Carroll, Cotton and Ehlers

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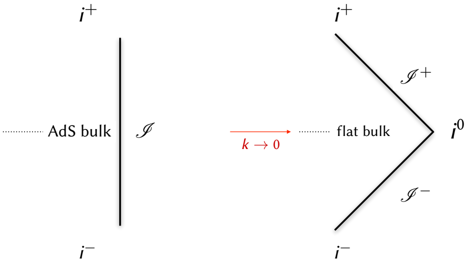
Highlights

Framework: asymptotically flat spacetimes

Emergence of Carroll: from AdS$_n$ to flat$_n$ asymptotics

$\Lambda = -\frac{(n-1)(n-2)}{2} k^2 \to 0$ $\qquad c_{\text{bry.}} = k \times R_{\text{celestial sphere}} \times c_{\text{bulk}}$



$\mathscr{I}^\pm$ is a null hypersurface $\to$ Carrollian geometry with conformal isometry group $\mathfrak{ccarr}(n-1) \equiv \text{BMS}_n$

If flat holography exists the dual theory should be defined on a *Carrollian spacetime* and be $\text{BMS}_n \equiv \mathfrak{ccarr}(n-1)$-invariant

Reconstructing the bulk from the boundary

Possible for Ricci-flat spacetimes – as for Einstein

- boundary data defined on null boundary
- Carrollian boundary dynamics from bulk Ricci flatness
- infinite number of data $\supset$ the Carrollian Cotton tensor
  - enters explicitly the bulk-metric expansion
  - contributes the losses via gravitational radiation
  - defines magnetic duals to charges (energy, momenta, etc.)

Hidden vs. visible symmetries in Ricci-flat spaces

Isometries or asymptotic isometries are *visible* and act *locally*

Reductions along isometric orbits exhibit *hidden* symmetries

- from 4 to 3 dimensions: *Ehlers' Möbius group* [Ehlers '62]
- from 4 to 2 dimensions: *Geroch' affine Möbius group* [Geroch '72]
- larger reduction: bigger / ∞-dim group / exceptional

acting *non-locally* in the parent space (phase-space symmetries)

Ehlers in a nutshell – Ricci flat [Ehlers '62; Geroch '71]

Bulk reduction along a Killing ξ: $\mathcal{M} \to \mathcal{S} = \mathcal{M}/_{\text{orb}(\xi)}$

$$\lambda = \xi^A \xi_A \Rightarrow \tilde{h}_{AB} = \lambda g_{AB} - \xi_A \xi_B$$

Further ingredients: twist and its on-shell potential

$$\mathrm{w} = \star(\xi \wedge \mathrm{d}\xi) \;\; \Rightarrow \mathrm{w} = \mathrm{d}\omega$$

Three-dimensional dynamics (sigma-model)

dofs. $\tau = \omega + \mathrm{i}\lambda \quad \tilde{h}_{AB}$

eqs. $\tilde{\mathcal{R}}_{AB} = -\frac{2}{(\tau-\bar{\tau})^2}\tilde{\mathcal{D}}_{(A}\tau\tilde{\mathcal{D}}_{B)}\bar{\tau} \quad \tilde{\mathcal{D}}^2\tau = \frac{2}{\tau-\bar{\tau}}\tilde{\mathcal{D}}_M\tau\tilde{\mathcal{D}}_N\tau\tilde{h}^{MN}$

inv. $\tau \to \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\,, \quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL(2,\mathbb{R})$: Ehlers' group

non-local for $\alpha = \delta = \cos\chi$ & $\beta = -\gamma = \sin\chi$

Punch line [Mittal et al. '22]

Ehlers group is realized on the Carrollian boundary and acts locally on the data including the Carrollian Cotton descendants

Carrollian geometry [Lévy-Leblond '65; Sen Gupta '66; Duval et al. '14; Bekaert et al. '16]

Basic ingredients in $d+1$ dimensions (coordinates $t, \mathbf{x}$)

- degenerate metric: $\mathrm{d}s^2 = \underbrace{-c^2}_{0} \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + \underbrace{a_{ij}\mathrm{d}x^i \mathrm{d}x^j}_{\mathrm{d}\ell^2}$
- field of observers: kernel $\frac{1}{\Omega}\partial_t$ (t should be spelled u)
- clock form: $\mu = -\Omega \mathrm{d}t + b_i \mathrm{d}x^i$ (Ehresmann connection)

General covariance (in the present parameterization)

Carrollian diffeomorphisms: $t' = t'(t, \mathbf{x}) \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$

Applications

- $\mathscr{I}^{\pm}$ in Ricci-flat spacetimes
- black-hole horizons – membrane paradigm

Carrollian dynamics

Carrollian-covariant action, energy and momenta

$$\begin{cases} \Pi = -\frac{1}{\sqrt{a}}\left(\frac{\delta S}{\delta \Omega} + \frac{b_i}{\Omega}\frac{\delta S}{\delta b_i}\right) & \text{energy density} \\ \Pi^i = \frac{1}{\sqrt{a}\Omega}\frac{\delta S}{\delta b_i} & \text{energy flux} \\ \Pi^{ij} = \frac{2}{\sqrt{a}\Omega}\frac{\delta S}{\delta a_{ij}} & \text{energy–stress tensor} \end{cases}$$

Conservation equations in Carrollian spacetimes

- Weyl covariance $\Rightarrow \Pi^i{}_i = \Pi$
- Carrollian covariance ($\xi = \xi^t(t, \mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i$ diffeos)

$$\Rightarrow \begin{cases} \frac{1}{\Omega}\hat{\mathscr{D}}_t\Pi + \hat{\mathscr{D}}_i\Pi^i + \Pi^{ij}\xi_{ij} = 0 & \text{time} \\ \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t\delta^i_j + \xi^i{}_j\right)P_i + \hat{\mathscr{D}}_i\Pi^i{}_j + 2\Pi^i\varpi_{ij} = 0 & \text{space} \end{cases}$$

$\rightarrow$ momentum P_i

Pure gravity – asymptotically flat or AdS

Solving Einstein' equations in $n = d + 2$ dim for G_{AB}

$\{r, t, x^i\}$, $i = 1, \ldots, d$ plus *gauge fixing* ($n = d + 2$ conditions)
$\rightarrow$ find G_{AB} as $\mathrm{O}\left(1/r^m\right)$ with coefficients $f(t, \mathbf{x})$ & dynamics

Solution space $\equiv$ collection of data $f(t, \mathbf{x})$

Desirable feature: organize $f(t, \mathbf{x})$ and their dynamics tensorially wrt a *covariant structure on the boundary*

Gauge choice

- Fefferman–Graham bry.-covariant but singular at $k \to 0$
- Bondi/Newman–Unti valid at $k \to 0$ but not bry.-covariant

From NU to boundary-covariant NU

Newman–Unti gauge in coordinates r, t, x^i

Gauge conditions: $G_{rr} = 0$, $G_{rt} = -1$, $G_{ri} = 0$ $(i = 1, \ldots, n-2)$

$$\mathrm{d}s^2_{\text{bulk}} = \frac{V}{r}\mathrm{d}t^2 - 2\mathrm{d}t\mathrm{d}r + G_{ij}\left(\mathrm{d}x^i - U^i\mathrm{d}t\right)\left(\mathrm{d}x^j - U^j\mathrm{d}t\right)$$

V, G_{ij}, U^i functions of *all* coordinates – expand in $r^m \to f(t, \mathbf{x})$

Good feature: ∂_r null affine geodesic with shear $\mathscr{C}_{ij}$

Unwanted feature: not stable under $t \to t(t, \mathbf{x})$ i.e. boundary Carrollian diffeomorphisms or under $r \to r\Omega(t, \mathbf{x})$ i.e. Weyl

Covariant Newman–Unti gauge: *incomplete* gauge fixing

$$G_{rt} \neq -1, \quad G_{ri} \neq 0$$

[see also Ciambelli et al. ’20; Geiller et al. ’22]

Einstein 4-dim spacetimes in a nutshell

Covariant Newman–Unti gauge in coordinates r, x^μ

boundary data

- boundary metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ (6)
- conformal boundary energy–momentum tensor $T_{\mu\nu}$ (5)

remaining Einstein' equations $\nabla_\mu T^{\mu\nu} = 0$ ("fluid")

The boundary Cotton tensor appears *explicitly* in G_{AB}

[de haro '08; Mansi et al. '09; de Freitas et al. '14; Gath et al. '15; Ciambelli et al. '18]

$C_{\mu\nu} = \eta_\mu{}^{\rho\sigma}\nabla_\rho\left(R_{\nu\sigma} - \frac{R}{4}g_{\nu\sigma}\right)$ symmetric traceless $\nabla_\mu C^{\mu\nu} = 0$

$C_{\mu\nu} \neq 0 \Leftrightarrow$ non-conformally flat bry. $\leftrightarrow$ asymptotically *locally* AdS bulk (e.g. Taub–NUT)

Ricci-flat in covariant Newman–Unti gauge

Full solution space in $n = 4$ [Brussels & Paris groups]

$\mathrm{d}s^2_{\text{Ricci-flat}}$ described in terms of $2+1$ *Carrollian boundary data*

- Carrollian geometry (6) with zero geometrical shear ξ_{ij}
  - degenerate metric (3) $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$
  - Ehresmann connection (3) $\mu = -\Omega \mathrm{d}t + b_i \mathrm{d}x^i$
- Carrollian conformal energy and momenta (5) "fluid"
  - energy (1) ε
  - momenta – heat current (2) and stress tensor (2) N_i & E_{ij}
- Carrollian dynamical shear (2) $\mathscr{C}_{ij}$
- *infinite* number of further Carrollian data – at every $\mathrm{O}\left(1/r^m\right)$: *Chthonian* challenges flat "holography"

obeying Carrollian dynamics ↔ flux-balance equations

Central ingredient: the Carrollian Cotton

Carrollian Cotton avatars

- reduce the Riemannian $C_{\mu\nu}$ wrt Carrollian diffeomorphisms
- expand wrt the speed of light $\rightarrow c, \psi_i, \chi_i, \Psi_{ij}, X_{ij}$
- $\nabla_\mu C^{\mu\nu} = 0 \rightarrow$ Carrollian dynamics for the descendants

$$\begin{cases} \frac{1}{\Omega}\hat{\mathscr{D}}_t c + \hat{\mathscr{D}}_i \chi^i = 0 \\ \frac{1}{2}\hat{\mathscr{D}}_j c + 2\chi^i \varpi_{ij} + \frac{1}{\Omega}\hat{\mathscr{D}}_t \psi_j - \hat{\mathscr{D}}_i \Psi^i{}_j = 0 \\ \frac{1}{\Omega}\hat{\mathscr{D}}_t \chi_j - \hat{\mathscr{D}}_i X^i{}_j = 0 \end{cases}$$

Bulk interpretation – e.g. the scalar c

$c/2$ “nut aspect” (magnetic mass) in contrast to $4\pi G\varepsilon$ “Bondi mass aspect” (electric mass) combined in $\hat{\tau} = -c + 8\pi \mathrm{i} G\varepsilon$

$3+1$ Ricci-flat bulk from $2+1$ Carrollian bry

Resummable under conditions → algebraic Petrov

→ remove shear and Chthonian & dualize the energy flux and stress

$$\mathrm{d}s^2_{\text{res. Ricci-flat}} = \mu\left[2\mathrm{d}r + 2\left(r\varphi_j - *\hat{\mathscr{D}}_j *\varpi\right)\mathrm{d}x^j - \left(r\theta + \hat{\mathscr{K}}\right)\mu\right] + \rho^2\mathrm{d}\ell^2 + \frac{\mu^2}{\rho^2}\left[8\pi G\varepsilon r + *\varpi c\right]$$

- $\rho^2 = r^2 + *\varpi^2$ (geometric series)
- $\mu = -\Omega\mathrm{d}t + b_i\mathrm{d}x^i$ and $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{2}{P^2}\mathrm{d}\zeta\mathrm{d}\bar{\zeta}$ (bry geometry)
- $-c + 8\pi\mathrm{i}G\varepsilon = \hat{\tau}$ (complex mass aspect)
- Einstein' eqs. & Cotton conservation → flux-balance equations

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\chi}_j - \hat{\mathscr{D}}_i\hat{X}^i{}_j = 0 \qquad \frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\tau} - \hat{\mathscr{D}}_j\hat{\chi}^j = 0$$
$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\psi}_i - \frac{1}{2}\hat{\mathscr{D}}_i\hat{\tau} - \hat{\mathscr{D}}^j\hat{\Psi}_{ij} + 2\mathrm{i} *\varpi *\hat{\chi}_i = 0$$

[Ciambelli et al. '18; Barnich et al. '19; Freidel et al. '21]

Contact with Newman–Penrose [NP '68]

Parallelly transported null tetrad $\mathbf{k} = \partial_r, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}}$

$$\text{here} \begin{cases} \Psi_0 = \Psi_1 = 0 & \mathscr{C}_{ij} = 0 \Rightarrow \text{Goldberg–Sachs} \\ \Psi_2 = \frac{i\hat{\tau}}{2(r - i{*}\varpi)^3} \\ \Psi_3 = \frac{iP\chi_\zeta}{(r - i{*}\varpi)^2} + O\left(1/(r - i{*}\varpi)^3\right) \\ \Psi_4 = \frac{iX_\zeta{}^{\bar{\zeta}}}{r - i{*}\varpi} + O\left(1/(r - i{*}\varpi)^2\right) \end{cases}$$

generally $\Psi_0^0 \propto iE_\zeta{}^{\bar{\zeta}}$, $\Psi_1^0 \propto iN_\zeta$, etc.

$$\dot{\psi}_3^0 = -\eth\psi_4^0,$$

$$\dot{\psi}_2^0 = -\eth\psi_3^0 + \sigma^0\psi_4^0,$$

$$\dot{\psi}_1^0 = -\eth\psi_2^0 + 2\sigma^0\psi_3^0,$$

Figure: NP '68 Eqs. (4.7)–(4.9)

Spin-off: elec. / magn. charges from the boundary

Riemannian boundary in asymptotically AdS bulk

ξ bry. conf. Killing (max 10) $\to I^\mu = \xi_\nu T^{\mu\nu}$ and $I^\mu_{\text{Cot}} = \xi_\nu C^{\mu\nu}$

$$Q_\xi = \int_{\Sigma_2} *\mathrm{I} \quad \text{and} \quad Q_{\text{Cot}\,\xi} = \int_{\Sigma_2} *\mathrm{I}_{\text{Cot}}$$

electric and *magnetic* conserved charges (bulk mass vs. nut)

Carrollian boundary for Ricci-flat spacetimes

- *infinite tower of conformal Killings* (asymptotic symmetries)

$$d+1=3 \to \mathfrak{ccarr}(3) \equiv \mathfrak{so}(3,1) \ltimes \text{supertransls.} \equiv \text{BMS}_4$$

- every Carrollian item (infinite number) & its magnetic dual produce towers of charges – *not always conserved*

Carrollian stationary boundary dynamics

Assuming ∂_t be a Killing $\to \theta = 0, \Omega = 1, \varphi_i = 0, \hat{\mathscr{K}} = K$

- $\mathrm{d}\ell^2 = \frac{2}{P^2(\zeta,\bar\zeta)}\mathrm{d}\zeta\mathrm{d}\bar\zeta$ $\quad K = \Delta \ln P$
- $\mu = -\mathrm{d}t + b_\zeta \mathrm{d}\zeta + b_{\bar\zeta}\mathrm{d}\bar\zeta$ $\quad *\varpi = \frac{\mathrm{i}P^2}{2}\left(\partial_\zeta b_{\bar\zeta} - \partial_{\bar\zeta} b_\zeta\right)$
- $\hat\tau = -c + 8\pi \mathrm{i} G\varepsilon$ $\quad$ complex mass aspect
- Flux-balance eqs. $\quad \Delta K = 0 \quad \partial_{\bar\zeta}\hat\tau = 0$

$$\boxed{2K = \hat{k}(\zeta) + \bar{\hat{k}}(\bar\zeta) \quad \hat\tau = \hat\tau(\zeta)}$$

Example: Kerr–Taub–NUT family (spherical case)

$$\hat{k} = K = 1 \quad \hat\tau = 2(\mathrm{i}M - n)$$
$$P = \tfrac{1}{2}\zeta\bar\zeta + 1 \quad *\varpi(\zeta,\bar\zeta) = n + a - \frac{2a}{P}$$

Boundary Möbius action [Mittal et al. '22]

Algebraic Ricci-flat spacetimes with Killing ∂_t

$$\tau\left(r,\zeta,\bar\zeta\right)=\frac{\hat\tau\left(\zeta\right)}{r+\mathrm{i}\ast\varpi\left(\zeta,\bar\zeta\right)}-\mathrm{i}\hat k\left(\zeta\right)$$

Carrollian boundary *local* transformations

$$\boxed{P'=\frac{P}{\left|\gamma\hat k+\mathrm{i}\delta\right|}\quad \hat k'=\mathrm{i}\frac{\alpha\hat k+\mathrm{i}\beta}{\gamma\hat k+\mathrm{i}\delta}\quad \hat\tau'=-\frac{\hat\tau}{\left(\gamma\hat k+\mathrm{i}\delta\right)^2}}$$

Example: Kerr–Taub–NUT family: $\hat\tau=2(\imath M-n)$, a

$$\hat\tau\to\hat\tau'=\hat\tau\,\mathrm{e}^{-2\mathrm{i}\chi}\quad a\to a'=a$$

electric & magnetic charges – mixed under Möbius

Quotable facts

- Ricci-flat spacetimes are reconstructed from boundary Carrollian-covariant data including the Cotton tensors
- part of the Newman–Penrose Ψs are Carrollian Cotton descendents
- a bulk isometry reveals into a boundary Ehlers' Möbius local invariance involving the Cotton as a magnetic facet

Further knowledge and investigation

- Towers of charges and dual charges
  - capture e.g. the multipolar moments [Geroch '70; Hansen '74]
  - organized wrt $SL(2,\mathbb{R})$
  - further comparison with bulk approaches for towers of charges and duality issues [Newman, Penrose '68; Godazgar^2, Pope '18–21]
- What could we learn for flat/celestial holography?

Starring

Lewis Carroll (1832 – 1898)

Poet and mathematician – Christ Church College, Oxford

Alice's Adventures in Wonderland & Through the Looking-Glass

Émile Cotton (1872 – 1950)

Professor of mathematics at the University of Grenoble

Cotton tensor

Jürgen Ehlers (1929 – 2008)

Max Planck Institute for Gravitational Physics – Potsdam

Ehlers group & Ehlers frame for the Newtonian limit of GR

$3+1$ Ricci-flat bulk from its $2+1$ Carrollian boundary

$\to$ add shear $\mathscr{C}_{ij}$ and Chthonian F_{ij}, …& no dualization

$$
\begin{aligned}
\mathrm{d}s^2_{\text{Ricci-flat}} \;=\; & 2\upmu\left[\mathrm{d}r - \frac{r\theta+\hat{\mathscr{K}}}{2}\upmu + \left(r\varphi_i - *\hat{\mathscr{D}}_i *\varpi - \frac{1}{2}\hat{\mathscr{D}}_j\mathscr{C}^j_{\ i}\right)\mathrm{d}x^i\right] \\
& +\mathscr{C}_{ij}\left(r\mathrm{d}x^i\mathrm{d}x^j - *\varpi *\mathrm{d}x^i\mathrm{d}x^j\right) + \left(r^2 + *\varpi^2 + \frac{\mathscr{C}_{kl}\mathscr{C}^{kl}}{8}\right)\mathrm{d}\ell^2 \\
& +\frac{1}{r}\left[8\pi G\varepsilon\upmu^2 - \frac{4}{3}\left(*\psi_i - 8\pi G\pi_i\right)\mathrm{d}x^i\upmu - \frac{16\pi G}{3}E_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] \\
& +\frac{1}{r^2}\left(*\varpi c\upmu^2 + F_{ij}\mathrm{d}x^i\mathrm{d}x^j + \cdots\right) + \mathrm{O}\left(1/r^3\right)
\end{aligned}
$$

- $\upmu = -\Omega\mathrm{d}t + b_i\mathrm{d}x^i$ and $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$
- $\hat{\mathscr{N}}^{ij} = \frac{1}{\Omega}\hat{\mathscr{D}}_t\mathscr{C}^{ij}$ covariant Bondi-like news
- $4\pi G\varepsilon - \frac{1}{8}\mathscr{C}^{jk}\hat{\mathscr{N}}_{jk} = M$ Bondi mass aspect
- $-\frac{c}{2} + 4\pi\mathrm{i}G\varepsilon = \hat{\tau}$ complex mass (electric *and* magnetic)
- $*\psi_i - 8\pi G\pi_i = N_i$ angular momentum aspect

Complete flux-balance equations & contact with NP

- $\frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\chi}_j - \hat{\mathscr{D}}_i\hat{X}^i_{\ j} = 0$
- $\frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\tau} - \hat{\mathscr{D}}_i\hat{\chi}^i = \frac{\mathrm{i}}{2}\left(\hat{\mathscr{D}}_i\hat{\mathscr{D}}_j\hat{\mathscr{N}}^{ij} + \mathscr{C}^{ij}\hat{\mathscr{D}}_i\hat{\mathscr{R}}_j + \frac{1}{2}\mathscr{C}_{ij}\frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\mathscr{N}}^{ij}\right)$
- $\frac{1}{\Omega}\hat{\mathscr{D}}_t\left(\mathrm{i}N_i + \hat{\psi}_i\right) - \frac{1}{2}\hat{\mathscr{D}}_i\hat{\tau} - \hat{\mathscr{D}}^j\hat{\Psi}_{ij} + 2\mathrm{i} *\varpi *\hat{\chi}_i = -\frac{\mathrm{i}}{2}\Big[\mathscr{C}^{ij}\hat{\mathscr{D}}_j\hat{\mathscr{K}} + *\mathscr{C}^{ij}\hat{\mathscr{D}}_j\hat{\mathscr{A}} - 4*\varpi*\mathscr{C}^{ij}\hat{\mathscr{R}}_j - \frac{1}{2}\hat{\mathscr{D}}^j\left(\hat{\mathscr{D}}_j\hat{\mathscr{D}}_k\mathscr{C}^{ik} - \hat{\mathscr{D}}^i\hat{\mathscr{D}}^k\mathscr{C}_{jk}\right) + \mathscr{C}^{ij}\hat{\mathscr{D}}^k\hat{\mathscr{N}}_{jk} + \frac{1}{2}\hat{\mathscr{D}}^j\left(\mathscr{C}^{ik}\hat{\mathscr{N}}_{jk}\right) - \frac{1}{4}\hat{\mathscr{D}}^i\left(\mathscr{C}^{jk}\hat{\mathscr{N}}_{jk}\right)\Big]$
- $\frac{1}{\Omega}\hat{\mathscr{D}}_t H_{ij} = \mathscr{H}_{ij}\left[\mathscr{C},\mathscr{N},\ldots\right]$ for E_{ij}, F_{ij}, …

$$\dot{\psi}_3^0 = -\eth\psi_4^0,$$
$$\dot{\psi}_2^0 = -\eth\psi_3^0 + \sigma^0\psi_4^0,$$
$$\dot{\psi}_1^0 = -\eth\psi_2^0 + 2\sigma^0\psi_3^0,$$
$$\dot{\psi}_0^0 = -\eth\psi_1^0 + 3\sigma^0\psi_2^0.$$