\sigma,1,\rho_o}} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n!^2(n+1)!^2} \int_0^{\rho_0} \partial^n_\rho f \partial^n_\rho \bar{g} d\rho$$

Stability of QNMs: we cannot choose the norm, the "physics" chooses for us

The system chooses what is a "big and small" perturbation: **here, the energy**.

$$||(\phi, \psi)||_E = E = \int_\Sigma T_{ab} t^a n^b d\Sigma \sim ||\phi||_{H^1_V} + ||\psi||_{L^2}$$

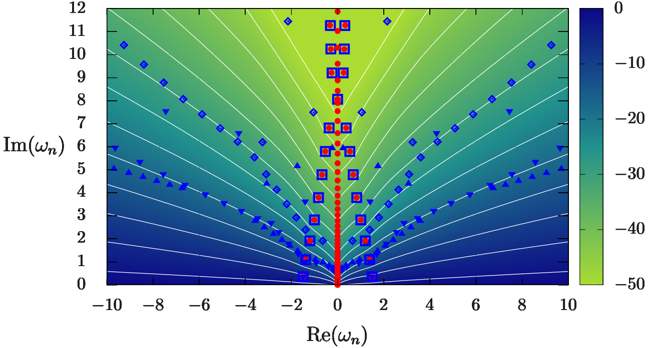
"Energy-Pseudospectrum = Chart of the Gevrey Ocean"

QNMs and norms: "Definition" versus "Stability" problems

Towards a reconciliation: the Gevrey Ocean

A small high-frequency perturbation in the energy, is a huge one in Gevrey terms.

- For a given V, **QNM eigenfunctions are defined to be in class Gevrey-2**.
- **Small (in the energy norm) perturbations** δV **of** V **lead to very different QNM** ω_n**'s**, respective eigenfunctions being indeed Gevrey-2.



"Energy-Pseudospectrum = Chart of the Gevrey Ocean"

The role of boundaries: the actual "infrared" phenomenon

The instability "is not" (only) in the potentials...

.

- **The Pseudospectrum is non-trivial for $V = 0$. That is: Minkowski.**
- The Pseudospectrum is trivial for $V = 0$ and $L_2 = 0$.

"A reading" of this:

- **Potential QNM instability is actually encoded in the "asymptotic hyperbolic structure"**, namely in the null asymptotic boundaries (both $\mathscr{I}^+$ and the BH horizon).
  (This "resonates" with Konoplya & Zhidenko's results: near BH horizon, but also $\mathscr{I}^+$).
- Stationary BHs and Minkowski: "zero-measure" solutions in an "enlarged phase space", with "frozen" small-scale (ultraviolet) degrees of freedom.
- Ultraviolet perturbations: "probe" generic solutions in such larger phase space, where QNM migrate to new universal QNM branches.
- (Effects of) additional "ultraviolet" degrees of freedom encoded in "infrared" asymptotic structures. **Which "infrared" asymptotic structures?**

Towards a geometric description of QNMs

Compactified hyperboloidal slicing: geometric data

i) Choose a slicing $\{\mathcal{S}^\infty_\tau\}$ of $\mathscr{I}^+$. Given the generator k^a along $\mathscr{I}^+$ consider, at each slice $\mathcal{S}^\infty_\tau$, the only null normal ℓ^a to $\mathcal{S}^\infty_\tau$ satisfying $k^a \ell_a = -1$. Repeat on the BH horizon.

ii) Choose a function γ on null infinity sections $\mathcal{S}^\infty_\tau$ (respectively, also a function γ on $\mathcal{S}^\mathcal{H}_\tau$ in BH spacetimes) and consider the null vector $\gamma^a = \gamma \ell^a$, outgoing from $\mathcal{M}$.

Repeat on the BH horizon.

iii) Extend the vector γ^a arbitrarily to the bulk (under appropriate conditions).

iv) Slices Σ_τ in $\tilde{\mathcal{M}}$ are compact manifolds with boundary, with boundaries given by spheres.

v) Resulting data:

$$(\{\Sigma_\tau\}, \gamma_{ab}, K_{ab}, w; \{\mathcal{S}^\infty_\tau\}, \{\mathcal{S}^\mathcal{H}_\tau\}, \gamma)$$

Evolution equations

$$\partial_\tau u = iLu \quad , \quad u = \begin{pmatrix} \phi \\ \psi = \partial_\tau \phi \end{pmatrix} \quad , \quad L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)$$

$$L_1 = \frac{1}{w}\big(\tilde{\Delta} - \tilde{V}(x)\big), \qquad L_2 = \frac{1}{w}\big(2\gamma \cdot \tilde{D} + \tilde{D} \cdot \gamma\big)$$

Adjoint infinitesimal evolution operator

$$L^\dagger = L + L^\partial \quad , \quad L^\partial = \frac{1}{i}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & L_2^\partial \end{array}\right)$$

$$L_2^\partial = -2\frac{\gamma^a k_a}{w}\left(\delta_{S^{\mathcal{H}}} - \delta_{S^\infty}\right) = -2\frac{(\gamma k^a)\ell_a}{w}\left(\delta_{S^{\mathcal{H}}} - \delta_{S^\infty}\right)$$

Missing degrees of freedom and BMS symmetry

Non-selfadjointness and missing degrees of freedom

- **Non-selfadjointness in dynamical problems generically reflects that some degrees of freedom are missing or are being lost**: the system is not isolated or is part of a larger system, leading to non-conservative dynamics.
- To cast such systems in terms of a selfadjoint problem, one must "complete" the system by adding an appropriate set of degrees of freedom.

How could we render our BH perturbation problem selfadjoint?

- In our scattering case, degrees of freedom flow to infinity and through the BH horizon.
- Consider adding formal degrees of freedom at future null infinity $\mathscr{I}^+$ and at the BH horizon $\mathcal{H}$, in such a way that they account for the degrees of freedom and energy leaving the system.
- Placing **"Geiger counters" at the infinite and horizon boundaries**, such that the total number of degrees of freedom is conserved.

Coupling boundary and bulk degrees of freedom

Being anchored to the boundary, such boundary degrees of freedom would be non-propagating ones (in the bulk)

- Flux at boundaries:

$$F(\tau) = \int_{\mathcal{S}^{\mathcal{H}}} \gamma |\partial_\tau \phi|^2 dS + \int_{\mathcal{S}^{\infty}} \gamma |\partial_\tau \phi|^2 dS$$

  Bondi-Sachs flux of energy-momentum at null infinity if: i) γ is built from the $\ell = 0, 1$ spherical harmonics and ii) we identify $\partial_\tau \phi$ as a news function $\mathcal{N}$.

- Here, γ arbitrary function on $\mathcal{S}^{\infty}_\tau$: supertranslation in the asymptotic BMS group.

$$\xi^a = \gamma k^a \quad , \quad \mathcal{L}_k \gamma = 0$$

  corresponds to a supertranslation in the asymptotic BMS group.

- Flux term (arbitrary $\gamma(x)$) [“Coupling” action: BCs as action boundary terms (Roberto!!!]:

$$H_\xi = \int_{\mathscr{I}^+} \left(\gamma \partial_\tau \phi \partial_\tau \phi + \gamma \partial_\tau \phi + \partial_\tau \phi \Delta_{\mathcal{S}^\infty} \gamma\right) d\tau dS$$

  corresponding to the **BMS supertranslation** ξ^a **and generating symplectomorphisms in the (appropriate) phase space of degrees of freedom that “live on the boundary”**.

Missing degrees of freedom and BMS symmetries

A "heuristic" proposal

Missing degrees of freedom can be encoded in BMS supertranslation degrees of freedom:

i) **BMS supertranslations** ξ^a **would stand as a dynamical symmetry**, acting on (and generating) the **phase space of degrees of freedom on the boundary** (the "clicks" in the Geiger counter analogy), with γ **a degree of freedom living on the null boundaries**.

ii) The whole geometric construction starts from the **choice of slicing of null boundaries**. This is **arbitrary**, but all choices are **related by an appropriate BMS supertranslation**.

iii) **The set of degrees of freedom** (ϕ, γ) **would be complete.** By this we mean that, as ϕ flows away, degrees of freedom γ are "activated" through a coupling controlled by the Hamiltonian (1), so **a total energy of the type** $E_o = E(\phi, \psi) + H_\xi$ **would be conserved**.

Missing degrees of freedom and BMS symmetries

"Hopes":

- From the perspective of the extended space of degrees of freedom (ϕ, γ), one could gain **insight into the observed instability of BH QNMs**.

  Paraphrasing this in terms of inverse scattering theory: **additional BMS degrees of freedom could provide necessary additional data at the boundaries** —complementary to (transmission and) reflection coefficients and (possible) bound energy levels— needed to determine the scattering potential.

- Conservative formulation of the dynamics can be constructed in the extended space of degrees of freedom (ϕ, γ), then a notion of **global spacetime normal modes**, as eigenfunctions of the time evolution generator, could be envisaged. This could be of **interest both from a phenomenological and from a fundamental (quantization) perspective.**

- Role of "metriplectic (Leibniz) structures" in geometry of dissipative systems? [Kabel & Wieland 22, ...]:

Conclusions and Perspectives

Conclusions

- Numerical evidence of **instability of QNM overtones** under high-frequency perturbations in the effective potential: **Nollert-Price BH QNM branches**.

  Fundamental QNM ultraviolet stable. Infrared unstable.

- Independent support from **Pseudospectrum** of non-perturbed operator.

- Hints towards **Universality of compact-object QNM branches** in the infinite high-frequency limit: a low regularity problem.
- **Strong modifications to BH QNMs in a purely classical GR setting.**

Others:

- ϵ**-dual QNM expansions**: from "energy-stability" of time-domain signal.
  - i) Keldysh expansion: on non-perturbed Kerr QNMs.
  - ii) Keldysh expansion: on perturbed BH QNMs.

- **BH QNM isospectrality loss**: patterns.

- ...

Conclusions and Perspectives

Perspectives: "Ultraviolet & Infrared" Conjectures

- **"Ultraviolet": Regge QNM branches conjecture.**

  Generic small-scale perturbations push QNM towards universal Regge branches, reaching them in the infinite "perturbation frequency" limit.

$$\mathrm{Re}(\omega_n) \sim \pm\left(\frac{\pi}{L}n + \frac{\pi\gamma_p}{2L}\right)$$

$$\mathrm{Im}(\omega_n) \sim \frac{1}{L}\left[\gamma \ln\left(\left(\frac{\pi}{L}n + \frac{\pi\gamma_p}{2L}\right) + \frac{\pi\gamma_\Delta}{2L}\right) - \ln S\right]$$

Can we measure the regularity of spacetime?

- **"Infrared": QNM Pseudospectrum encoded in asymptotic symmetries**

  "Missing" degrees of freedom are accounted for by asymptotic symmetry charges at $\mathscr{I}^+$ (BMS, ...?) and the BH horizon (extended-BMS [Ashtekar, Lewandowski et al.. 22]). The universal features of QNM branches can be understood in terms of the phase space of these dynamical symmetries.

- **"Infrared-ultraviolet": BMS-charges as inverse scattering data**

  Asymptotic (BMS?) charges provide necessary scattering data for Inverse-Scattering reconstruction, in particular the (low-regularity) small-scale structure of the spacetime [cf. Petropoulos, Mason, Taujanskas, Geiler, Taujanskas; Damour, Barnich].

Conclusions and Perspectives

Perspectives: Open Problems

- ϵ**-dual QNM expansions and BH spectroscopy**:
  - i) On non-perturbed Kerr QNMs: "average" BH background properties.
  - ii) On perturbed BH QNMs: properties of perturbations (astrophysical perturbations, **small-scale fundamental gravity physics**...)
- Relation to **"Flea on the Elephant"** and Konoplya & Zhidenko [cf. Konoplya].
- **Cosmic Censorship Conjecture in Reissner-Nordström de-Sitter**: Assessment of Cauchy horizon instability and fundamental QNM instability.
- Pseudospectrum with H^p **Sobolev** (and Gevrey-2) norms. Stability?
- QNM instability in **Kerr**: specific features. Role of superradiance?
- **Metriplectic** framework of dissipative systems for QNM description.
- Isospectrality loss: **integrability and QNM instability universality**.
- **Proofs:** from heuristic and numerical evidences to **Theorems**.

Conclusions and Perspectives

Universality in Binary Black Hole Dynamics:
An Integrability Conjecture

- José Luis Jaramillo, Badri Krishnan, Carlos F. Sopuerta, "Universality in Binary Black Hole Dynamics: An Integrability Conjecture"; arXiv:2305.08554
  Honorable Mention in 2023 Gravitational Research Foundation competition essay.
- José Luis Jaramillo, Badri Krishnan, "Airy-function approach to binary black hole merger waveforms: The fold-caustic diffraction model"; arXiv:2206.02117
- José Luis Jaramillo, Badri Krishnan, "Painlevé-II approach to binary black hole merger dynamics: universality from integrability"; arXiv:2211.03405

An "Asymptotic Reasoning" [Batterman] Bottom-Up hierarchy to BBH mergers: from Diffraction Caustics to Integrability.

Asymptotic BBH Model	Mathematical/Physical Framework	Key Structures/Mechanisms
Fold-caustic model	Geometric Optics Catastrophe (singularity) Theory	Arnol'd-Thom's Theorem Classification of Stable Caustics
Airy function model	Fresnel's Diffraction, Semiclassical Theory	Universal Diffraction Patterns in Caustics
Painlevé-II model	Painlevé Transcendents and Integrability Self-force calculations and EMRBs	Painlevé property Non-linear Turning Points
KdV-like model	Inverse Scattering Transform and Integrability Dispersive Non-linear PDEs Critical Phenomena in Dispersive PDEs	Painlevé test, Lax pairs Darboux transformations, Soliton Scattering Universal Wave Patterns, Dubrovin's Conjecture
Propagation models on (anti)-Self-Dual backgrounds	Ward's Conjecture and Integrability Twistorial techniques	(anti-)Self-Dual DoF, Instantons, Tunneling Penrose Transform, 'Twistor' BBH data

Conclusions and Perspectives

References

Basic elements:

- Jaramillo J. L., Macedo R. P. and Al Sheikh L., "Pseudospectrum and black hole quasinormal mode instability", Phys. Rev. X 11 031003 (2021); arXiv:2004.06434
- Jaramillo J.L., Macedo R.P. and Al Sheikh L., "A Gravitational wave signatures of black hole quasinormal mode instability", Phys. Rev. Lett. 128 211102 (2022); arXiv:2105.03451
- Gasperín E. and Jaramillo J.L., "Energy scales and black hole pseudospectra: the structural role of the scalar product", Class. Quantum Grav. 39 115010 (2022); arXiv:2107.12865
- Jaramillo, J.L., "Pseudospectrum and binary black hole merger transients", Class. Quantum Grav. 39 217002 (2022); arXiv:2206.08025

Applications:

- Destounis K., Macedo R. P., Berti E., Cardoso V. and Jaramillo J. L., "Pseudospectrum of Reissner-Nordström black holes: quasinormal mode instability and universality Phys. Rev. D 104 084091 (2021); arXiv:2107.09673
- Boyanov V., Destounis K., Jaramillo J.L., Macedo R.P. and Cardoso V., "Pseudospectrum of horizonless compact objects: a bootstrap instability mechanism", Phys.Rev.D 107, 6, 064012 (2023); arXiv:2209.12950.

Conclusions and Perspectives

Vishveshwara, C.V., "On the black hole trail...: a personal journey"

18th Conference of the Indian Association for General Relativity and Gravitation, pp. 11-22 (1996)

wave during the coalscence of binary black holes[18]. Recently Aguirregabiria and I have studied the sensitivity of the quasinormal modes to the scattering potential[19]. The motivation is to understand how any perturbing influence, such as another gravitating source, that might alter the effective potential would thereby affect the quasinormal modes. Interestingly, we find that the fundamental mode is, in general, insensitive to small changes in the potential, whereas the higher modes could alter drastically. The fundamental mode would therefore carry the imprint of the black hole, while higher modes might indicate the nature of the perturbing source.

Quasinormal modes are perhaps the rebuttal to the criticism of my thesis examiner regarding the nonobservability of black holes.

Conclusions and Perspectives

The "Gevrey Ocean" and the Dijon Legacy...

Conclusions and Perspectives

"Truth suffers from much Analysis"

(Ancient Fremen saying) Dune Messiah, Frank Herbert

A 'hint of Geometry' perhaps...?