Partition function for a volume of space

Manus Visser

University of Cambridge

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Horizon entropy as a probe of QG

- Bekenstein-Hawking entropy provides a low-energy window into the realm of quantum gravity

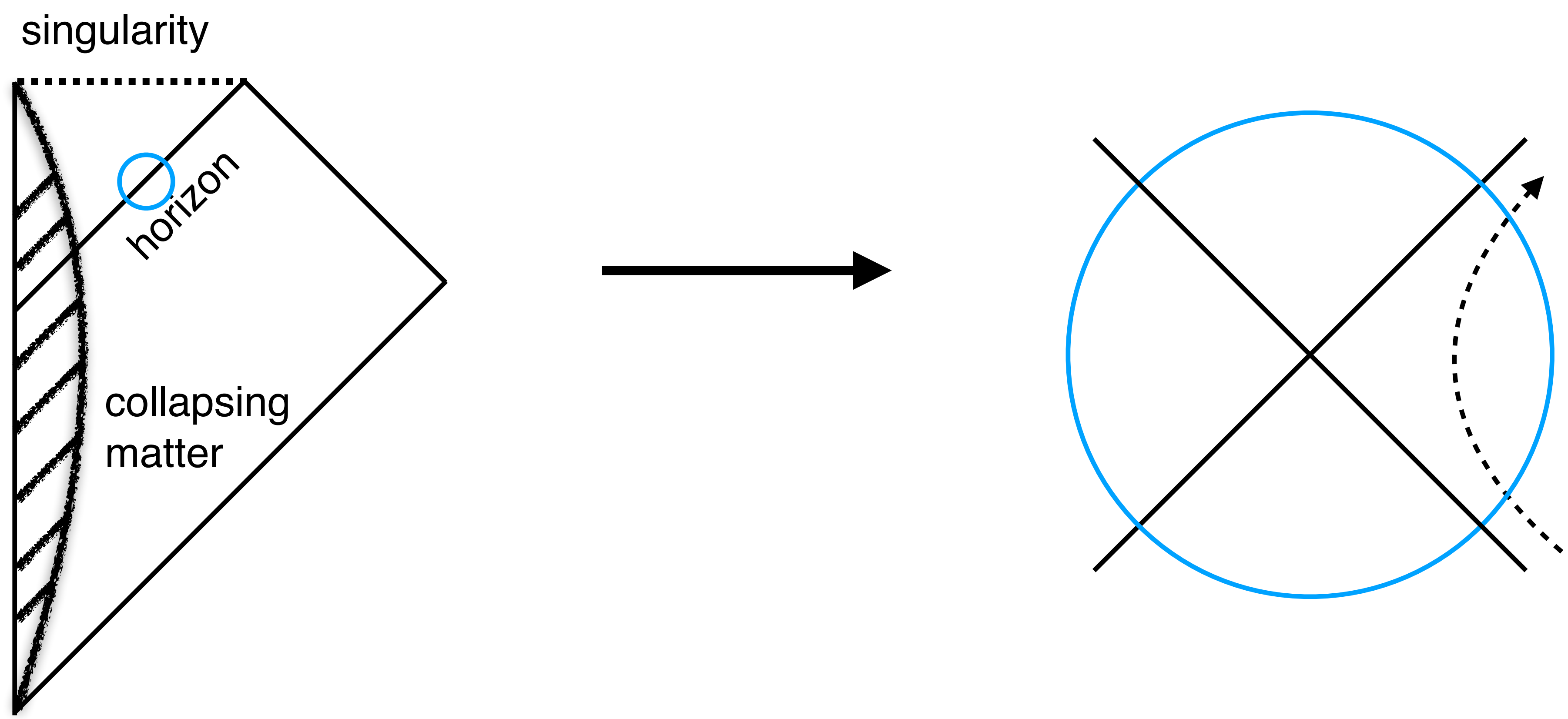
$$S = k_B \frac{Ac^3}{4\hbar G}$$

A = horizon area

- This beautiful formula combines all fundamental constants of Nature.

Horizon entropy

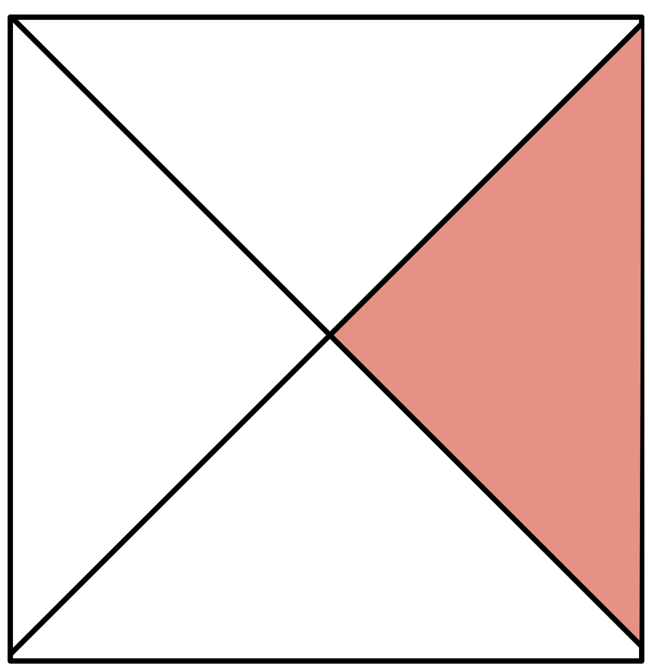
- By zooming in on the event horizon of a black hole we find the horizon of an accelerating observer in flat space. That local Rindler horizon again has thermodynamic properties. Unruh, Jacobson



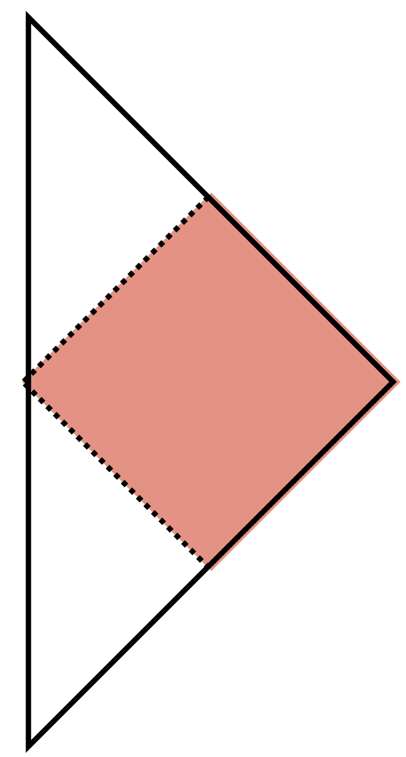
- All horizon entropies should have the same statistical origin (entanglement entropy?). Sorkin, Frolov, Novikov, Srednicki, Solodukhin, Fursaev, Zelnikov, Susskind, Uglum, Jacobson, …

Horizon entropy

- Bekenstein-Hawking entropy is a universal formula: applies not only to black hole horizon, but also to cosmological and acceleration horizons.



Static patch of de Sitter spacetime



Rindler wedge of Minkowski spacetime

- The de Sitter entropy is set by the cosmological constant, and the Rindler entropy is infinite.

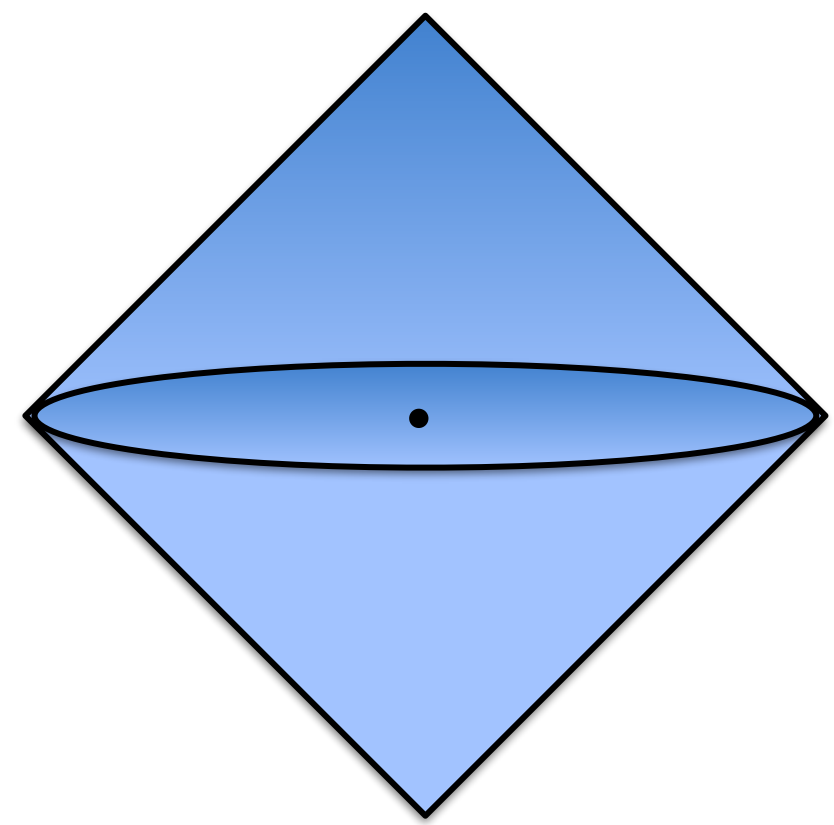
Entropy of causal diamonds

- The de Sitter static patch and the Rindler wedge are examples of causal diamonds.

- A *causal diamond* is the intersection of the future of one point with the past of another point.

- It is expected that causal diamonds of any size in any Lorentzian spacetime have an associated Bekenstein-Hawking entropy.

Banks, Fischler, Bousso, Draper, Farkas, Jacobson, MV, …



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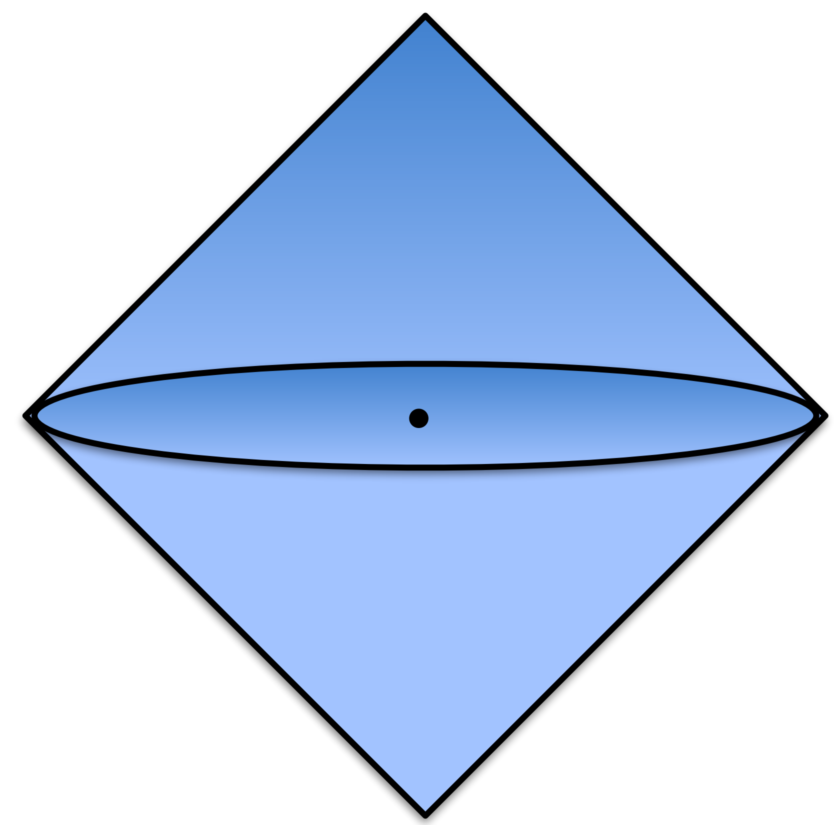

- *How to justify this?*

Gibbons-Hawking partition function

- Gibbons and Hawking derived the entropy of black holes and de Sitter horizons from a Euclidean saddle approximation of the gravitational partition function.

Q: can the entropy of causal diamonds be derived from a saddle approximation to a partition function?

See also Banks-Draper-Farkas '20
and our recent statistical interpretation (Jacobson-MV '22)

Gibbons-Hawking partition function

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Q: can the entropy of causal diamonds be derived from a saddle approximation to a partition function?

A: Yes! Using the method of constrained instantons

Outline

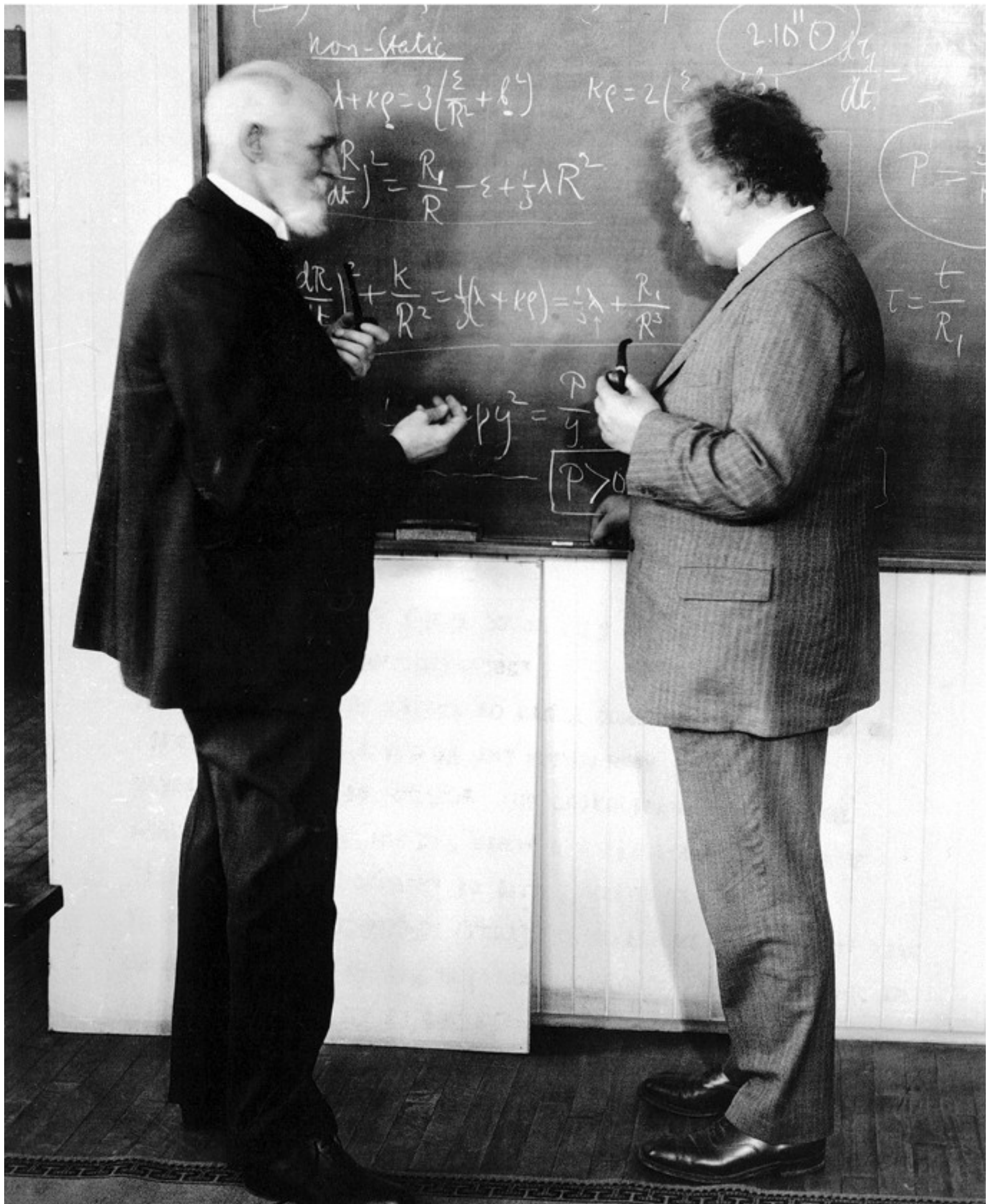
1. Gravitational partition function for de Sitter space

   - Review of Gibbons-Hawking, Action integrals and partition functions in quantum gravity, PRD, 1977.

2. Constrained sphere partition function

   - T. Jacobson & M.V. *Partition function of a volume of space,* (2212.10607, PRL)

Willem de Sitter (1872-1934)

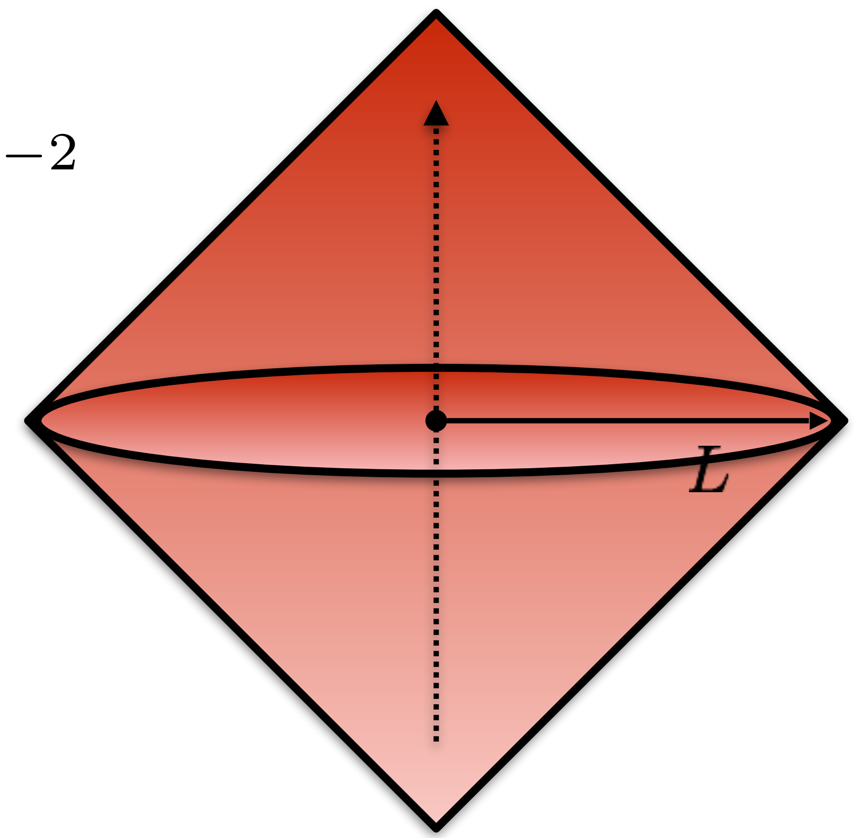


- Director of the Leiden Observatory from 1919.
- Had vivid discussions and correspondence with Einstein on cosmological spacetimes.

Thermodynamics of de Sitter space

- Static coordinates for de Sitter

$$ds^2 = -\left(1 - R^2/L^2\right) dt^2 + \frac{dR^2}{1 - R^2/L^2} + R^2 d\Omega_{d-2}^2$$



- Cosmological horizon at: $R = L$.

- De Sitter entropy and temperature

$$T_{\rm GH} = \frac{\hbar\kappa}{2\pi} \qquad S_{\rm BH} = \frac{A(L)}{4\hbar G}$$

$\kappa = 1/L$

Gibbons-Hawking (1977)

PHYSICAL REVIEW D VOLUME 15, NUMBER 10 15 MAY 1977

Cosmological event horizons, thermodynamics, and particle creation

G. W. Gibbons* and S. W. Hawking
D.A.M.T.P., University of Cambridge, Silver Street, Cambridge, United Kingdom
(Received 4 March 1976)

It is shown that the close connection between event horizons and thermodynamics which has been found in the case of black holes can be extended to cosmological models with a repulsive cosmological constant. An observer in these models will have an event horizon whose area can be interpreted as the entropy or lack of information of the observer about the regions which he cannot see. Associated with the event horizon is a surface gravity κ which enters a classical "first law of event horizons" in a manner similar to that in which temperature occurs in the first law of thermodynamics. It is shown that this similarity is more than an analogy: An observer with a particle detector will indeed observe a background of thermal radiation coming apparently from the cosmological event horizon. If the observer absorbs some of this radiation, he will gain energy and entropy at the expense of the region beyond his ken and the event horizon will shrink. The derivation of these results involves abandoning the idea that particles should be defined in an observer-independent manner. They also suggest that one has to use something like the Everett-Wheeler interpretation of quantum mechanics because the back reaction and hence the spacetime metric itself appear to be observer-dependent, if one assumes, as seems reasonable, that the detection of a particle is accompanied by a change in the gravitational field.

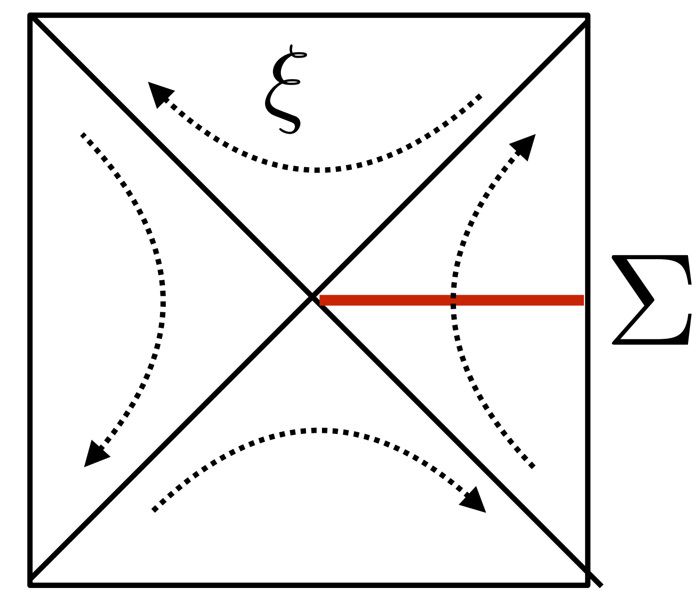
Thermodynamics of de Sitter space

- What evidence do we have for the thermodynamics of the static patch of de Sitter?

  1. QFT in fixed de Sitter background

  2. De Sitter horizon satisfies thermodynamic laws

  3. Euclidean sphere partition function



Thermodynamics of de Sitter space

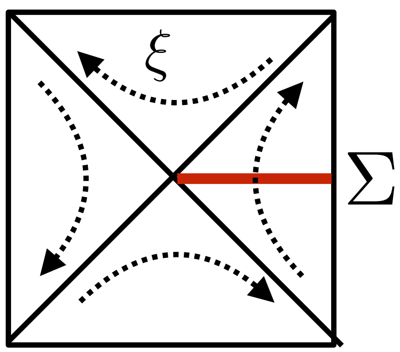
1. The de Sitter vacuum state when restricted to one static patch is thermal with respect to the Hamiltonian K generating the Killing flow

Gibbons-Hawking (1977)

$$\rho_{\rm vac} = \frac{1}{Z}\exp(-K/T_{\rm GH}) \qquad T_{\rm GH} = \kappa\hbar/2\pi$$

2. Gravitational first law of de Sitter static patch

$$0 = \frac{\kappa}{8\pi G}\delta A + \int_\Sigma \delta T_{\mu\nu}\xi^\mu d\Sigma^\nu$$



NB The static patch has no boundary where ADM energy can be defined.

Gibbons-Hawking partition function

PHYSICAL REVIEW D VOLUME 15, NUMBER 10 15 MAY 1977

Action integrals and partition functions in quantum gravity

G. W. Gibbons* and S. W. Hawking
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England
(Received 4 October 1976)

One can evaluate the action for a gravitational field on a section of the complexified spacetime which avoids the singularities. In this manner we obtain finite, purely imaginary values for the actions of the Kerr-Newman solutions and de Sitter space. One interpretation of these values is that they give the probabilities for finding such metrics in the vacuum state. Another interpretation is that they give the contribution of that metric to the partition function for a grand canonical ensemble at a certain temperature, angular momentum, and charge. We use this approach to evaluate the entropy of these metrics and find that it is always equal to one quarter the area of the event horizon in fundamental units. This agrees with previous derivations by completely different methods. In the case of a stationary system such as a star with no event horizon, the gravitational field has no entropy.

3. Gibbons-Hawking represented the canonical partition function in gravity as a Euclidean path integral over metrics

$$Z = \mathrm{Tr}\, e^{-\beta H} \quad \longleftrightarrow \quad Z = \int \mathcal{D}g\, e^{-I_E[g]/\hbar}$$

Gibbons-Hawking partition function

- If the action is very large compared to Planck's constant, the path integral can perhaps be estimated as:

$$Z \sim \exp\left(-I_E^{\mathrm{saddle}}/\hbar\right)$$

- From the canonical partition $Z = \mathrm{Tr}\, e^{-\beta H}$ one usually gets thermodynamic quantities for the system

$$\ln Z = -\beta F$$

$$-\frac{d}{d\beta}\ln Z = \langle H \rangle_\beta$$

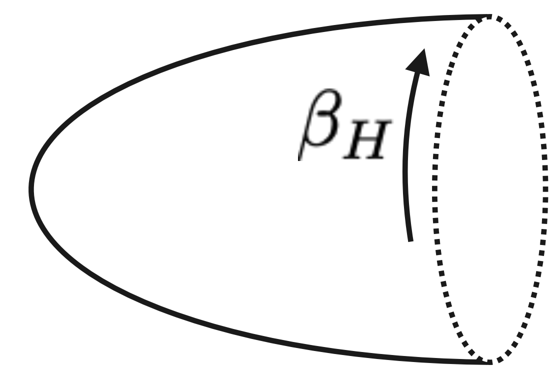
$$\left(1 - \beta\frac{d}{d\beta}\right)\ln Z = S$$

Entropy from the partition function

- If the saddle geometry is a Euclidean black hole spacetime, then

$$I_E/\hbar = \beta F = \beta M - S$$

$$S = \left(\beta \frac{d}{d\beta} - 1\right) I_E/\hbar = \frac{\text{horizon area}}{4\hbar G}$$

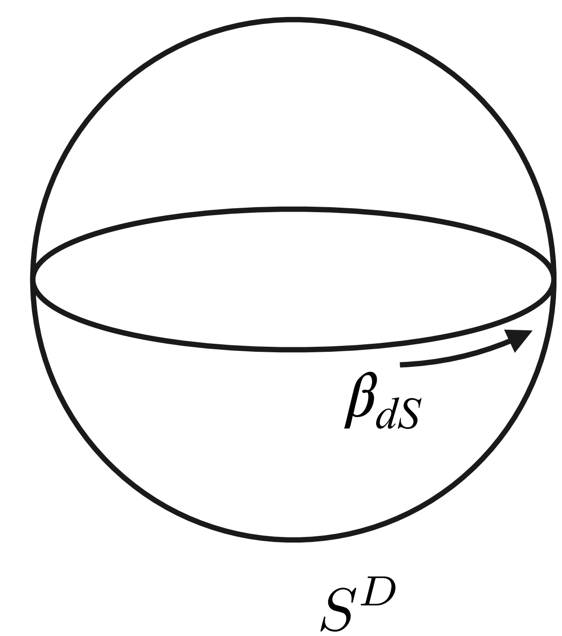


- If the saddle geometry is Euclidean de Sitter space (a round sphere whose radius is *L*), then

  - *M = 0* (since the saddle has no boundary)

  - the entropy is

$$S_{dS} = -I_E/\hbar = \frac{A(L)}{4\hbar G}$$



Gibbons-Hawking partition function

- Since the energy vanishes for Euclidean de Sitter space, the partition function seems to count the dimension of the quantum gravity Hilbert space

Banks, Fischler

$$Z = \mathrm{Tr}\, e^{-\beta H} \quad \& \quad H = 0$$

Jacobson, Banihashemi '22

$\longrightarrow$

$$Z \to \mathrm{Tr}_{\mathcal{H}}\, 1 = e^{S_{dS}}$$

= dimension of Hilbert space
of states surrounded by a horizon,
i.e. states of a ball

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   *(*2212.10607, PRL)

Partition function for a volume of space

- Should not "area = entropy" apply to *any* causal diamond?
- To specify a generic diamond, one must somehow fix its size:

  a) Fixed edge area: no saddle exists, since fluctuations are too large.

  b) Fixed Euclidean spacetime volume: not a constraint on the states.

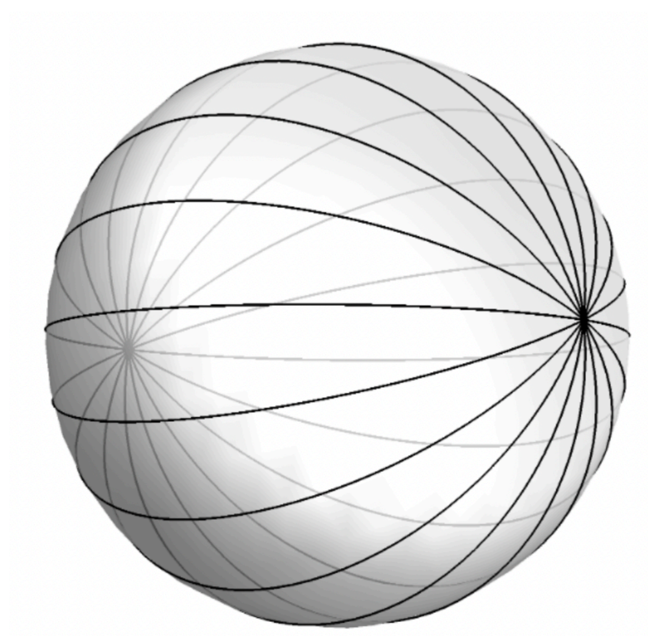
  c) Fixed spatial volume: this works!

$$Z[V] = \mathrm{Tr}_{\mathcal{H}} 1 \quad \longleftrightarrow \quad Z[V] = \int \mathcal{D}g \delta(\mathcal{C}[g] - V) e^{-I_E[g]}$$

Euclidean sphere geometry

- Counting the dimension of the Hilbert space of a diamond is equivalent to counting the states of a spatial slice at one time.
- We consider a spatial topological *(D-1)*-ball whose boundary has topology S^{D-2}.
- The Euclidean manifold generated by rotating the ball through a complete circle about the ball boundary is a topological *D*-sphere

  e.g. D=2 version:



$$S^D$$

Constrained path integral

- In the gravitational path integral we can introduce a constraint as follows

Affleck, Cotler-Jensen

$$\int [dg]\, e^{-I_E[g]} = \int dV \int [dg]\delta(\mathcal{C}[g] - V)e^{-I_E[g]}$$

- Introducing a Lagrange multiplier to impose the constraint the integral can be rewritten as

$$\int [dg]e^{-I_E[g]} = \int dV \int d\lambda \int [dg]e^{-I_E[g]+\lambda(\mathcal{C}[g]-V)}$$

- Saddle-point equations of motion

$$\delta I_E[g] + \lambda\, \delta\mathcal{C}[g] = 0\,, \quad \mathcal{C}[g] = V$$

Constrained sphere partition function

Method of constrained instantons Affleck, Cotler-Jensen

$$Z[V,\Lambda] = \int \mathcal{D}g\delta(\mathcal{C}[g]-V)\exp\left[\frac{1}{16\pi\hbar G}\int d^Dx\sqrt{g}(R-2\Lambda)\right]$$
$$= \int \mathcal{D}\lambda\,\mathcal{D}g\,\exp\left[\frac{1}{16\pi\hbar G}\int d^Dx\sqrt{g}(R-2\Lambda)+\frac{1}{\hbar}\int d\phi\,\lambda(\phi)\left(\int d^{D-1}x\sqrt{\gamma}-V\right)\right]$$

- Foliate S^D by *(D-1)*-balls at constant ϕ with induced metric $\gamma_{ab} = g_{ab} - N^2\phi_{,a}\phi_{,b}$

$$N \equiv (g^{ab}\phi_{,a}\phi_{,b})^{-1/2}$$

- The saddle point equations are the Einstein equations sourced by an effective perfect fluid with vanishing energy density,

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad \text{with} \quad T_{ab} = \frac{\lambda}{N}\gamma_{ab} \equiv P\gamma_{ab}$$

Static, spherically symmetric saddle

$$ds^2 = N^2(r)d\phi^2 + h(r)dr^2 + r^2 d\Omega_{D-2}^2$$

- *N(r)* is determined by the Tolman-Oppenheimer-Volkoff-equation, with boundary conditions:

  1) $N = 0$ at $r = R_V$, the horizon

  2) $N' = -\sqrt{h}$ at $r = R_V$, to remove conical singularity

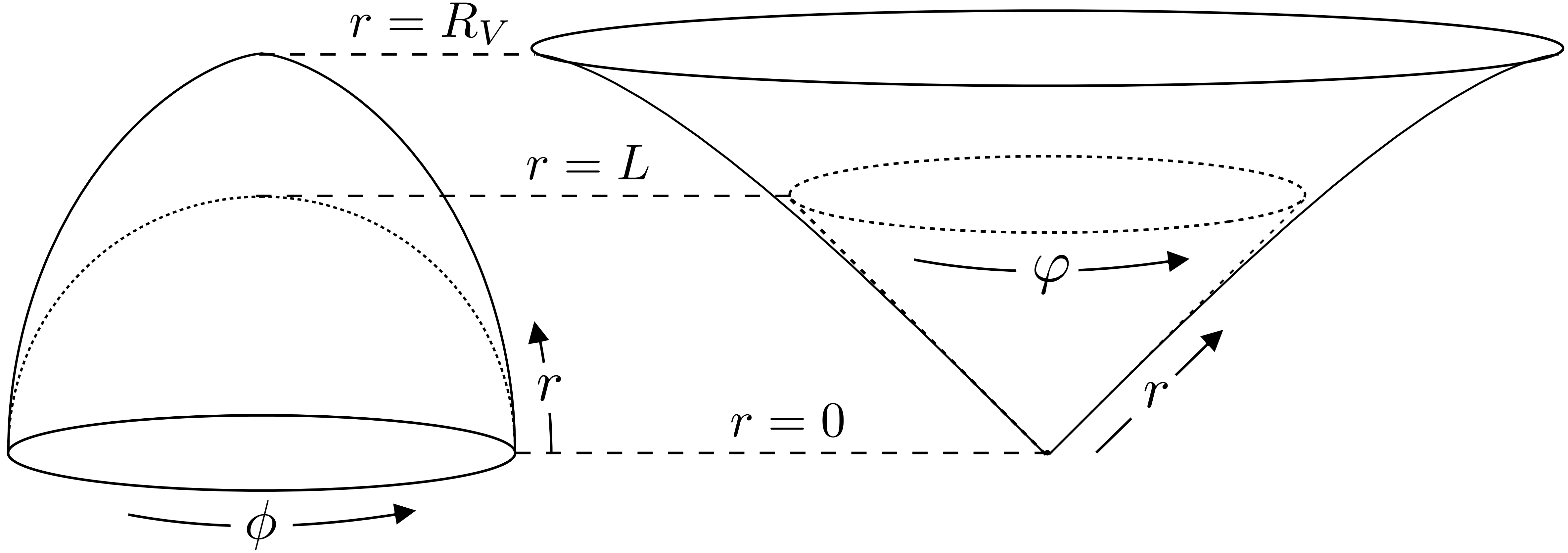
- $\Lambda = 0$ solution:

$$R_V = [(D-1)V/\Omega_{D-2}]^{1/(D-1)}$$

$$ds^2 = \frac{1}{4R_V^2}(R_V^2 - r^2)^2 d\phi^2 + dr^2 + r^2 d\Omega_{D-2}^2$$

Euclidean saddle

- Comparison between diamond saddle and spherical dS saddle



- Diamond saddle has topology S^D, is conformally flat, and has a curvature singularity at the horizon

Euclidean action

- Even though the saddle has a 1/(r-R_V) curvature singularity at the horizon, the action is finite.

- On-shell Euclidean action

$$I_{\text{saddle}} = -\frac{1}{16\pi G}\int d^D x\sqrt{g}R = -\frac{A_V}{4G}$$

- Hence, in the zero-loop saddle-point approximation:

$$\boxed{Z[V] \approx \exp(A_V/4\hbar G)}$$

This generalizes the GH partition function to a finite volume of space!

$\Lambda > 0$ saddle

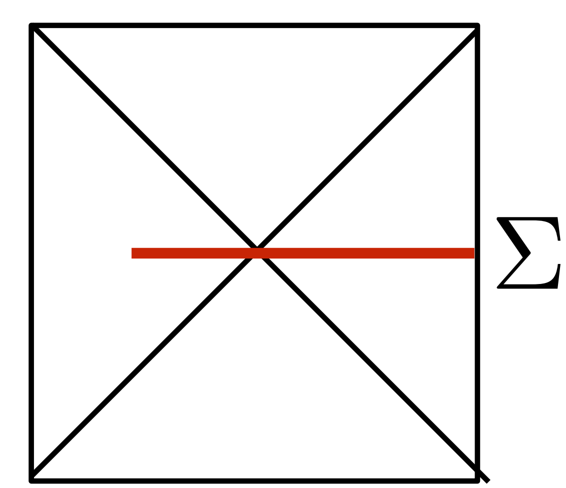
- The saddle solution with positive cosmological constant is similar, BUT:

  1) If $V = V_{dS}$ (static patch spatial volume), then the saddle is dS, which is smooth

  2) If V is larger than the dS spatial hemisphere, the entropy *decreases* as volume increases

  2) There is no saddle if V is larger than the full de Sitter spatial sphere.

  3) The integral over all V is dominated by the de Sitter saddle.



Conclusions

- *Partition function of a volume of space* = dimension of the quantum gravity Hilbert space of a topological ball with fixed proper volume.

- The Hilbert space dimension matches with the semiclassical *horizon entropy* attributed to diamonds, and exhibits the *holographic* nature of nonperturbative quantum gravity in finite volumes of space.

Future directions:

- *Higher curvature corrections:* determine whether they smoothen out the curvature singularity at the horizon.

- *Volume constraint:* relation with York Hamiltonian?