Conformal Scattering of Maxwell Potentials

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Outline

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- Fundamentals of Lax–Phillips scattering
- Conformal scattering
- Maxwell potentials on curved spacetimes
- Decay rates

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[based on arXiv:2211.14579 with J.-P. Nicolas & arXiv:2304.08270 with J. Valiente Kroon]

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- has spectrum $= \mathbb{R}$
- for each eigenvalue $\sigma \in \mathbb{R} \setminus \{0\}$ has a 2-sphere of generalized eigenfunctions

$$W_\sigma(x, \omega) = (e^{-i\sigma x \cdot \omega}, i\sigma e^{-i\sigma x \cdot \omega})$$

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Taking the Fourier transform of $\tilde{\Phi}$ in σ, one obtains a new representation of Φ,

$$\mathfrak{T}^+\Phi = (\mathcal{F}_\sigma \circ S_\sigma)\Phi.$$

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Evolution of ϕ is just translation in time of initial data.

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Upside: neat story;
Downside: heavy reliance on spectral theory; only valid on static backgrounds; technical restriction;

Friedlander's radiation field

Theorem (Friedlander 1967, 1980)

In Minkowski spacetime the solution ϕ *to* (1) *can be recovered from either of the "radiation fields"*

$$\hat{\phi}^{+}(u,\omega) = \sqrt{1+u^2}\lim_{r\to\infty} r\phi(r+u,r,\omega),$$
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$$\text{translation representation at infinity} \iff \text{radiation fields}$$

Remark

In fact Friedlander considered curved, static backgrounds with strong decay at i^0*, among which the only solution to Einstein's equations is Minkowski; seems Friedlander was motivated by recovering the full richness of the Lax–Phillips theory.*

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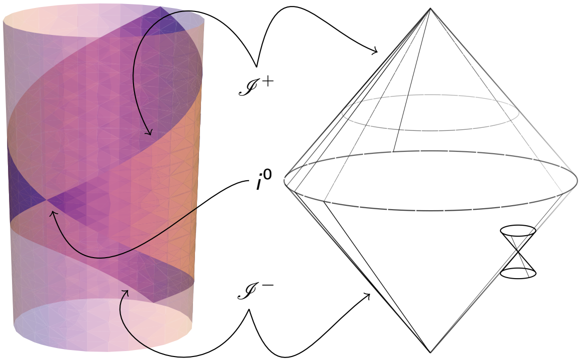
$(\mathbb{R}^{1+3}, \eta)$ can be smoothly embedded into $\mathbb{R} \times \mathbb{S}^3$ using a conformal rescaling: $\exists\, \Omega > 0$ smooth such that

$$\hat{\eta} = \Omega^2 \eta = \mathrm{d}\tau^2 - g_{\mathbb{S}^3}.$$



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Problem :
If (M, g) asymptotically flat with non-zero ADM mass $m \neq 0$, then the eigenvalues of the Weyl tensor of $\hat{g}_{ab} = \Omega^2 g_{ab}$ are proportional to

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Workaround : use incomplete compactification which leaves i^0 at infinity; is there a class of spacetimes for which this works?

Corvino–Schoen–Chruściel–Delay spacetimes

[*cf.* talk by P. LeFloch]

Theorem (Corvino '00, Chruściel–Delay '02, '03, Corvino–Schoen '06)

There exists an infinite dimensional class of spacetimes (M, g) *such that*

- *there exists a smooth function* $\Omega > 0$ *on* M *and another spacetime* $(\hat{M}, \hat{g})$, *where* $\hat{g}_{ab} = \Omega^2 g_{ab}$, *such that* $\hat{M} \setminus \partial\hat{M}$ *is diffeomorphic to* M, $\Omega = 0$ *on* $\partial\hat{M}$, *and* $\mathrm{d}\Omega \neq 0$ *on* $\partial\hat{M}$,
- *every inextendible null geodesic acquires two distinct endpoints on* $\partial\hat{M}$,
- $\partial\hat{M} = \mathscr{I}^+ \cup \mathscr{I}^- \cup i^+ \cup i^-$, *where* $\mathscr{I}^\pm$ *is the past (future) lightcone of* $i^\pm$,
- *the rescaled metric* $\hat{g}_{ab}$ *is* C^k *at* $i^\pm$ *and* $\mathscr{I}^\pm$ *for any fixed* k,
- *the spacetime* (M, g) *satisfies Einstein's equations* $R_{ab} = 0$,
- M *is diffeomorphic to the Schwarzschild solution outside the domain of influence of a given compact subset* K *of a Cauchy surface* Σ

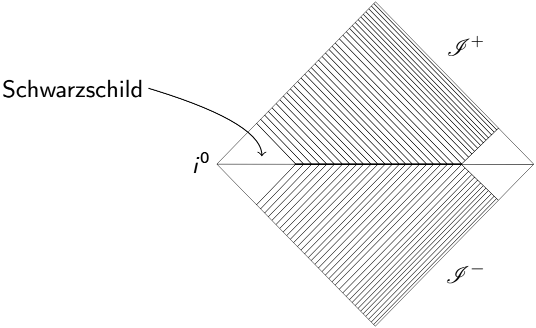
Corvino–Schoen–Chruściel–Delay spacetimes

Theorem (Penrose, 1965)

For CSCD spacetimes the topology of $\mathscr{I}^{\pm}$ is given by

$$\mathscr{I}^{+} \simeq \mathscr{I}^{-} \simeq \mathbb{R} \times \mathbb{S}^2,$$

and the $\mathbb{R}$ factors correspond to the rays generating $\mathscr{I}^{\pm}$.



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- the physical stress-energy tensor decays like $\mathcal{O}(\Omega^2)$ towards $\mathscr{I}^{\pm}$; $\implies$ $\hat{\nu} \approx \hat{\lambda} \approx \hat{\mu} \approx \hat{\pi} \approx \hat{\gamma} \approx \hat{\tau} \approx 0$
- the rescaled Weyl tensor vanishes on $\mathscr{I}^{+}$; $\implies$ kills some components of the Ricci curvature ($\hat{\Phi}_{22} \approx 0 \approx \hat{\Phi}_{21}$)
- o these two assumptions allow us to construct a conformal factor Ω in which $\mathscr{I}^{\pm}$ is essentially "flat" (analogue of r^{-1} in Minkowski, and here $\Omega = r^{-1}$ near i^0)
- also assume that there exists a Cauchy surface Σ on which

$$\|r^2 \mathrm{Ric}_{\Sigma}\|_{L^\infty} < C$$

  for some constant $0 < C < 1$ (more later)
- o "small" matter is allowed; spacetime not stationary
- o do not expect "no-scattering" condition to hold ($\therefore$ no translation representation)

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Seek to construct isomorphisms $\mathfrak{T}^{\pm}$ between function spaces on initial surface Σ and $\mathscr{I}^{\pm}$. Need:

- energy estimates
- gauge choice ((2) not hyperbolic a priori)
- solve characteristic Cauchy problem without loss of regularity

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We have the estimate

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3. Can patch these together as a result of $\hat{\lambda} \approx 0 \approx \hat{\nu}$ (always true on $\mathscr{I}^+$)

□

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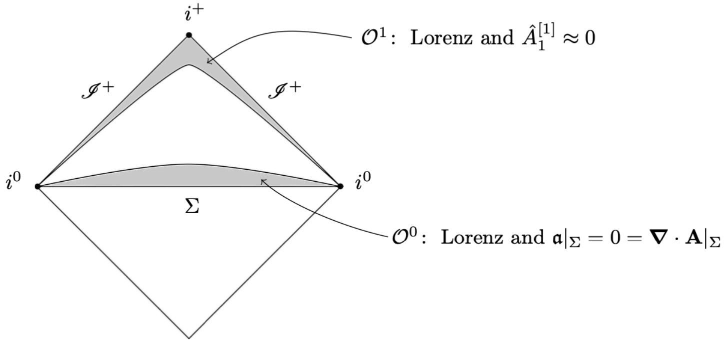
This works:



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and (2) becomes

$$\widehat{\Box}\hat{A}_a - \hat{\nabla}_a(2f\hat{A}_1^{[1]}) + \hat{R}_{ab}\hat{A}^b = 0 \implies \text{hyperbolic, non-singular.}$$

Construction for Maxwell potentials

Physical Lorenz gauge $\nabla_a A^a = 0$ has expansion in powers of Ω near $\mathscr{I}^+$:

$$\Omega^{-2}\nabla_a A^a = -2\Omega^{-1} f \hat{A}_1 + \hat{\text{þ}}\hat{A}_1 - 2\hat{A}_1 \operatorname{Re}(\hat{\rho}) + \hat{\text{þ}}'\hat{A}_0 - 2\operatorname{Re}(\hat{\eth}\bar{\hat{A}}_2) + \mathcal{O}(\Omega).$$

At order $\mathcal{O}(\Omega^{-1})$ obtain:

$$\hat{A}_1 \approx 0.$$

Follows that

$$\hat{A}_1 = \Omega \hat{A}_1^{[1]}, \qquad \hat{F}_2 \approx -\partial_u \bar{\hat{A}}_2.$$

and (2) becomes

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At order $\mathcal{O}(1)$ then obtain:

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whence

$$\int_{\mathscr{I}^+} |\hat{F}_2|^2 \, \mathrm{d}u \wedge \mathrm{dv}_{\mathbb{S}^2} = \int_{\mathscr{I}^+} |\partial_u \hat{A}_2|^2 \, \mathrm{d}u \wedge \mathrm{dv}_{\mathbb{S}^2}.$$

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Definition

The space of scattering data for the Maxwell potential is

$$\dot{\mathcal{H}}^1(\mathscr{I}^+) = \left\{ (\hat{A}_0, 0, \hat{A}_2) \in \dot{H}^2(\mathbb{R}; H^{-1}(\mathbb{S}^2)) \times \mathcal{C}_c^\infty(\mathscr{I}^+) \times \dot{H}^1(\mathbb{R}; L^2(\mathbb{S}^2)) \right\} / \sim$$
$$\simeq \dot{H}^1(\mathbb{R}; L^2(\mathbb{S}^2)).$$

The equivalence relation $\sim$ identifies $\hat{A}_2$'s that differ by a constant on $\mathscr{I}^+$.

Construction for Maxwell potentials

Gauge near Σ

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- There is enough residual gauge freedom in $\nabla_a A^a = 0$ to set

$$\mathfrak{a}|_\Sigma = 0 = \boldsymbol{\nabla} \cdot \boldsymbol{A}|_\Sigma,$$

where $\mathfrak{a} = T^a A_a$, $\boldsymbol{A}$ is the projection of A_a to Σ, and $\boldsymbol{\nabla}$ is the connection on Σ.

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Solution : show $(\boldsymbol{E}, \boldsymbol{B}) \in L^2(\Sigma)^2$ in one-to-one correspondence with $(\boldsymbol{A}, \nabla_T \boldsymbol{A})$ in suitable Hilbert space; this will be the space of initial data.

Construction for Maxwell potentials

Space of initial data

Need to solve $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ for $\boldsymbol{A}$ when $\boldsymbol{B} \in L^2(\Sigma)$.

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So cannot easily use standard elliptic theory. [Σ unbounded, so $\dot{H}^1(\Sigma)$ does not compactly embed into $\dot{H}^{-1}(\Sigma)$. Do not understand $\ker \boldsymbol{P}$.]

Workaround:

For

$$\Delta \boldsymbol{A}_k + \boldsymbol{R}_{kj}\boldsymbol{A}^j = -(\boldsymbol{\nabla} \times \boldsymbol{B})_k$$

have the estimate

$$\|\boldsymbol{A}\|^2_{\dot{H}^1} \leqslant \|\boldsymbol{B}\|^2_{L^2} + \int_\Sigma |\boldsymbol{R}_{ij}\boldsymbol{A}^i\boldsymbol{A}^j| \, \mathrm{dv}_\Sigma \leqslant \|\boldsymbol{B}\|^2_{L^2} + C\delta\|\boldsymbol{A}\|^2_{\dot{H}^1}$$

using Hardy's inequality:

$$\int_\Sigma \frac{|\boldsymbol{A}|^2}{r^2} \, \mathrm{dv}_\Sigma \leqslant C \int_\Sigma |\boldsymbol{\nabla}\boldsymbol{A}|^2 \, \mathrm{dv}_\Sigma$$

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The space of initial data $(\boldsymbol{A}, \nabla_T \boldsymbol{A})$ on Σ is given by

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Invertibility:

Theorem (Hörmander '90, Bär–Wafo '15)

For $\hat{M}$ a globally hyperbolic Lorentzian manifold and $\mathcal{S} \subset \hat{M}$ a characteristic (partial) Cauchy hypersurface, for any $f \in L^2_{loc,sc}(\hat{M})$ and any $u_0 \in H^1_c(\mathcal{S})$ there exists a unique solution

$$u \in \mathcal{C}^0_{sc}(t(\hat{M}); H^1(\mathcal{S}_\circ)) \cap \mathcal{C}^1_{sc}(t(\hat{M}); L^2(\mathcal{S}_\circ))$$

to

$$Pu = f,$$

where P is a linear wave operator on $\hat{M}$.

Remark

Applies to systems of hyperbolic equations and non-compact $\mathcal{S}$ with possibly Lipschitz singularities (e.g. lightcone or intersection of null planes). *No loss of regularity*.

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We therefore obtain the scattering operator

$$\mathscr{S} = \mathfrak{T}^+ \circ (\mathfrak{T}^-)^{-1} : \dot{\mathcal{H}}^1(\mathscr{I}^-) \longrightarrow \dot{\mathcal{H}}^1(\mathscr{I}^+)$$

which is an isomorphism of Hilbert spaces.

Construction for Maxwell potentials

Minkowski: role of symmetries

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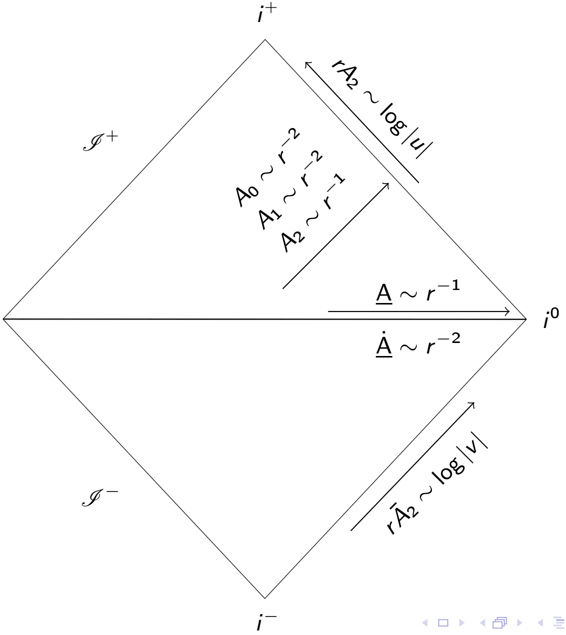
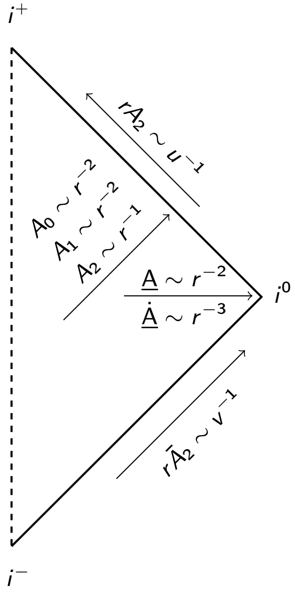
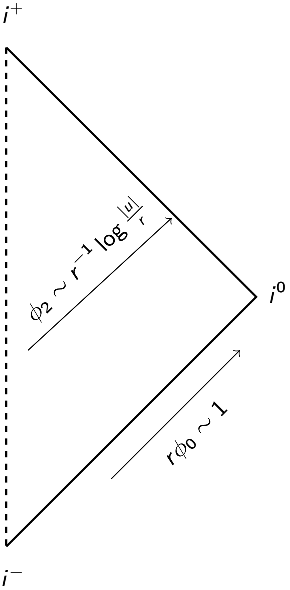
Decay rates obtained using ∂_t as multiplier



Decay rates obtained using K_0 as multiplier



Weaker decay towards i^0



Thank you !