Computer algebra on manifolds with applications to gravity

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Outline

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SageMath in a few words

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1. everybody can use it, by download from `https://www.sagemath.org`
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SageMath is based on Python

- no need to learn any specific syntax to use it
- Python is a powerful *object oriented language*, with a neat syntax
- SageMath benefits from the Python ecosystem (e.g. Jupyter notebook, NumPy, Matplotlib)

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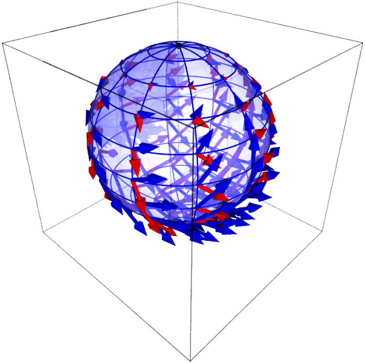
SageMath is developed by an enthusiastic community

- mostly composed of mathematicians
- welcoming newcomers

Differential geometry with SageMath

SageManifolds project: extends SageMath towards **differential geometry** and **tensor calculus**



Stereographic-coordinate frame on $\mathbb{S}^2$

- https://sagemanifolds.obspm.fr
- $\sim$ 119,000 lines of Python code
- fully included in SageMath (after review process)
- $\sim$ 30 contributors (developers and reviewers) cf. https://sagemanifolds.obspm.fr/authors.html
- dedicated mailing list
- help desk: https://ask.sagemath.org

Everybody is welcome to contribute
$\Longrightarrow$ visit https://sagemanifolds.obspm.fr/contrib.html

Current status

Already present (SageMath 10.0):

- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators, submanifolds
- vector bundles (tangent bundle, tensor bundles)
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds, and with all monoterm tensor symmetries taken into account
- Lie derivative along a vector field
- differential forms: exterior and interior products, exterior derivative, Hodge duality
- multivector fields: exterior and interior products, Schouten-Nijenhuis bracket
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- computation of geodesics (numerical integration)

Current status

Already present *(cont'd)*:

- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations
- extrinsic geometry of pseudo-Riemannian submanifolds
- series expansions of tensor fields
- symplectic manifolds
- 2 symbolic backends: Pynac/Maxima (SageMath's default) and SymPy

Future prospects:

- more symbolic backends (Giac, FriCAS, ...)
- more graphical outputs
- spinors, integrals on submanifolds, variational calculus, etc.
- connection with numerical relativity: use SageMath to explore numerically-generated spacetimes

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Vector fields on a smooth manifold

The set $\mathfrak{X}(M)$ of vector fields on a smooth manifold M over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is endowed with two algebraic structures:

1. $\mathfrak{X}(M)$ is an infinite-dimensional vector space over $\mathbb{K}$, the scalar multiplication $\mathbb{K} \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(\lambda, \boldsymbol{v}) \mapsto \lambda\boldsymbol{v}$ being defined by

$$\forall p \in M, \quad (\lambda\boldsymbol{v})\big|_p = \lambda\boldsymbol{v}\big|_p \, ,$$

2. $\mathfrak{X}(M)$ is a module over the commutative algebra $C^\infty(M)$, the scalar multiplication $C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(f, \boldsymbol{v}) \mapsto f\boldsymbol{v}$ being defined by

$$\forall p \in M, \quad (f\boldsymbol{v})\big|_p = f(p)\boldsymbol{v}\big|_p \, ,$$

the right-hand side involving the scalar multiplication by $f(p) \in \mathbb{K}$ in the vector space T_pM.

$\mathfrak{X}(M)$ as a $C^\infty(M)$-module

$\mathfrak{X}(M)$ is a ***free module*** over $C^\infty(M) \iff \mathfrak{X}(M)$ admits a basis

If this occurs, then $\mathfrak{X}(M)$ is actually a ***free module of finite rank*** over $C^\infty(M)$ and $\operatorname{rank}\mathfrak{X}(M) = \dim M = n$.
One says then that M is a ***parallelizable*** manifold.
A basis $(\boldsymbol{e}_a)_{1\le a\le n}$ of $\mathfrak{X}(M)$ is called a ***vector frame***

Basis expansion[1]:

$$\forall \boldsymbol{v} \in \mathfrak{X}(M), \quad \boldsymbol{v} = v^a \boldsymbol{e}_a, \quad \text{with } v^a \in C^\infty(M) \tag{1}$$

At each point $p \in M$, (1) gives birth to an identity in the tangent space T_pM:

$$\boldsymbol{v}|_p = v^a(p)\, \boldsymbol{e}_a|_p\,, \quad \text{with } v^a(p) \in \mathbb{K},$$

which is nothing but the expansion of the tangent vector $\boldsymbol{v}|_p$ on the basis $(\boldsymbol{e}_a|_p)_{1\le a\le n}$ of the vector space T_pM.

[1] Einstein's convention for summation on repeated indices is assumed.

Parallelizable manifolds

M is **parallelizable** $\iff$ $\mathfrak{X}(M)$ is a free $C^\infty(M)$-module of rank n
$\iff$ M admits a global vector frame
$\iff$ the tangent bundle is trivial: $TM \simeq M \times \mathbb{K}^n$

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Examples of parallelizable manifolds

- $\mathbb{R}^n$ (global coordinate chart $\Rightarrow$ global vector frame)
- the circle $\mathbb{S}^1$ (*rem:* no global coordinate chart)
- the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$
- the 3-sphere $\mathbb{S}^3 \simeq \mathrm{SU}(2)$, as any Lie group
- the 7-sphere $\mathbb{S}^7$
- any orientable 3-manifold (Steenrod theorem)

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Examples of non-parallelizable manifolds

- the sphere $\mathbb{S}^2$ (hairy ball theorem!) and any n-sphere $\mathbb{S}^n$ with $n \notin \{1, 3, 7\}$
- the real projective plane $\mathbb{RP}^2$

SageMath implementation of vector fields

Choice of the $C^\infty(M)$-module point of view for $\mathfrak{X}(M)$, instead of the infinite-dimensional $\mathbb{K}$-vector space one

$\Longrightarrow$ implementation advantages:

- reduction to finite-dimensional structures: free $C^\infty(U)$-modules of rank n on parallelizable open subsets $U \subset M$
- for tensor calculus on each parallelizable open set U, use of exactly the same `FiniteRankFreeModule` code as for the tangent spaces

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Decomposition of M into parallelizable parts

Assumption: the smooth manifold M can be covered by a finite number m of parallelizable open subsets U_i $(1 \leq i \leq m)$

Example: this holds if M is compact (finite atlas)

More details on the implementation:

[E. Gourgoulhon & M. Mancini, Les cours du CIRM **6**, 1 (2018)]

Tensor field storage

`TensorField` T

dictionary `TensorField._restrictions`

domain 1: U_1 — `TensorFieldParal` $T|_{U_1} = T^a{}_b e_a \otimes e^b = T^{\hat{a}}{}_{\hat{b}} \varepsilon_{\hat{a}} \otimes \varepsilon^{\hat{b}} = \ldots$

domain 2: U_2 — `TensorFieldParal` $T|_{U_2}$

$\ldots$

dictionary `TensorFieldParal._components`

frame 1: (e_a) — `Components` $(T^a{}_b)_{1 \leq a,\, b \leq n}$

frame 2: $(\varepsilon_{\hat{a}})$ — `Components` $(T^{\hat{a}}{}_{\hat{b}})_{1 \leq \hat{a},\, \hat{b} \leq n}$

$\ldots$

dictionary `Components._comp`

$(1,1)$: `DiffScalarField` $T^1{}_1$

$(1,2)$: `DiffScalarField` $T^1{}_2$

$\ldots$

dictionary `DiffScalarField._express`

chart 1: (x^a) — `ChartFunction` $T^1{}_1\left(x^1, \ldots, x^n\right)$

chart 2: (y^a) — `ChartFunction` $T^1{}_1\left(y^1, \ldots, y^n\right)$

$\ldots$

dictionary `ChartFunction._express`

SR: `Expression` $x^1 \cos x^2$

SymPy: `Basic` $x^1 \cos x^2$

$\ldots$

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Near-horizon geometry of the extremal Kerr black hole

Extremal Kerr black hole: $a = m \iff \kappa = 0$ (degenerate horizon)
2-dimensional isometry group: $(\mathbb{R}, +) \times \mathrm{U}(1)$

Near-horizon geometry of extremal Kerr BH is similar to $\mathrm{AdS}_2 \times \mathbb{S}^2$
$\Longrightarrow$ 4-dimensional isometry group: $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{U}(1)$
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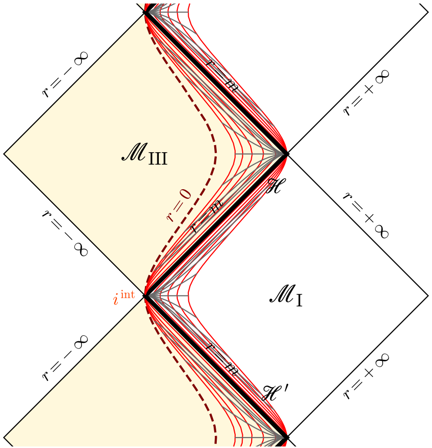
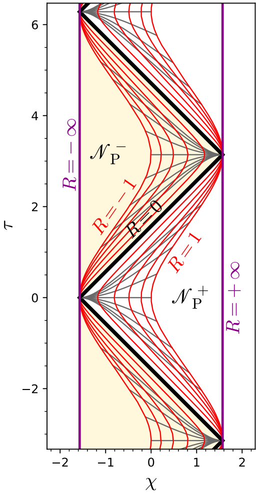
Let us explore this geometry with a SageMath notebook:
https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_extremal_Kerr_near_horizon.ipynb

(In the nbviewer menu, click on to run an interactive version on a Binder server)

Carter-Penrose diagrams generated with SageMath



Extremal Kerr



NEHK spacetime

https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Kerr_extremal_extended.ipynb
https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/NHEK_spacetime.ipynb

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Other examples

- **Schwarzschild spacetime** (static black hole):
  https://nbviewer.org/github/egourgoulhon/SageMathTour/blob/master/Notebooks/demo_pseudo_Riemannian_Schwarzschild.ipynb
- **Computation of geodesics in Kerr spacetime** (rotating black hole):
  https://nbviewer.jupyter.org/github/BlackHolePerturbationToolkit/kerrgeodesic_gw/blob/master/Notebooks/Kerr_geodesics.ipynb
- **The spheres** $\mathbb{S}^2$ **and** $\mathbb{S}^3$:
  https://nbviewer.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_sphere_S2.ipynb

  https://nbviewer.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_sphere_S3_Hopf.ipynb

Image of an accretion disk surrounding a Schwarzschild BH

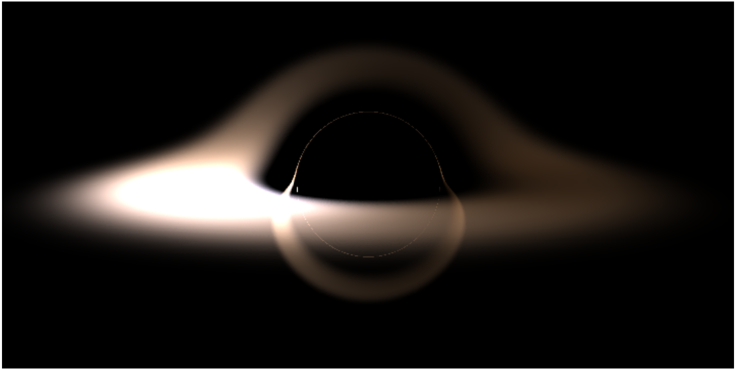

Image computed with SageMath by integrating null geodesics, cf. the notebook https://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Notebooks/SM_black_hole_rendering.ipynb

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Conclusions

Symbolic calculus on manifolds in the free Python-based system `SageMath`

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Many examples available at
https://sagemanifolds.obspm.fr/examples.html

Want to join the SageManifolds project or to simply stay tuned?

visit https://sagemanifolds.obspm.fr/
(download, documentation, example notebooks, mailing list)