Carroll Symmetry & Memory Effect for gravitational waves

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Abstract: Observing the motion of test particles in a Gravitational Wave could provide a way to detect the latter. Following Souriau (1973), the geodesic equations can be integrated using the 5-parameter isometries of plane gravitational waves, identified as Lévy-Leblond 's (1965) "**Carroll**" group in 2+1 dimensions with no rotations. The group acts as symmetry for the particle subject to the wave; the associated conserved quantities determine the trajectories.

Based on:

- *"Carroll symmetry of gravitational plane waves,"* Class. Quant. Grav. **34** (2017) 175003 [arXiv:1702.08284 [gr-qc]]

- *"The Memory Effect for Plane Gravitational Waves,"* Phys. Lett. B **772** (2017) 743. [arXiv:1704.05997 [gr-qc]].

- *"Soft gravitons and the memory effect for plane gravitational waves,"* Phys. Rev. D **96** (2017) no.6, 064013 [arXiv:1705.01378 [gr-qc]].

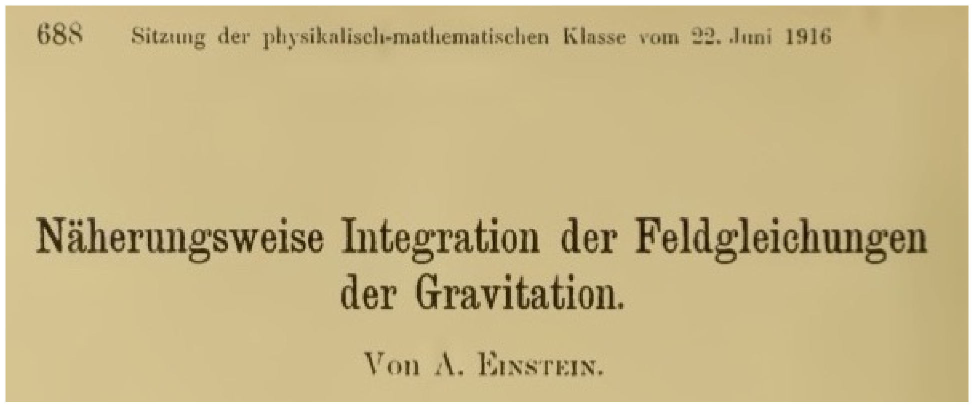
- *"Sturm-Liouville and Carroll: at the heart of the Memory Effect,"* Gen. Rel. Grav. **50** (2018) no.9, 107 [arXiv:1803.09640 [gr-qc]].

Road map:

- Gravitational waves from Einstein to LIGO
- The Memory Effect (Zel'dovich-Polnarev)
- Isometries & Carroll (Souriau & Lévy-Leblond)
- Geodesics in Brinkmann coordinates
- Example: gravitational collapse
- Isometries (in Brinkmann)
- Isometries (in BJR)
- Isometries & geodesics

Gravitational Waves

from
Einstein to LIGO


688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Einstein (1916) predicts GWs. However,

A. Einstein and N. Rosen *"On Gravitational waves,"* J. Franklin Inst. **223** (1937) 43. casts doubt on their physical existence ⇝ controversy, long debate. e.g.

H. Bondi *"Plane Gravitational Waves in General Relativity,"* Nature, **179** (1957) 1072-1073.

Ya. B. Zel'dovich and A. G. Polnarev "Radiation of gravitational waves by a cluster of superdense stars," Astron. Zh. **51**, 30 (1974)

resonant "Weber bar"

$\rightsquigarrow$ bombshell report:

J. Weber, Phys. Rev. Lett. **22** (1969) 1320

EVIDENCE FOR DISCOVERY OF GRAVITATIONAL RADIATION*

J. Weber
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 29 April 1969)

Coincidences have been observed on gravitational-radiation detectors over a base line of about 1000 km at Argonne National Laboratory and at the University of Maryland. The probability that all of these coincidences were accidental is incredibly small. Experiments imply that electromagnetic and seismic effects can be ruled out with a high level of confidence. These data are consistent with the conclusion that the detectors are being excited by gravitational radiation.

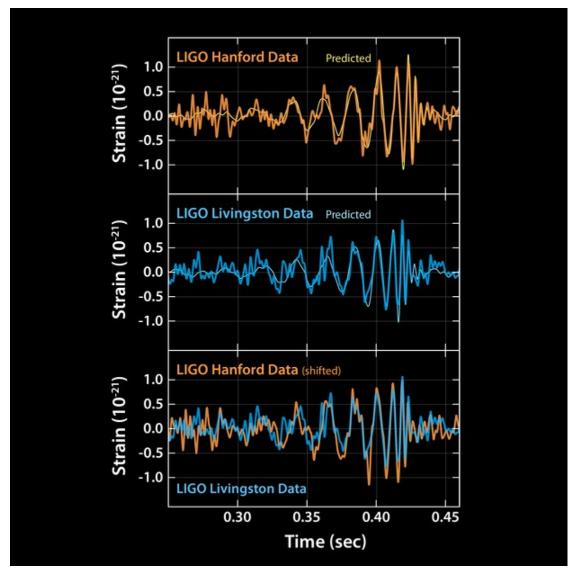
Weber's claims NOT confirmed, though

https://aeon.co/essays/how-joe-weber-s-gravity-ripples-turned-out-to-be-all-noise

Attracts lots of attention, marks whole generation of physicists.

e.g. G. W. Gibbons S. W. Hawking *"Theory of the detection of short bursts of gravitational radiation,"* Phys. Rev. D **4** (1971) 2191. . . . *no* funding from SRC ⇝ return to theory . . . ⇝ . . .

Research taken up by Kip Thorn (team of Wheeler & team of Zeldovich) ⇝ LIGO, VIRGO ≈ 1500 people, 1 billion $ . . .



Thorn-Weiss-Barish [+ Ron Drever] Nobel 2017

"For as long as 40 years, people have been thinking about this, trying to make a detection, sometimes failing in the early days, and then slowly but surely getting the technology together to be able to do it" (Weiss)

Memory

Ya. B. Zel'dovich and A. G. Polnarev, *"Radiation of gravitational waves by a cluster of superdense stars,"* Astron. Zh. **51**, 30 (1974)

> *. . . detector, consisting of two noninteracting bodies (such as satellites). [. . .] the distance between a pair of free bodies should change, and in principle this effect might possibly serve as a nonresonance detector. [. . .] although distance between free bodies will change, their relative velocity will become vanishingly small*

(gives *no* proof)

Elaborated by V.B. Braginsky & L. P. Grishchuk

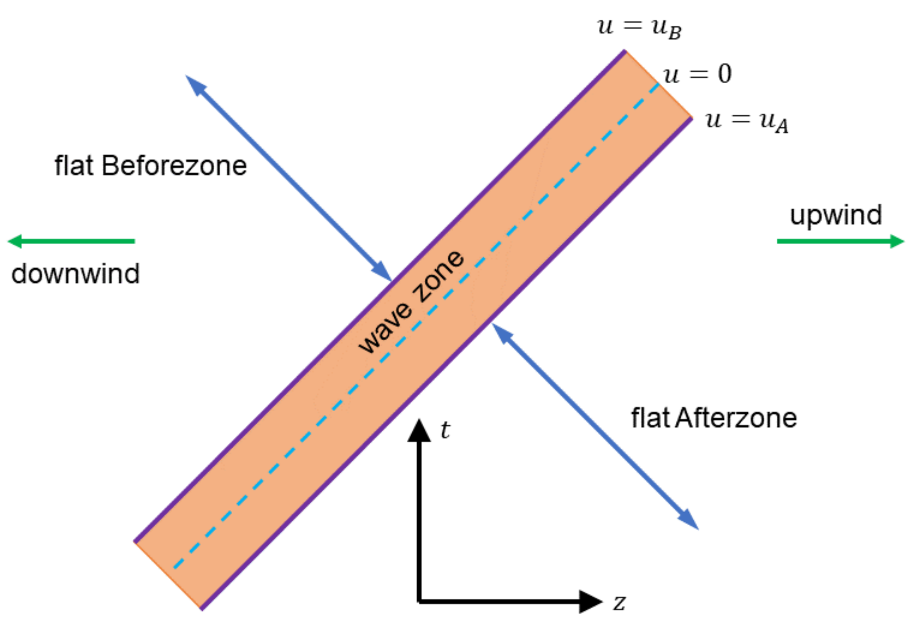
"Kinematic resonance and the memory effect in free mass gravitational antennas," Zh. Eksp. Teor. Fiz. **89** *744-750 (1985)* who introduce

"memory effect"

> *"distance between a pair of bodies is different from the initial distance in the presence of a gravitational radiation pulse. . . . possible application to detect gravitational radiation . . ."*

Assumption (GR) : particles follow geodesics.

Sandwich wave: burst of gravitational wave. Spacetime non-flat only in short interval $u_B \leq u \leq u_A$ of retarded time [Wavezone]. Flat both in Beforezone $u < u_B$ that the wave has not reached yet, and in Afterzone $u_A < u$ where has already passed, see fig.



u flows from left to the right, whereas wave advances from right to left.

Isometries & Carroll
Souriau & Lévy-Leblond

Assumption: for very large distances GW approximated with exact plane GW

H. Bondi, F. A. E. Pirani and I. Robinson, *"Gravitational waves in general relativity. 3. Exact plane waves,"* Proc. Roy. Soc. Lond. A **251** (1959) 519 : plane GWs have 5-parameter group of isometry: 3 translations + 2 WHAT ?

Souriau *"Ondes et radiations gravitationnelles,"* Colloques Internationaux du CNRS No 220, pp. 243-256. Paris (1973): symmetry $\rightsquigarrow$ integration of geodesic eqns.

C. Duval, et al. *"Carroll symmetry of gravitational plane waves,"* Class. Quant. Grav. **34** (2017) 175003 [`arXiv:1702.08284 [gr-qc]`] : in Baldwin-Jeffery-Rosen (BJR) coords isometries $\equiv$ "Carroll" group Lévy-Leblond 1965 : $c \to 0$ contraction of Poincaré

Geodesics in Brinkmann* coordinates

plane GWs

$$\delta_{ij}dX^i dX^j + 2dUdV + K_{ij}(U)X^iX^jdU^2 \qquad (1a)$$

profile $K_{ij}(U)X^iX^j =$

$$\tfrac{1}{2}\mathcal{A}_+(U)\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_\times(U)\,X^1X^2 \qquad (1b)$$

where $\mathcal{A}_+$ and $\mathcal{A}_\times$ $+$ and $\times$ polarization-state amplitudes. $\boldsymbol{X} = (X^i)$ transverse, U, V light-cone coords. $(X^\mu) = (U, \boldsymbol{X}, V)$ global

Vacuum Einstein solutions : Ricci flat

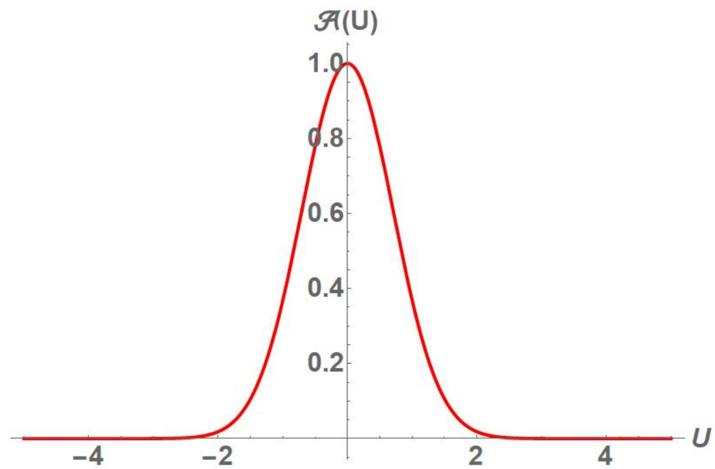
$$R_{\mu\nu} = 0 \Leftrightarrow \mathrm{Tr}(K_{ij}) = 0. \qquad (2)$$

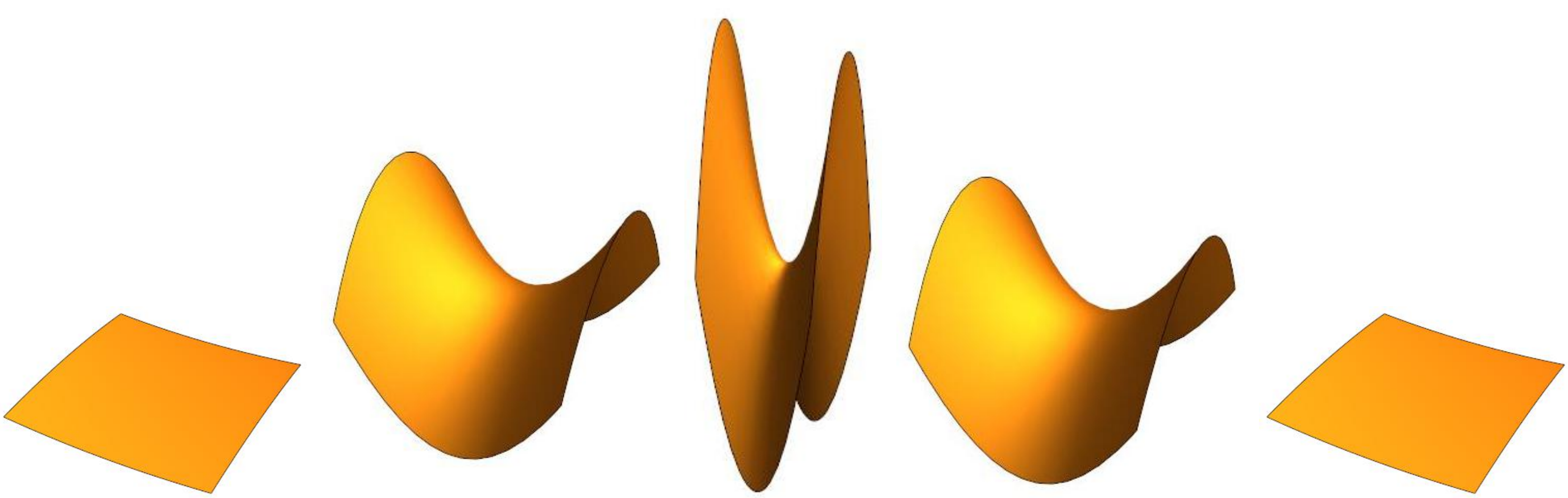
Sandwich wave: $K(U) \neq 0$ only in "wave zone" $U_B < U < U_A$. Assumption : metric Minkowski in "Beforezone" $U < U_B$, flat in "Afterzone" $U_A < U$.

* M. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94** (1925) 119–145.

• Linearly polarized burst $\mathcal{A}_\times = 0$ with Gaussian profile

$$K_{ij}(U)X^iX^j = \frac{e^{-U^2}}{\sqrt{\pi}}\Big((X^1)^2 - (X^2)^2\Big). \qquad (3)$$





Gaussian profile $\mathcal{A}_+(u) = \exp[-u^2]$.

Geodesics : solution of

$$\frac{d^2X^1}{dU^2} - \frac{1}{2}\mathcal{A}_+ X^1 = 0, \tag{4a}$$

$$\frac{d^2X^2}{dU^2} + \frac{1}{2}\mathcal{A}_+ X^2 = 0, \tag{4b}$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}_+}{dU}\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_+\Big(X^1\frac{dX^1}{dU} - X^2\frac{dX^2}{dU}\Big) = 0\,. \tag{4c}$$

$X^{1,2}$-components decoupled. Projection of $4D$ worldline to transverse $(X^1 - X^2)$ plane independent of $V(U_0)$ & $\dot{V}(U_0)$.

Assumption: particle at rest in Beforezone:

$$\boldsymbol{X}(U) = \boldsymbol{X}_0, \quad \dot{\boldsymbol{X}}(U) = 0 \quad U \leq U_B. \tag{5}$$

N.B. For affine parameter ($\sim$ "dot") $-g_{\mu\nu}\dot{X}^\mu\dot{X}^\mu = m^2$ const of the motion. For $m = 0$ null lift. For $m^2 \neq 0$ shift $m^2U \Rightarrow$ restrict to $m = 0 \Rightarrow$ enough to solve transverse eqn (4a)-(4b).

N.B. GW plane wave fits to **Eisenhart-Duval framework** $\equiv$ relativistic ("Kaluza-Klein") desription for non-relativistic physics

Profile $-\frac{1}{2}K_{ij}(U)X^iX^j$ of Brinkmann (1) $\sim$ Newton potential

$$\delta_{ij}dX^idX^j + 2dUdV + \underbrace{K_{ij}(U)X^iX^j}_{-2(\text{Newton potential})} dU^2$$

Framework originally proposed by

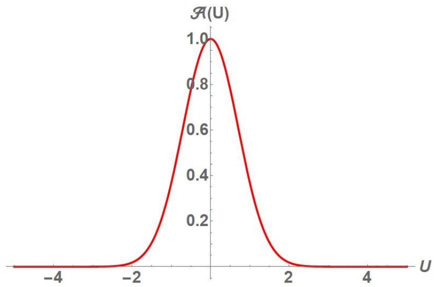
L. P. Eisenhart, *"Dynamical trajectories and geodesics"*, Annals. Math. **30** 591-606 (1928).

- forgotten - then rediscovered independently :

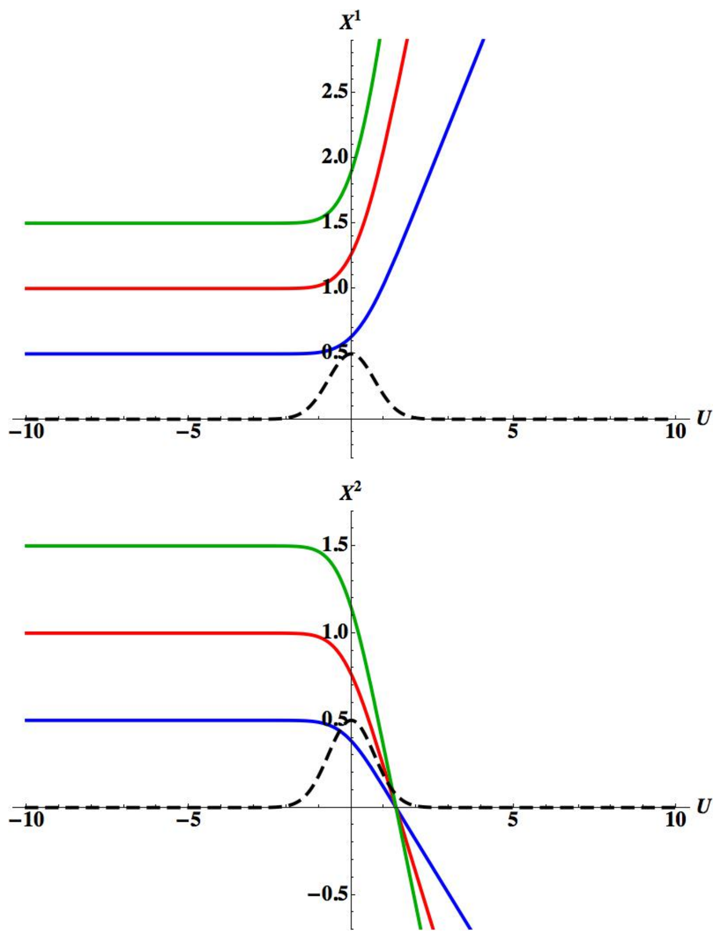
C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, *"Bargmann Structures and Newton-Cartan Theory,"* Phys. Rev. D **31** (1985) 1841.

C. Duval, G. W. Gibbons, and P. A. Horvathy, *"Celestial Mechanics, Conformal Structures and Gravitational Waves,"* Phys. Rev. **D43**, 3907 (1991)

• linearly polarized GW with Gaussian profile (1a-b) with $\mathcal{A}_{+} = e^{-U^2}$ $\mathcal{A}_{\times} = 0$

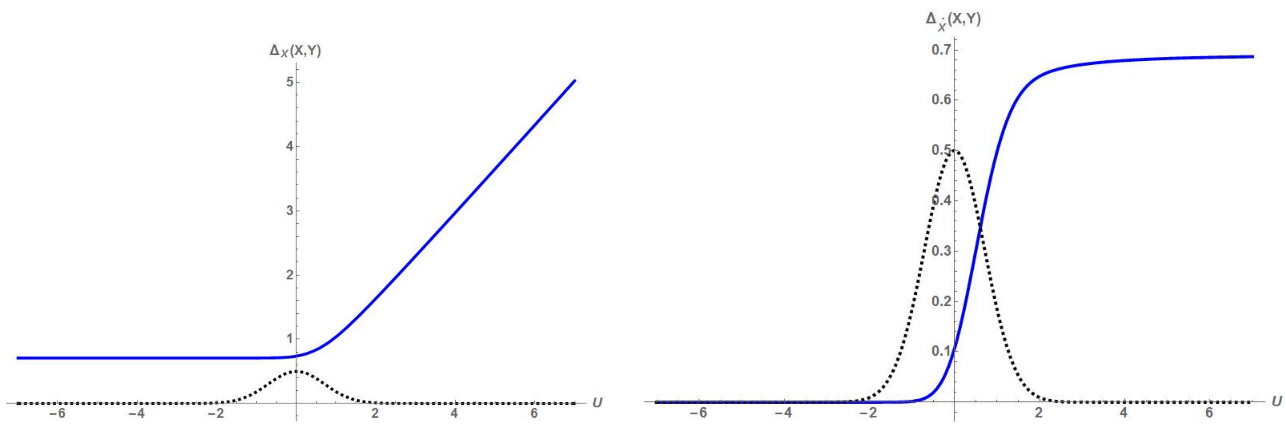


$K_{ij}(U)X^iX^j = e^{-U^2}\big((X^1)^2 - (X^2)^2\big)$ repulsive in X^1 attractive in X^2 cf. (4a-b) $\rightsquigarrow$



Geodesics for Gaussian burst for blue/red/green positions in Beforezone. X^2 *focuses for all initial positions* $(0, X^2_0)$ *!*

Variation of relative (euclidean) distance $\Delta_X(\boldsymbol{X},\boldsymbol{Y}) = |\boldsymbol{X}-\boldsymbol{Y}|$ and of relative velocity $\Delta_{\dot{X}} = |\dot{\boldsymbol{X}}-\dot{\boldsymbol{Y}}|$. Latter could (in principle) be observed through the Doppler effect (Braginski-Grishchuk).



(a) (b)

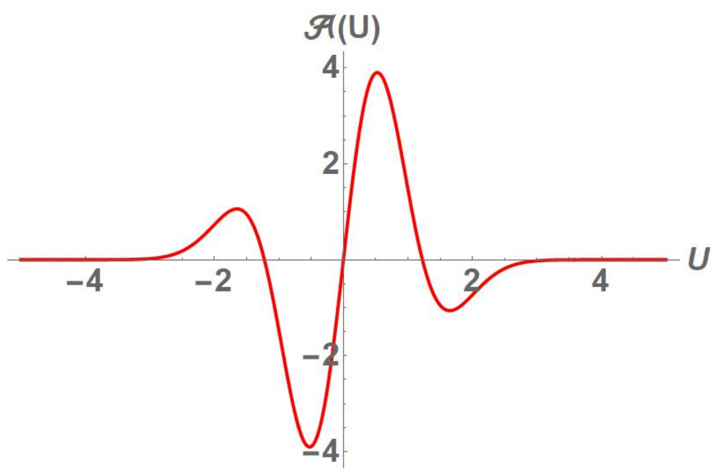
In Gaussian case (a) Two particles initially at rest recede from each other after wave has passed. Their distance, Δ_X, increases roughly linearly in the after-zone. (b) The relative velocity, $\Delta_{\dot{X}}$, jumps rapidly to approximately constant but non-zero value.

disproves Zel'dovich-Polnarev 1974 **NO** simple displacement !

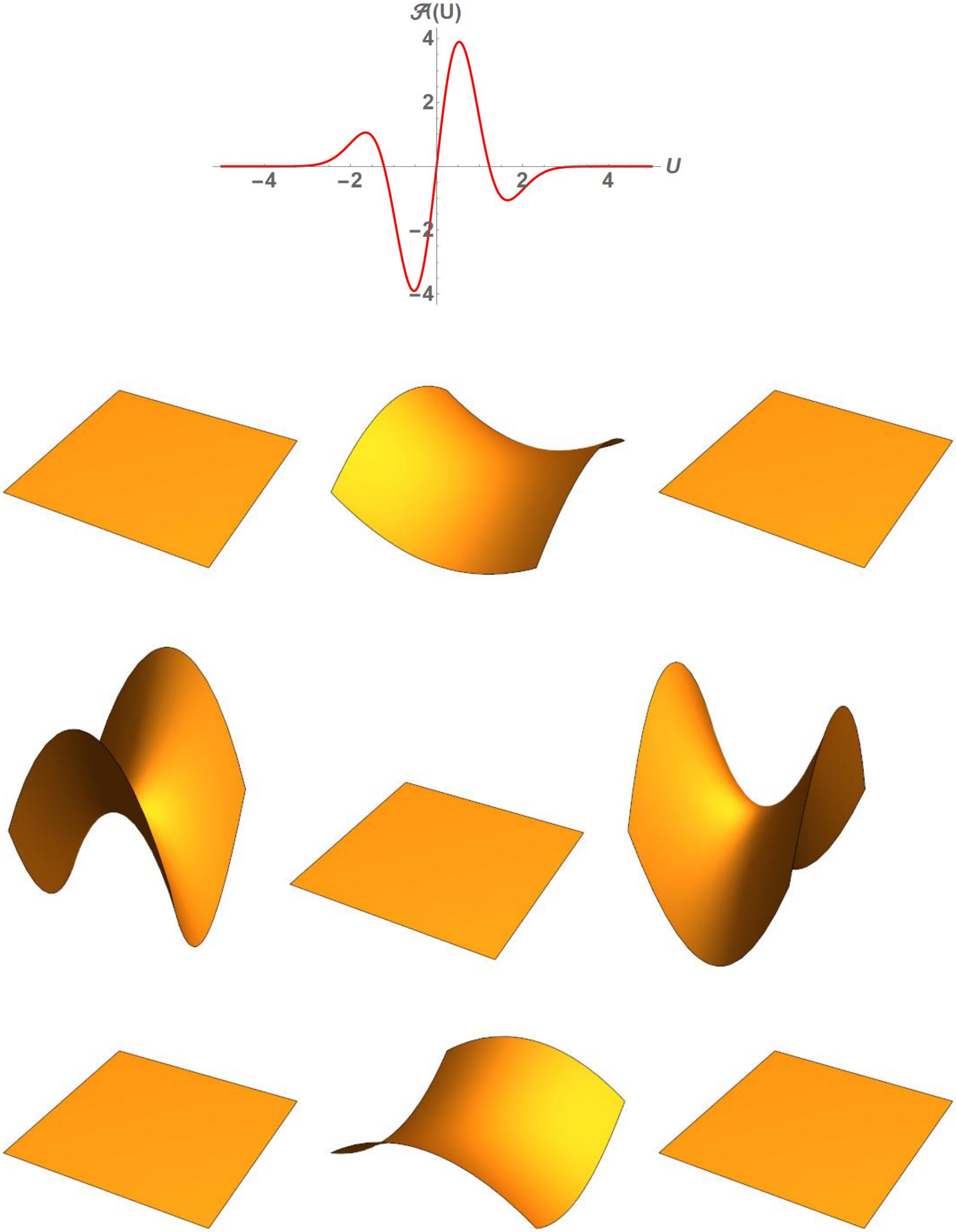
Example: gravitational collapse

Gibbons & Hawking (G-H) 1971 : **Gravitational collapse** $\rightsquigarrow$ linearly polarized GH profile

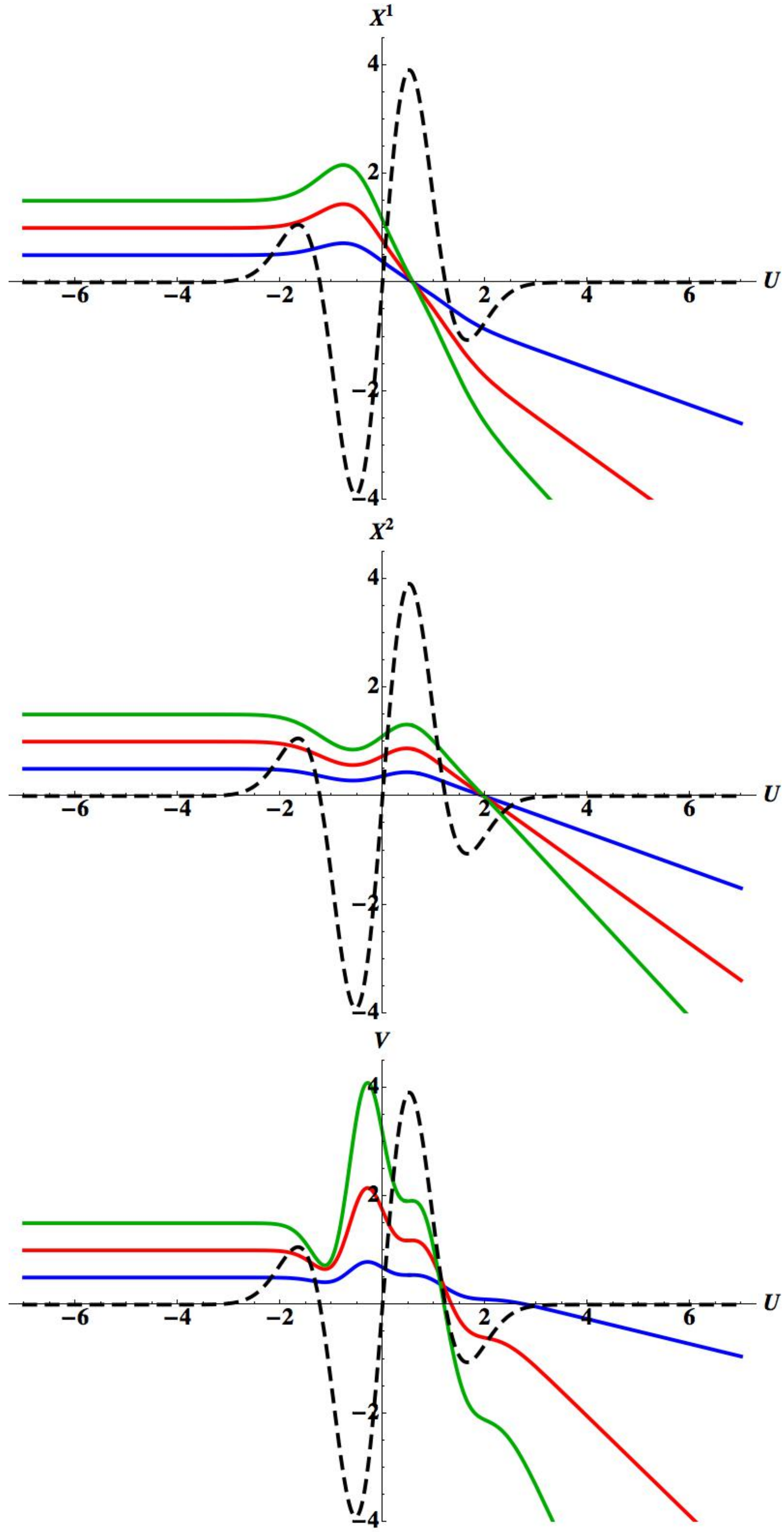
$$\mathcal{A}_+(U) = \frac{1}{2}\frac{d^3(e^{-U^2})}{dU^3} \tag{6}$$



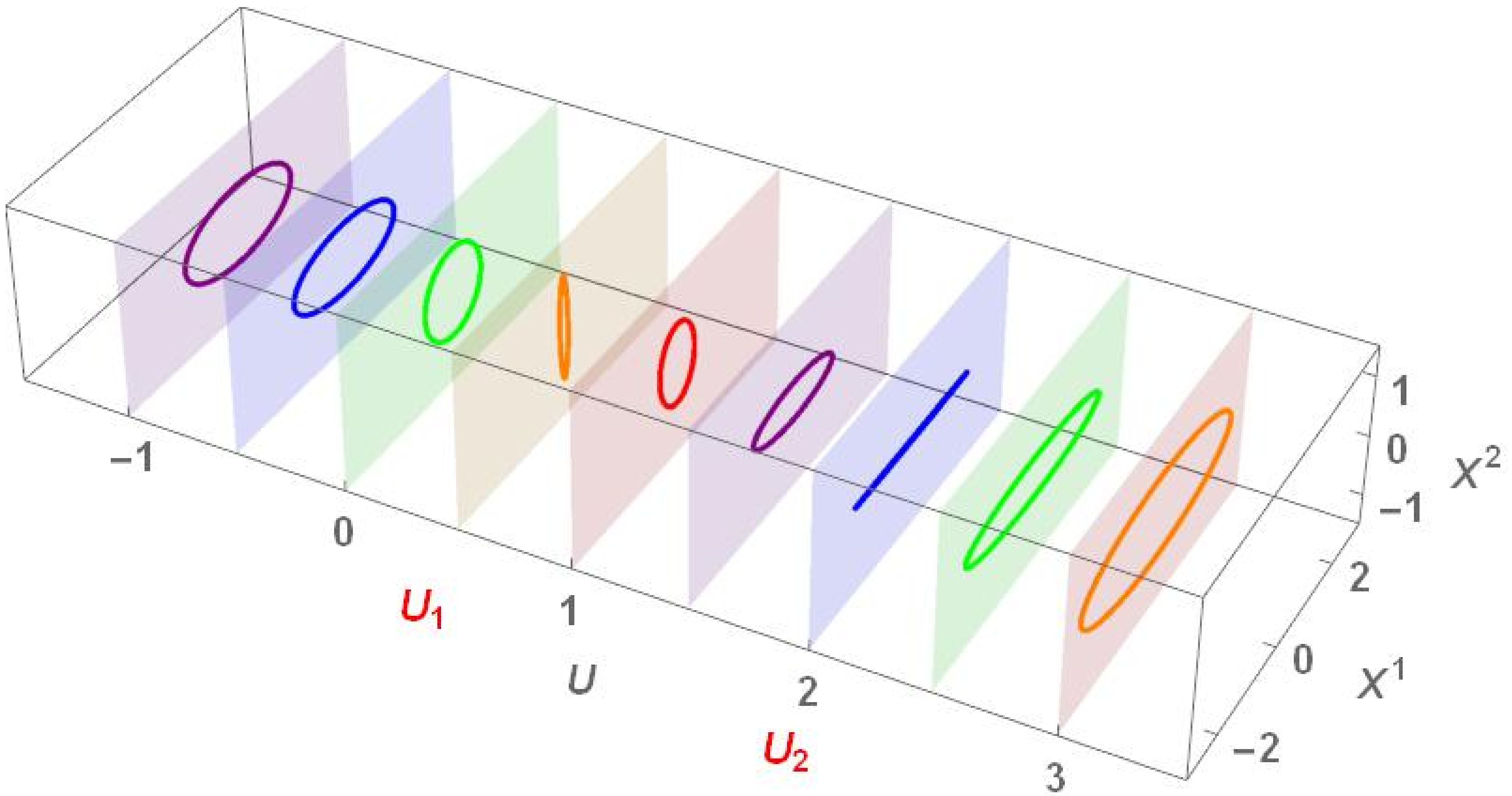
Attractive/repulsive directions alternate with sign.



"Time" evolution of G-H *wave profile* $\mathcal{A}_+(U) = (\exp[-U^2])'''$



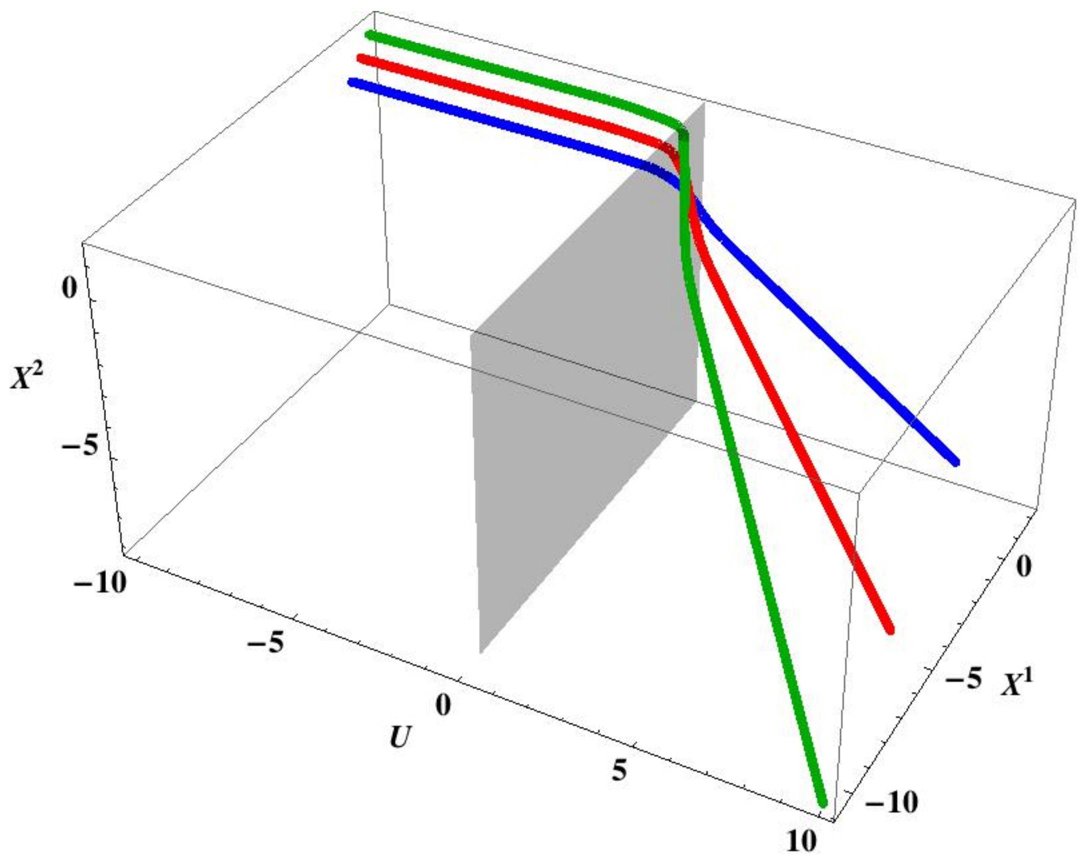
Geodesics for particles at rest in Beforezone ∼ *gravitational collapse profile* G-H*.* X^1 *focused to* $U_1 = 0.593342$ *and* X^2 *to* $U_2 = 1.97472$*.*



"Tissot" diagram for "collapse profile"* $\frac{1}{2}\, d^3(e^{-U^2})/dU^3$ *(6). Circle at* $u = u_0 = 0$ *is deformed to ellipse, which at* U_1, U_2 *circle degenerates to vertical/horizontal segment.*

* Nicolas-Auguste Tissot (1824–1897) cartographer. Tissot indicatrix is graphical representation that describes its distortion on a map.

- For $K_{ij} \neq 0$???



Motion for collapse profile $\frac{1}{2}d^3(e^{-U^2})/dU^3$. *In flat Afterzone motion is (approximately) along diverging straight lines* $\sim$ Bargmann $\rightsquigarrow$ Newton's 1st law !!!

VELOCITY EFFECT

Non-vanishing constant velocity in Afterzone $\rightsquigarrow$ disproves (again) Zel'dovich-Polnarev 1974

agrees with Ehlers-Kundt 1962, Souriau 1973 Braginsky-Thorn 1987, Bondi-Pirani 1988 Grishchuk-Polnarev 1989, . . .

Isometries (in Brinkmann)

Bondi-Pirani-Robinson 1959: metric (11) has 5-dim isometry group. In Brinkmann coords $(\boldsymbol{X}, U, V)$

$$\delta_{ij} dX^i dX^j + 2dUdV + K_{ij}(U) X^i X^j dU^2$$

cf. in (1).

Torre *"Gravitational waves: Just plane symmetry,"* Gen. Rel. Grav. **38** (2006) 653 : Killing vectors

$$S_i(U)\partial_i + \dot{S}_i(U) X^i \partial_V\,, \quad \partial_V, \tag{7}$$

"dot" $= d/dU$. S_i, $i = 1, 2$ solution of vector Sturm-Liouville eqn

$$\ddot{S}_i(U) = K_{ij}(U) S_j(U)\,. \tag{8}$$

• In Minkowski $K_{ij} \equiv 0$, (8) solved by

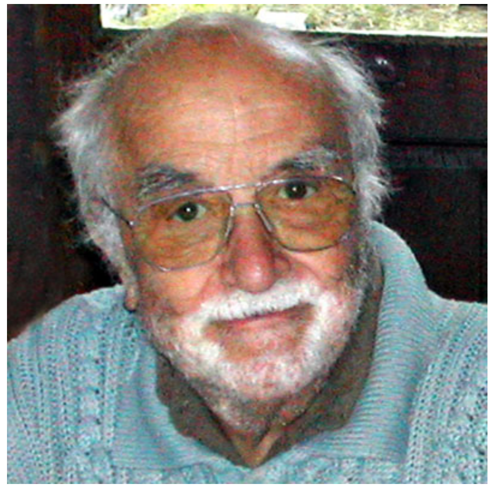
$$S_i = \gamma_i + \beta_i U \tag{9}$$

combination of translations in transverse plane X^1–X^2 +Galilei boosts lifted to Bargmann space,

$$Y = (\gamma_i + U\beta_i)\partial_i + \left(\delta + X^i \beta_i\right)\partial_V. \tag{10}$$

$i = 1, 2$, $\delta =$ const. (5th isometry = "vertical translation" generated by ∂_V).

Isometries & geodesics (in BJR)



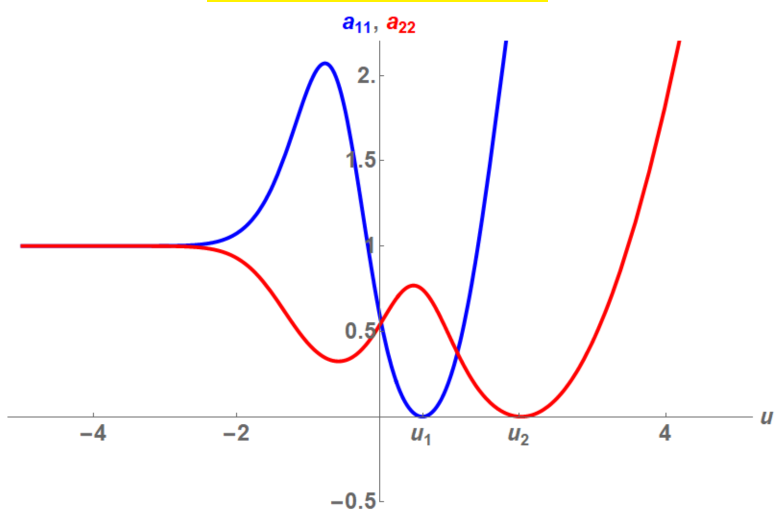
J-M. Souriau 1973

Using Baldwin-Jeffery-Rosen (BJR) coordinates $(\boldsymbol{x}, u, v)$ metric takes form,

$$a_{ij}(u)\, dx^i dx^j + 2du\, dv\,. \qquad (11)$$

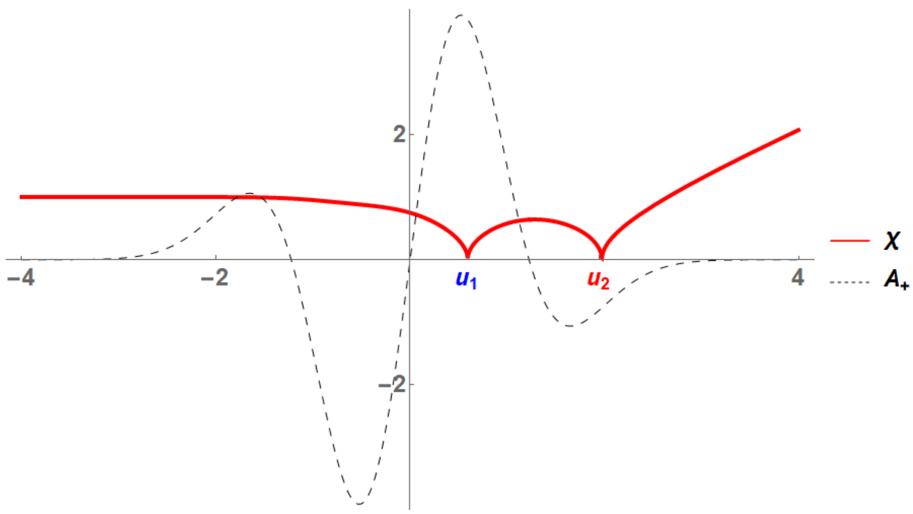
$a(u) \equiv (a_{ij}(u))$ positive 2×2 matrix.
Brinkmann/Bargmann "potential" $K_{ij}X^iX^j$ traded for transverse metric $a_{ij}(u)$.

BJR coords $(u, \boldsymbol{x}, v)$ "no global" $\equiv$ coordinate



singularities

coord sing detected by $\det(a) = 0$. Souriau: ALWAYS exist u_1 where $\det(a)(u_1) = 0$



$\chi = (\det(a))^{1/4}$ *for "collapse" wave. Zeros of* χ *coincide with points* u_i, $i = 1, 2$, *where Brinkmann trajectories are focused. In flat Outside regions* χ *approximately linear.*

Souriau: in coordinate patch $u_1 < u < u_2$ isometries implemented on space-time

$$\begin{array}{lcl} u & \to & u, \\ \boldsymbol{x} & \to & \boldsymbol{x} + H(u)\,\mathbf{b} + \mathbf{c}, \\ v & \to & v - \mathbf{b}\cdot\boldsymbol{x} - \frac{1}{2}\mathbf{b}\cdot H(u)\,\mathbf{b} + \nu, \end{array} \tag{12}$$

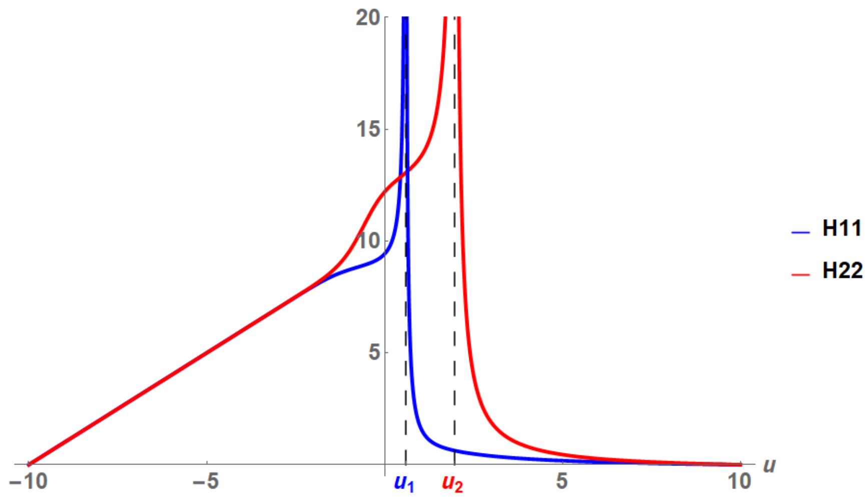
Acts on $u =$ const slice.

where $H(u)$ is symmetric 2×2 Souriau matrix,

$$\boxed{H(u) = \int_{u_0}^{u} a(t)^{-1} dt} \qquad u_1 < u_0 < u_1 \tag{13}$$

• for $a = \mathrm{Id}$ (Minkowski) $\Rightarrow$ $H(u) = u - u_0$ $\Rightarrow$ Galilei.

• for collapse profile

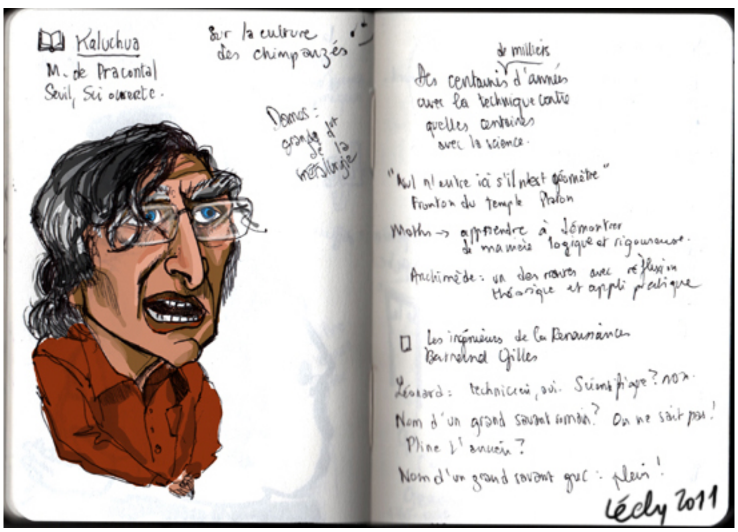


Souriau matrix for collapse profile. In Beforezone, $H(u) \approx u\,\mathrm{Id}$. In Afterzone $H(u)$ falls off rapidly.

Restriction to $u = u_0 = 0 \Rightarrow H(u) = 0 \rightsquigarrow$ boost implemented by

$$\boxed{\begin{cases} \boldsymbol{x}' &= \boldsymbol{x} \\ v' &= v - \boldsymbol{b}\cdot\boldsymbol{x} \end{cases}} \qquad (14)$$

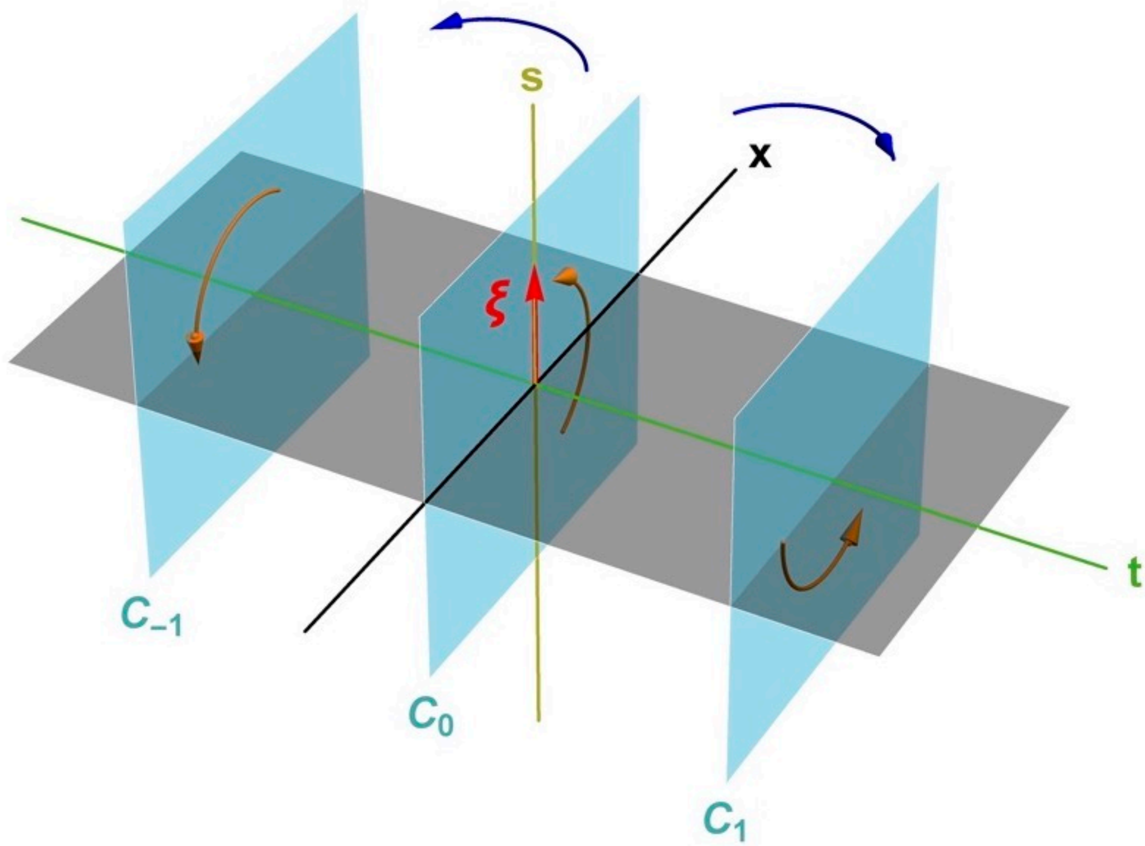
$\equiv$ Lévy-Leblond's "Carroll" boost with broken rotations.



J.-M. Lévy-Leblond, *"Une nouvelle limite non-relativiste du group de Poincaré,"* Ann. Inst. H Poincaré **3** (1965) 1

NB: Relation not realized by Souriau ... recalled by Duval in 2017 ...

implementation on $u-$const slice obtained by "exporting" from u_0 using Souriau matrix $H(u)$.

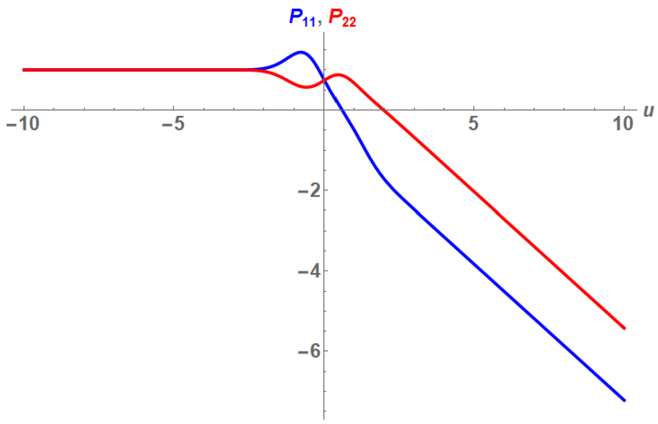


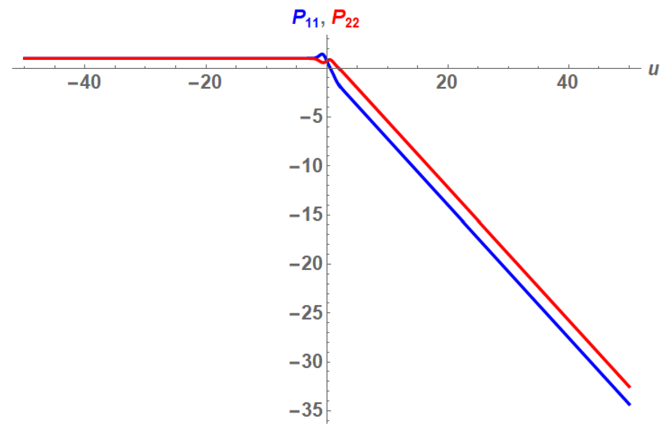
Combining with BJR $\to$ Brinkman $\rightsquigarrow$ boosts in Brinkmann:

$$\boldsymbol{X} \to \boldsymbol{X} + Q(u)\,\mathbf{b} \qquad Q(u) = P(u)H(u). \tag{15}$$

In afterzone

- numerical solution for P:





$P_{ii} \approx$ linear & *have same slope* C *for both components,*

$$P_{11}(u) \approx Cu + B \quad \& \quad P_{22}(u) \approx Cu + D\,. \qquad (16)$$

- Souriau matrix can be integrated,

$$H_{11}(u) \approx -\frac{1}{C(Cu+B)},\ H_{22}(u) \approx -\frac{1}{C(Cu+D)}\,. \qquad (17)$$

H and P combine approx to const $\neq 0$ matrix,

$$Q = HP \approx \frac{1}{C}\mathrm{diag}(1,1) \qquad (18)$$

$\Rightarrow$ boosts act in Afterzone as translations ?,

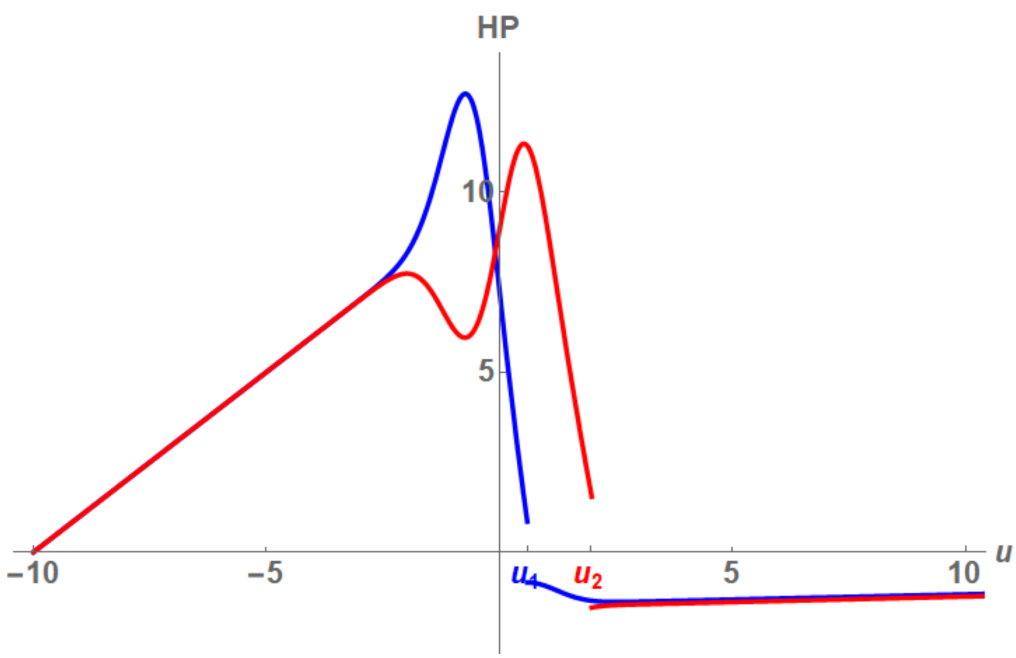
$$\boldsymbol{X} \to \boldsymbol{X} + \frac{1}{C}\,\mathbf{b}\,. \qquad (19)$$

N.B. Elbistan et al $Q = PH$ satisfies same Sturm-Liouville eqn (8), as P ,

$$\ddot{Q} = K(u)Q\,, \qquad Q^\dagger\dot{Q} = \dot{Q}\,Q^\dagger\,. \qquad (20)$$

Afterzone: $K \approx 0 \;\Rightarrow\; Q = Au+B$ but why $A \approx 0$?

However confirmed numerically :



Matrix $Q = HP$ *is usual Galilean expression* $(u-u_0)\mathrm{Id}$ *in Beforezone, but* $\approx$ *constant* (18) *in Afterzone. (At* u_i *have numerical uncertainty* $\sim$ *singularity of* H *and vanishing of* P*).*

Isometries & geodesics

Brinkmann $\Leftrightarrow$ (BJR) coords $(\boldsymbol{X}, U, V)$? $\Leftrightarrow (\boldsymbol{x}, u, v)$:

G. W. Gibbons *"Quantized Fields Propagating in Plane Wave Space-Times,"* Commun. Math. Phys. **45** (1975) 191.

$$U = u\,, \quad \boldsymbol{X} = P(u)\boldsymbol{x}\,, \quad V = v - \frac{1}{4}\boldsymbol{x}\cdot\dot{a}(u)\boldsymbol{x} \qquad (21)$$

where 2×2 matrix $P = (P_{ij})$ is solution of matrix Sturm-Liouville pb cf. (8)

$$\boxed{a_{ij} = \left(P^\dagger P\right)_{ij}, \quad \ddot{P} = K(u)P\,, \quad P^\dagger\dot{P} = \dot{P}^\dagger P} \qquad (22)$$

Noether $\Rightarrow$ 5 isometries $\Rightarrow$ conserved quantities. In BJR (from (12))

$$\mathbf{p} = a(u)\,\dot{\boldsymbol{x}}\,, \quad \mathbf{k} = \boldsymbol{x}(u) - H(u)\,\mathbf{p}\,, \qquad (23)$$

interpreted as conserved linear & boost-momentum, supplemented by $m = \dot{v} = 1$.

Extra const of motion $e = \frac{1}{2}\mathrm{g}_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu$. Geodesics timelike/lightlike/ spacelike if e negative/zero/positive.

Conversely, geodesics determined by Noether quantities,

$$\boldsymbol{x}(u) = H(u)\,\mathbf{p} + \mathbf{k}, \tag{24a}$$

$$v(u) = -\tfrac{1}{2}\mathbf{p}\cdot \boxed{H(u)}\,\mathbf{p} + e\,u + d, \tag{24b}$$

Only quantity to calculate is Souriau matrix $\boxed{H(u)}$.

• In flat Minkowski $a = \mathbf{1} \Rightarrow H(u) = u\,\mathbf{1}$, yields free motion

$$\boldsymbol{x}(u) = u\,\mathbf{p} + \mathbf{k}, \tag{25a}$$

$$v(u) = \Big(-\tfrac{1}{2}|\mathbf{p}|^2 + e\Big)\,u + v_0. \tag{25b}$$

usual boosts / usual motions.

Consider sandwich wave. In Beforezone $u < u_B$ $K = 0 \Rightarrow$ SL eqn. solved by $P(u) = \mathbf{1} \Rightarrow$ Brinkmann and BJR coords coincide.

Crucial fact : by (23) momentum of particle at rest vanishes, $\mathbf{p} = 0$ for $u \leq u_B$ because of initial condition $\dot{\boldsymbol{x}}(u) = 0$. $\mathbf{p}$ conserved $\Rightarrow$

$$\mathbf{p} = 0 \quad \text{for all } u \tag{26}$$

for any H i.e. for any metric a.

$$\boxed{\boldsymbol{x}(u) = \boldsymbol{x}_0, \quad v(u) = e\,u + v_0\,.} \tag{27}$$

In BJR coords particles initially at rest remain at rest during and after passage of wave !!

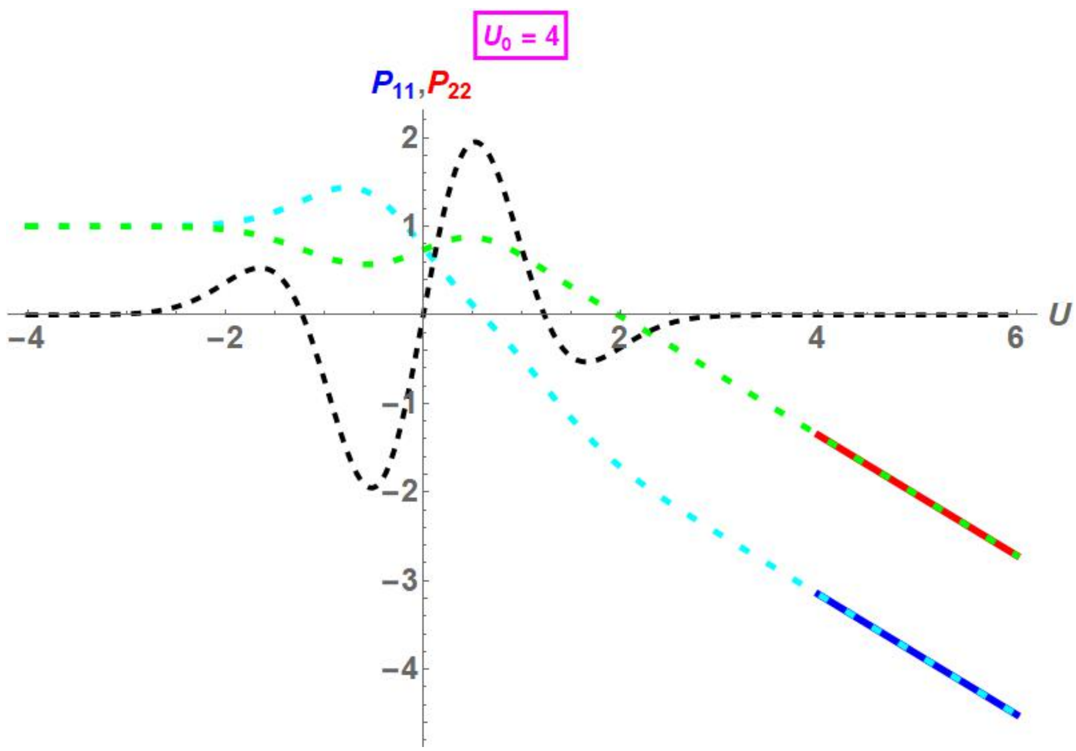
In Brinkmann coords both GWs and geodesics are global with no singularity. Solving SL eqns (22) [e.g. numerically] for P,

$$\boxed{\boldsymbol{X}(U) = P(u)\,\boldsymbol{x}^0, \quad \boldsymbol{x}^0 = \text{const}} \tag{28}$$

Complicated trajectory comes from $\mathbf{P(u)}$!!!

In flat afterzone $u \geq U_A$ exact analytic solution

Complicated-looking trajectories in B coords recovered: plots overlap perfectly *up to point where BJR coords becomes singular*.



Analytic (heavy line in **red**/**blue***) and numerical (dashed line in cyan) solutions overlap perfectly in (approx) Afterzone* $u \geq u_0 = 4$.