Journées Relativistes de Tours

GRAVITATIONAL WAVES FROM COMPACT BINARIES BEYOND THE EINSTEIN QUADRUPOLE FORMULA

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Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1945]

$$4\pi R^2 \bar{\mathfrak{J}} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{\mathfrak{J}}_{\mu\nu}^{2} - \frac{1}{3}\left(\sum_{\mu} \dddot{\mathfrak{J}}_{\mu\mu}\right)^{2}\right].$$

1. Einstein quadrupole formula

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\mathsf{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}}$$

2. Amplitude quadrupole formula

$$\boxed{h_{ij}^{\mathsf{TT}} = \frac{2G}{c^4 R}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t - \frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\mathsf{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)}$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

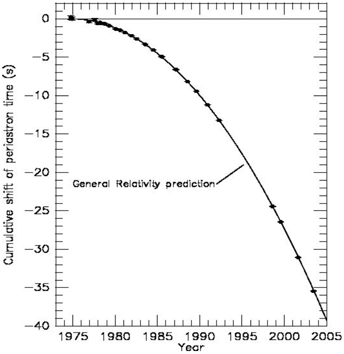
$$\boxed{F_i^{\mathsf{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7}$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

The orbital decay of the Hulse-Taylor binary pulsar

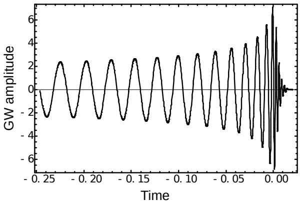
$$\dot{P} = -\frac{192\pi}{5c^5}\frac{m_1 m_2}{M^2}\left(\frac{2\pi G\,M}{P}\right)^{5/3}\underbrace{\frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}}}_{\text{eccentricity enhancement factor}}$$

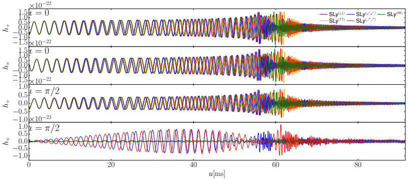
[Peters & Mathews 1963]



- Derivation based on flux-balance equation [Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]
- Derivation based on EoM including the radiation reaction term at 2.5PN [Damour & Deruelle 1981; Damour 1982]
- Resolution of the radiation reaction controversy [Ehlers, Rosenblum, Goldberg & Havas 1976; Will & Walker 1980]

The gravitational chirp of compact binaries

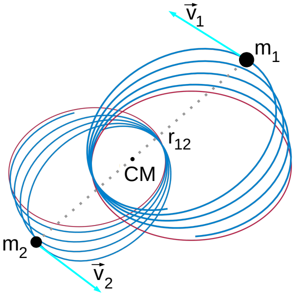




- Inspiralling phase
  - Post-Newtonian theory
  - Point-particle approximation
  - Dependence on spin precession
  - Universality of the signal in GR
  - Effacing of the internal structure

  [Brillouin 1922; Damour 1982]
- Late inspiral
  - Post-Newtonian + Effective theory
  - Effects due to tidal interactions
  - Dependence on the internal structure (EoS)
- Merger and post-merger
  - Numerical relativity
  - Strong dependence on internal structure
  - Phenomenological models (EOB, IMR)

Post-Newtonian equations of motion



$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN}}$$
$$+ \underbrace{\frac{1}{c^4}\left[\cdots\right]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}\left[\cdots\right]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^6}\left[\cdots\right]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}\left[\cdots\right]}_{\substack{\text{3.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^8}\left[\cdots\right]}_{\substack{\text{4PN}\\\text{conservative \& dissipative (tail)}}} + \mathcal{O}\left[\left(\frac{v}{c}\right)^9\right]$$

Methods to compute PN equations of motion

1. Traditional methods in classical GR
   - ADM Hamiltonian canonical formalism in GR
   - Fokker EH action in harmonic coordinates
   - Surface-integral approach *à la* EIH
   - Extended fluids in the compact body limit
2. QFT inspired methods
   - Effective-field theory
   - Scattering amplitude approach
3. Dimensional regularization is the common tool ['t Hooft & Veltman 1972]
   - UV divergences: point particles modelling compact objects
   - IR divergences: integration over all space of formal PN expansion

4PN: state-of-the-art on equations of motion

	References	Method
3PN	[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
	[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002] [Blanchet, Damour & Esposito-Farèse 2004]	Harmonic EoM
	[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
	[Foffa & Sturani 2011]	Effective field theory
4PN	[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014, 2016]	ADM Hamiltonian
	[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
	[Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019]	Effective field theory
	[Blümlein, Maier, Marquard & Schäfer 2020]	EFT Hamiltonian

- ADM Hamiltonian: One regularization ambiguity left at 4PN order and fixed by comparison with GSF calculations
- Fokker Lagrangian: First complete derivation of the EoM at 4PN order without regularization ambiguities

[See also talk by Thibault Damour]

Conservative 4PN binding energy for circular orbits

With $x = (\frac{Gm\omega}{c^3})^{3/2}$ the orbital frequency and $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$ the mass ratio

$$
\begin{aligned}
E = -\frac{m\nu c^2 x}{2}\Bigg\{ & 1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2 \\
& + \left[-\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\right]x^3 \\
& + \Bigg[-\frac{3969}{128} + \left(-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}\gamma_E + \frac{448}{15}\ln(16x)\right)\nu \\
& \quad + \left(-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4\Bigg]x^4\Bigg\}.
\end{aligned}
$$

3.5PN: state-of-the-art on gravitational wave field

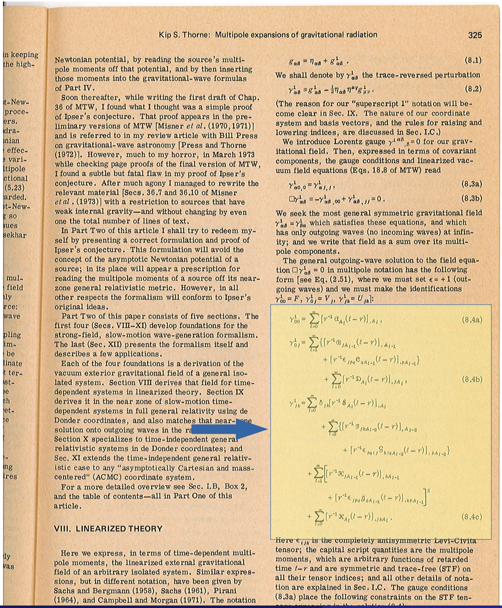
(beyond the Einstein quadrupole formula)

$$4\pi R^2 \mathfrak{J} = \frac{\varkappa}{40\pi}\left[\sum_{\mu\nu} \dddot{J}_{\mu\nu}^2 - \frac{1}{3}\left(\sum_{\mu} \dddot{J}_{\mu\mu}\right)^2\right].$$

1PN	[Epstein & Wagoner 1975; Wagoner & Will 1976]	EW moments
	[Blanchet & Damour 1989; Blanchet & Schäfer 1989]	BD moments
2PN	[Blanchet, Damour & Iyer 1995]	MPM-PN formalism
	[Will & Wiseman 1996]	DIRE formalism
	[BDIWW 1995; BIWW 1996]	

- Further developments up to 3.5PN order done using the MPM-PN formalism
- The DIRE formalism is equivalent to the MPM-PN formalism for general isolated matter systems

The Multipolar-post-Minkowskian formalism



Newtonian potential, by reading the source's multipole moments off that potential, and by then inserting those moments into the gravitational-wave formulas of Part IV.

Soon thereafter, while writing the first draft of Chap. 36 of MTW, I found what I thought was a simple proof of Ipser's conjecture. That proof appears in the preliminary versions of MTW [Misner *et al.* (1970, 1971)] and is referred to in my review article with Bill Press on gravitational-wave astronomy [Press and Thorne (1972)]. However, much to my horror, in March 1973 while checking page proofs of the final version of MTW, I found a subtle but fatal flaw in my proof of Ipser's conjecture. After much agony I managed to rewrite the relevant material [Secs. 36.7 and 36.10 of Misner *et al.* (1973)] with a restriction to sources that have weak internal gravity—and without changing by even one the total number of lines of text.

In Part Two of this article I shall try to redeem myself by presenting a correct formulation and proof of Ipser's conjecture. This formulation will avoid the concept of the asymptotic Newtonian potential of a source; in its place will appear a prescription for reading the multipole moments of a source off its near-zone general relativistic metric. However, in all other respects the formalism will conform to Ipser's original ideas.

Part Two of this paper consists of five sections. The first four (Secs. VIII–XI) develop foundations for the strong-field, slow-motion wave-generation formalism. The last (Sec. XII) presents the formalism itself and describes a few applications.

Each of the four foundations is a derivation of the vacuum exterior gravitational field of a general isolated system. Section VIII derives that field for time-dependent systems in linearized theory. Section IX derives it in the near zone of slow-motion time-dependent systems in full general relativity using de Donder coordinates, and also matches that near-[illegible] solution onto outgoing waves in the ra[illegible] Section X specializes to time-independent general relativistic systems in de Donder coordinates; and Sec. XI extends the time-independent general relativistic case to any "asymptotically Cartesian and mass-centered" (ACMC) coordinate system.

For a more detailed overview see Sec. I.B, Box 2, and the table of contents—all in Part One of this article.

VIII. LINEARIZED THEORY

Here we express, in terms of time-dependent multipole moments, the linearized external gravitational field of an arbitrary isolated system. Similar expressions, but in different notation, have been given by Sachs and Bergmann (1958), Sachs (1961), Pirani (1964), and Campbell and Morgan (1971). The notation

$$g_{\alpha\beta} = \eta_{\alpha\beta} + g^1_{\alpha\beta} . \quad (8.1)$$

We shall denote by $\gamma^1_{\alpha\beta}$ the trace-reversed perturbation

$$\gamma^1_{\alpha\beta} \equiv g^1_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}g^1_{\mu\nu} . \quad (8.2)$$

(The reason for our "superscript 1" notation will become clear in Sec. IX. The nature of our coordinate system and basis vectors, and the rules for raising and lowering indices, are discussed in Sec. I.C.)

We introduce Lorentz gauge $\gamma^{1\,\alpha\beta}{}_{,\beta}=0$ for our gravitational field. Then, expressed in terms of covariant components, the gauge conditions and linearized vacuum field equations (Eqs. 18.8 of MTW) read

$$\gamma^1_{\alpha 0,0} = \gamma^1_{\alpha j,j} , \quad (8.3a)$$

$$\Box\gamma^1_{\alpha\beta} \equiv -\gamma^1_{\alpha\beta,00} + \gamma^1_{\alpha\beta,jj} = 0 . \quad (8.3b)$$

We seek the most general symmetric gravitational field $\gamma^1_{\alpha\beta} = \gamma^1_{\beta\alpha}$ which satisfies these equations, and which has only outgoing waves (no incoming waves) at infinity; and we write that field as a sum over its multipole components.

The general outgoing-wave solution to the field equation $\Box\gamma^1_{\alpha\beta} = 0$ in multipole notation has the following form [see Eq. (2.51), where we must set $\epsilon = +1$ (outgoing waves) and we must make the identifications $\gamma^1_{00} = F$, $\gamma^1_{0j} = V_j$, $\gamma^1_{jk} = U_{jk}$]:

$$\gamma^1_{00} = \sum_{l=0}^{\infty} [r^{-1}\mathcal{A}_{A_l}(t-r)]_{,A_l} , \quad (8.4a)$$

$$\gamma^1_{0j} = \sum_{l=1}^{\infty} \{ [r^{-1}\mathcal{B}_{jA_{l-1}}(t-r)]_{,A_{l-1}} + [r^{-1}\epsilon_{jpq}\mathcal{C}_{qA_{l-1}}(t-r)]_{,pA_{l-1}} \} + \sum_{l=0}^{\infty} [r^{-1}\mathcal{D}_{A_l}(t-r)]_{,jA_l} , \quad (8.4b)$$

$$\gamma^1_{jk} = \sum_{l=0}^{\infty} \delta_{jk}[r^{-1}\mathcal{E}_{A_l}(t-r)]_{,A_l} + \sum_{l=2}^{\infty} \{ [r^{-1}\mathcal{F}_{jkA_{l-2}}(t-r)]_{,A_{l-2}} + [r^{-1}\epsilon_{pq(j}\mathcal{G}_{k)qA_{l-2}}(t-r)]_{,pA_{l-2}} \} + \sum_{l=1}^{\infty} \Big[[r^{-1}\mathcal{H}_{jA_{l-1}}(t-r)]_{,kA_{l-1}} + [r^{-1}\epsilon_{jpq}\mathcal{I}_{qA_{l-1}}(t-r)]_{,kpA_{l-1}} \Big]^S + \sum_{l=0}^{\infty} [r^{-1}\mathcal{K}_{A_l}(t-r)]_{,jkA_l} . \quad (8.4c)$$

Here ϵ_{ijk} is the completely antisymmetric Levi-Civita tensor; the capital script quantities are the multipole moments, which are arbitrary functions of retarded time $t-r$ and are symmetric and trace-free (STF) on all their tensor indices; and all other details of notation are explained in Sec. I.C. The gauge conditions (8.3a) place the following constraints on the STF ten-

- Most general solution of the Einstein vacuum linearized field equations in harmonic coordinates [Thorne 1980]

$$G h_1^{\mu\nu}\big[\underbrace{I_L(u), J_L(u)}_{\text{multipole moments}}\big]$$

- Iterate that linear solution in a post Minkowskian expansion [Blanchet & Damour 1986]

$$h^{\mu\nu}[I_L, J_L] = G h_1^{\mu\nu} + G^2 h_2^{\mu\nu} + \cdots$$

Linearized multipolar vacuum solution [Pirani 1964; Thorne 1980]

Solution of linearized vacuum field equations in harmonic coordinates

$$\boxed{\Box h^{\alpha\beta}_{(1)} = \partial_\mu h^{\alpha\mu}_{(1)} = 0}$$

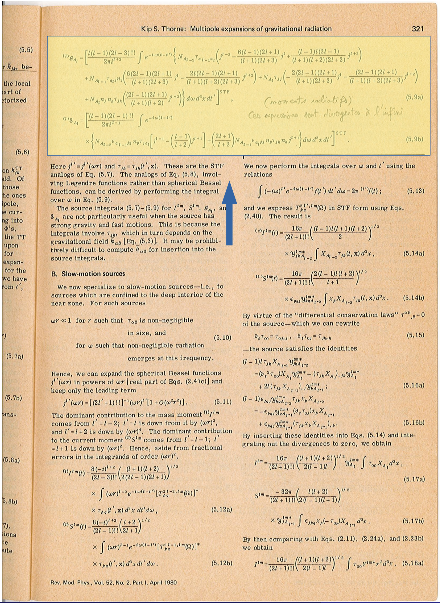
$$h^{00}_{(1)} = -\frac{4}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{1}{r}I_L\right) \qquad \boxed{L = i_1 i_2 \cdots i_\ell}$$

$$h^{0i}_{(1)} = \frac{4}{c^3}\sum_{\ell=1}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-1}\left(\frac{1}{r}I^{(1)}_{iL-1}\right) + \frac{\ell}{\ell+1}\varepsilon_{iab}\partial_{aL-1}\left(\frac{1}{r}J_{bL-1}\right)\right\}$$

$$h^{ij}_{(1)} = -\frac{4}{c^4}\sum_{\ell=2}^{+\infty}\frac{(-)^\ell}{\ell!}\left\{\partial_{L-2}\left(\frac{1}{r}I^{(2)}_{ijL-2}\right) + \frac{2\ell}{\ell+1}\partial_{aL-2}\left(\frac{1}{r}\varepsilon_{ab(i}J^{(1)}_{j)bL-2}\right)\right\}$$

- multipole moments $I_L(u)$ and $J_L(u)$ are arbitrary functions of $u = t - r/c$
- mass $M \equiv I = \mathrm{const}$, center-of-mass position $G_i \equiv I_i = \mathrm{const}$
  linear momentum $P_i \equiv I_i^{(1)} = 0$, angular momentum $J_i = \mathrm{const}$

Sad situation of the field in the 1980's



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$$
{}^{(l)}\mathcal{I}_{A_l} = \Big[\frac{l(l-1)(2l-3)!!}{2\pi i^{l+2}} \int e^{-i\omega(t-t')}\Big\{ N_{A_{l-2}}\tau_{a_{l-1}a_l}\Big(j^{l-2} - \frac{6(l-1)(2l+1)}{(l+1)(2l+3)} j^l + \frac{(l-1)l(2l-1)}{(l+1)(l+2)(2l+3)} j^{l+2}\Big)
+ N_{A_{l-1}}\tau_{a_l j} n_j \Big(\frac{6(2l-1)(2l+1)}{(l+1)(2l+3)} j^l - \frac{2l(2l-1)(2l+1)}{(l+1)(l+2)(2l+3)} j^{l+2}\Big) + N_{A_l}\tau_{jj}\Big(-\frac{2(2l-1)(2l+1)}{(l+1)(2l+3)} j^l - \frac{(2l-1)(2l+1)}{(l+1)(l+2)(2l+3)} j^{l+2}\Big)
+ N_{A_l} n_j n_k \tau_{jk}\Big(\frac{(2l-1)(2l+1)}{(l+1)(l+2)} j^{l+2}\Big)\Big\} d\omega\, d^3x\, dt'\Big]^{\rm STF}, \quad (5.9a)
$$

(moments radiatifs)

Ces expressions sont divergentes à l'infini

$$
{}^{(l)}\mathcal{S}_{A_l} = \Big[\frac{(l-1)(2l-1)!!}{2\pi i^{l-1}} \int e^{-i\omega(t-t')} \times \Big\{ N_{A_{l-2}}\epsilon_{a_{l-1}pj} n_p \tau_{j a_l}\Big[j^{l-1} - \Big(\frac{l-1}{l+2}\Big) j^{l+1}\Big] + \Big(\frac{2l+1}{l+2}\Big) N_{A_{l-1}}\epsilon_{a_l pj} n_p \tau_{jk} n_k j^{l+1}\Big\} d\omega\, d^3x\, dt'\Big]^{\rm STF}. \quad (5.9b)
$$

Here $j^{l'} \equiv j^{l'}(\omega r)$ and $\tau_{jk} \equiv \tau_{jk}(t', \mathbf{x})$. These are the STF analogs of Eq. (5.7). The analogs of Eq. (5.8), involving Legendre functions rather than spherical Bessel functions, can be derived by performing the integral over ω in Eq. (5.9).

The source integrals (5.7)–(5.9) for I^{lm}, S^{lm}, $\mathcal{I}_{A_l}$, and $\mathcal{S}_{A_l}$ are not particularly useful when the source has strong gravity and fast motions. This is because the integrals involve τ_{jk}, which in turn depends on the gravitational field $\bar{h}_{\alpha\beta}$ [Eq. (5.3)]. It may be prohibitively difficult to compute $\bar{h}_{\alpha\beta}$ for insertion into the source integrals.

B. Slow-motion sources

We now specialize to slow-motion sources—i.e., to sources which are confined to the deep interior of the near zone. For such sources

$\omega r \ll 1$ for r such that $\tau_{\alpha\beta}$ is non-negligible in size, and for ω such that non-negligible radiation emerges at this frequency. (5.10)

Hence, we can expand the spherical Bessel functions $j^{l'}(\omega r)$ in powers of ωr [real part of Eqs. (2.47c)] and keep only the leading term

$$j^{l'}(\omega r) = [(2l'+1)!!]^{-1}(\omega r)^{l'}[1 + O(\omega^2 r^2)]. \quad (5.11)$$

The dominant contribution to the mass moment ${}^{(l)}I^{lm}$ comes from $l' = l-2$; $l' = l$ is down from it by $(\omega r)^2$, and $l' = l+2$ is down by $(\omega r)^4$. The dominant contribution to the current moment ${}^{(l)}S^{lm}$ comes from $l' = l-1$; $l' = l+1$ is down by $(\omega r)^2$. Hence, aside from fractional errors in the integrands of order $(\omega r)^2$,

$$
{}^{(l)}I^{lm}(t) = \frac{8(-i)^{l+2}}{(2l-3)!!}\Big(\frac{(l+1)(l+2)}{2(2l-1)(2l+1)}\Big)^{1/2} \times \int (\omega r)^{l-2} e^{-i\omega(t-t')}[T_{pq}^{2\,l-2,lm}(\Omega)]^* \times \tau_{pq}(t', \mathbf{x})\, d^3x\, dt'\, d\omega, \quad (5.12a)
$$

$$
{}^{(l)}S^{lm}(t) = \frac{8(-i)^{l+2}}{(2l-1)!!}\Big(\frac{l+2}{2l+1}\Big)^{1/2} \times \int (\omega r)^{l-1} e^{-i\omega(t-t')}[T_{pq}^{2\,l-1,lm}(\Omega)]^* \times \tau_{pq}(t', \mathbf{x})\, d^3x\, dt'\, d\omega. \quad (5.12b)
$$

We now perform the integrals over ω and t' using the relations

$$\int (-i\omega)^{l'} e^{-i\omega(t-t')} f(t')\, dt'\, d\omega = 2\pi\, {}^{(l')}f(t); \quad (5.13)$$

and we express $T_{pq}^{2\,l',lm}(\Omega)$ in STF form using Eqs. (2.40). The result is

$$
{}^{(2)}I^{lm}(t) = \frac{16\pi}{(2l+1)!!}\Big(\frac{(l-1)l(l+1)(l+2)}{2}\Big)^{1/2} \times \mathcal{Y}^{lm*}_{jkA_{l-2}} \int X_{A_{l-2}}\tau_{jk}(t, \mathbf{x})\, d^3x, \quad (5.14a)
$$

$$
{}^{(1)}S^{lm}(t) = \frac{16\pi}{(2l+1)!!}\Big(\frac{2(l-1)l(l+2)}{l+1}\Big)^{1/2} \times \epsilon_{pqj}\mathcal{Y}^{lm*}_{kqA_{l-2}} \int x_p X_{A_{l-2}}\tau_{jk}(t, \mathbf{x})\, d^3x. \quad (5.14b)
$$

By virtue of the "differential conservation laws" $\tau^{\alpha\beta}{}_{,\beta} = 0$ of the source—which we can rewrite

$$\partial_t \tau_{00} = \tau_{0j,j}, \quad \partial_t \tau_{0j} = \tau_{jk,k} \quad (5.15)$$

—the source satisfies the identities

$$
(l-1)l\,\tau_{jk}X_{A_{l-2}}\mathcal{Y}^{lm*}_{jkA_{l-2}} = (\partial_t{}^2\tau_{00})X_{A_l}\mathcal{Y}^{lm*}_{A_l} - (\tau_{jk}X_{A_l})_{,jk}\mathcal{Y}^{lm*}_{A_l} + 2l(\tau_{jk}X_{A_{l-1}})_{,j}\mathcal{Y}^{lm*}_{kA_{l-1}}; \quad (5.16a)
$$

$$
(l-1)\epsilon_{pqj}\mathcal{Y}^{lm*}_{kqA_{l-2}}\tau_{jk}x_p X_{A_{l-2}} = -\epsilon_{pqj}\mathcal{Y}^{lm*}_{qA_{l-1}}(\partial_t\tau_{0j})x_p X_{A_{l-1}} + \epsilon_{pqj}\mathcal{Y}^{lm*}_{qA_{l-1}}(\tau_{jk}x_p X_{A_{l-1}})_{,k}. \quad (5.16b)
$$

By inserting these identities into Eqs. (5.14) and integrating out the divergences to zero, we obtain

$$
I^{lm} = \frac{16\pi}{(2l+1)!!}\Big(\frac{(l+1)(l+2)}{2(l-1)l}\Big)^{1/2}\mathcal{Y}^{lm*}_{A_l}\int \tau_{00}X_{A_l}\, d^3x, \quad (5.17a)
$$

$$
S^{lm} = \frac{-32\pi}{(2l+1)!!}\Big(\frac{l(l+2)}{2(l-1)(l+1)}\Big)^{1/2} \times \mathcal{Y}^{lm*}_{jA_{l-1}}\int \epsilon_{jpq}x_p(-\tau_{0q})X_{A_{l-1}}\, d^3x. \quad (5.17b)
$$

By then comparing with Eqs. (2.11), (2.24a), and (2.23b) we obtain

$$
I^{lm} = \frac{16\pi}{(2l+1)!!}\Big(\frac{(l+1)(l+2)}{2(l-1)l}\Big)^{1/2}\int \tau_{00}Y^{lm*}r^l\, d^3x, \quad (5.18a)
$$

Rev. Mod. Phys., Vol. 52, No. 2, Part I, April 1980

- Multipole moments given by divergent integrals [Epstein & Wagoner 1975; Thorne 1980]

$$\sim \int \mathrm{d}^3\mathbf{x}\, r^\ell \hat{n}^L(\theta,\varphi)\, \underbrace{\tau^{\mu\nu}(\mathbf{x},u)}_{\text{pseudo-tensor}}$$

- PN iteration yields divergent Poisson like integrals from 3PN [Anderson & DeCanio 1975; Kerlick 1980]
- Treatment of point-particles in non-linear GR poorly understood [Infeld & Plebański 1960]
- Tails, memory, tails-of-tails, $\cdots$ completely ignored

Multipolar-post-Minkowskian expansion

[Blanchet & Damour 1986; Blanchet, Damour & Iyer 1995; Blanchet 1987, 1998]

- Look for the general multipolar expansion $\mathcal{M}(h^{\mu\nu})$ generated outside the source in the form [Bonnor 1959; Bonnor & Rotenberg 1961]

$$\mathcal{M}(h^{\mu\nu}) = \underbrace{G\,h_1^{\mu\nu} + G^2\,h_2^{\mu\nu} + \cdots + G^n\,h_n^{\mu\nu} + \cdots}_{\text{formal post-Minkowskian expansion}}$$

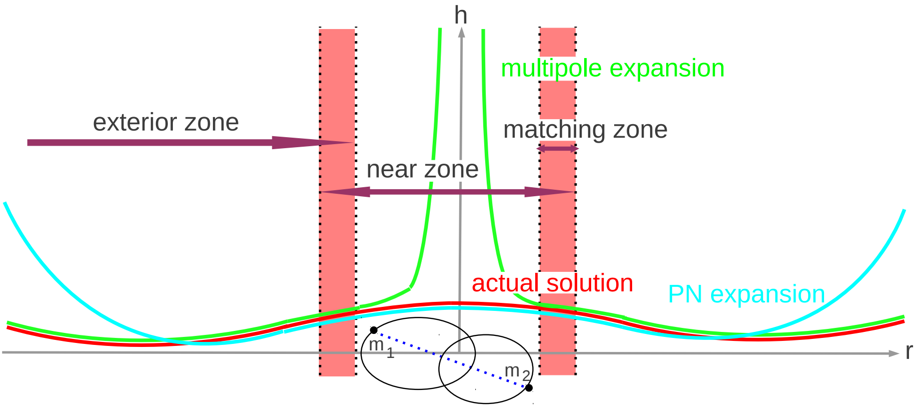
- Start from Thorne's multipolar linearized solution $h_1^{\mu\nu}[I_L, J_L]$
- Iterate that solution using a regularization scheme based on analytic continuation in $B \in \mathbb{C}$ to treat the singularity of the multipole expansion when $r \to 0$

$$\boxed{\underset{B=0}{\text{Finite Part}}\,\Box^{-1}_{\text{ret}}\left[(r/r_0)^B f\right]}$$

- A particular MPM construction obtains the metric in radiative coordinates such that the retarded time $u = t - r$ is a null coordinate and the metric admits an expansion in powers of $1/r$ at future null infinity

The MPM-PN formalism

The MPM outer metric is matched to the PN inner field of the source



matching equation $\Longrightarrow$ $\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})$

Field equations and Green's function in d dimensions

- Einstein's field equations in harmonic (de Donder) coordinates

$$\partial_\nu h^{\mu\nu} = 0 \qquad \text{(harmonic gauge condition)}$$

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4}\,\tau^{\mu\nu} \qquad \text{(wave equation in } D = d+1 \text{ dimensions)}$$

$$\tau^{\mu\nu} = |g| T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu} \qquad \text{(matter + gravitation pseudo tensor)}$$

- The Green's function is implemented in the real space-time domain

$$G_{\text{ret}}(\mathbf{x},t) = -\frac{\tilde{k}}{4\pi}\frac{\theta(t-r)}{r^{d-1}}\,\gamma_{\frac{1-d}{2}}\left(\frac{t}{r}\right)$$

$$\gamma_{\frac{1-d}{2}}(z) \equiv \frac{2\sqrt{\pi}}{\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2}-1)}\left(z^2-1\right)^{\frac{1-d}{2}}$$

The multipole expansion outside the matter source

- The multipole expansion $\mathcal{M}(h^{\mu\nu})$ is a retarded solution the *vacuum* field equations $\Box\mathcal{M}(h^{\mu\nu}) = \mathcal{M}(\Lambda^{\mu\nu})$ valid formally everywhere except at $r = 0$

$$\mathcal{M}(h^{\mu\nu}) = \overbrace{\mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\mathrm{ret}}\left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda^{\mu\nu})\right]}^{\text{regularization when } r\to 0} \underbrace{-\frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\,\hat{\partial}_L\widetilde{\mathcal{F}}^{\mu\nu}_L}_{\text{retarded homogeneous solution}}$$

$$\Box\widetilde{\mathcal{F}}^{\mu\nu}_L(r,t) = 0 \quad \text{in } d \text{ dimensions}$$

- The multipole moment functions $\mathcal{F}^{\mu\nu}_L(t)$ are symmetric-trace-free (STF) with respect to their ℓ indices $L \equiv i_1 \cdots i_\ell$

$$\widetilde{\mathcal{F}}^{\mu\nu}_L(r,t) = \frac{\tilde{k}}{r^{d-2}}\int_1^{+\infty} \mathrm{d}z\,\gamma_{\frac{1-d}{2}}(z)\,\mathcal{F}^{\mu\nu}_L(t - zr)$$

The multipole expansion matched to the PN source

- Explicit matching to a general extended PN isolated source gives

$$\mathcal{F}^{\mu\nu}_L(t) = \overbrace{\underset{B=0}{\mathrm{FP}} \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^B}^{\text{IR regularization}} \hat{x}_L \int_{-1}^{1} \mathrm{d}z\, \delta^{(d)}_\ell(z)\ \underbrace{\overline{\tau}^{\mu\nu}(\mathbf{x}, t+zr)}_{\text{PN expansion of the pseudo-tensor}}$$

$$\delta^{(d)}_\ell(z) \equiv \frac{\Gamma\left(\frac{d}{2}+\ell\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}+\ell\right)} \left(1-z^2\right)^{\frac{d-3}{2}+\ell}$$

- The $B\varepsilon$ regularization
  - first apply the limit $B \to 0$ in generic dimensions $d = 3 + \varepsilon$
  - then the usual dimensional regularization when $\varepsilon \to 0$

Mass and current irreducible multipole moments

[Henry, Faye and Blanchet 2020]

- The irreducible decomposition of $\mathcal{F}_L^{\mu\nu}$ reads (with $\langle \cdots \rangle$ the STF projection)

$$\begin{aligned}
\mathcal{F}_L^{00} &= R_L \\
\mathcal{F}_L^{0i} &= T_{iL}^{(+)} + T_{i|\langle i_\ell L-1\rangle}^{(0)} + \delta_{i\langle i_\ell} T_{L-1\rangle}^{(-)} \\
\mathcal{F}_L^{ij} &= U_{ijL}^{(+2)} + \underset{L}{\mathrm{STF}}\,\underset{ij}{\mathrm{STF}} \Big[U_{i|i_\ell jL-1}^{(+1)} + \delta_{ii_\ell} U_{jL-1}^{(0)} + \delta_{ii_\ell} U_{j|i_{\ell-1}L-2}^{(-1)} \\
&\qquad + \delta_{ii_\ell}\delta_{ji_{\ell-1}} U_{L-2}^{(-2)} + W_{ij|i_\ell i_{\ell-1}L-2} \Big] + \delta_{ij} V_L
\end{aligned}$$

- The "mass-type" contributions R_L, $T_{L+1}^{(+)}$, $T_{L-1}^{(-)}$, $U_{L+2}^{(+2)}$, $U_L^{(0)}$, $U_{L-2}^{(-2)}$, V_L are STF in the ordinary sense
- The "current-type" contributions $T_{i|\langle i_\ell L-1\rangle}^{(0)}$, $U_{i|i_{\ell+1}L}^{(+1)}$, $U_{i|i_{\ell-1}L-2}^{(-1)}$ have more complicated symmetries

Mass and current irreducible multipole moments

[Henry, Faye and Blanchet 2020]

- The mass moment I_L is given by the usual STF moment, but the generalization of the current moment involves two tensors $J_{i|L}$ and $K_{ij|L}$ having the symmetries of mixed Young tableaux

$$I_L = \begin{array}{|c|c|c|}\hline i_\ell & \dots & i_1 \\ \hline \end{array}$$

$$J_{i|L} = \begin{array}{|c|c|c|c|}\hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline i \\ \cline{1-1} \end{array} \qquad K_{ij|L} = \begin{array}{|c|c|c|c|c|}\hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i \\ \cline{1-2} \end{array}$$

- The tensor $K_{ij|L}$ is absent in 3 dimensions

$$\sharp(\text{components}) = \frac{(d-3)d(d-1)_{\ell-2}(2\ell+d-2)(\ell+d-1)}{2\ell(\ell+1)(\ell-2)!}$$

and plays no role with dimensional regularization

The irreducible mass quadrupole moment

- Posing

$$\overline{\Sigma} \equiv \frac{2}{d-1}\,\frac{(d-2)\overline{\tau}^{00}+\overline{\tau}^{ii}}{c^2} \qquad \overline{\Sigma}^i \equiv \frac{\overline{\tau}^{i0}}{c} \qquad \overline{\Sigma}^{ij} \equiv \overline{\tau}^{ij}$$

$$\overline{\Sigma}_{[\ell]}(\mathbf{x},t) = \int_{-1}^{1} \mathrm{d}z\, \delta_\ell^{(d)}(z)\, \overline{\Sigma}(\mathbf{x},t+zr)$$

$$\boxed{\begin{aligned} I_{ij} = \frac{d-1}{2(d-2)} \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^d\mathbf{x} \left(\frac{r}{r_0}\right)^B \Bigg\{ &\hat{x}^{ij}\,\overline{\Sigma}_{[2]} - \frac{4(d+2)}{d(d+4)}\,\hat{x}^{ijk}\,\dot{\overline{\Sigma}}^k_{[3]} \\ &+ \frac{2(d+2)}{d(d+1)(d+6)}\,\hat{x}^{ijkl}\,\ddot{\overline{\Sigma}}^{kl}_{[4]} \\ &- \frac{4(d-3)(d+2)}{d(d-1)(d+4)}\,B\,\hat{x}^{ijk}\,\frac{x^l}{r^2}\,\overline{\Sigma}^{kl}_{[3]} \Bigg\} \end{aligned}}$$

- The $B\varepsilon$ regularization is systematically applied (the limit $B \to 0$ is finite)

Techniques to compute the 4PN mass quadrupole

- Method of super-potentials

$$\int \mathrm{d}^3\mathbf{x}\, r^B\, \hat{x}_L \overbrace{\phi}^{\text{linear potential}} \underbrace{P}_{\text{difficult potential}} = \int \mathrm{d}^3\mathbf{x}\, r^B \Big(\Psi_L^\phi \, \Delta P + \underbrace{\partial_i \Big[\partial_i \Psi_L^\phi P - \Psi_L^\phi \partial_i P \Big]}_{\text{yields a surface term}} \Big)$$

where Ψ_L^ϕ is obtained from the super-potentials ϕ_{2k} of $\phi = \phi_0$ as

$$\boxed{\Psi_L^\phi = \Delta^{-1}\big(\hat{x}_L\, \phi\big) = \sum_{k=0}^{\ell} \frac{(-2)^k \ell!}{(\ell-k)!} x_{\langle L-K}\partial_{K\rangle} \overbrace{\phi_{2k+2}}^{\Delta\phi_{2k+2} = \phi_{2k}}}$$

- Method of surface integrals

$$\boxed{\mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x}\, r^B \hat{x}_L\, \Delta G = -(2\ell+1) \int \mathrm{d}\Omega\, \hat{n}_L X_\ell(\boldsymbol{n})}$$

where X_ℓ is the coefficient of $r^{-\ell-1}$ in the expansion of G when $r \to +\infty$

- Schwartz distributional derivatives in d dimensions systematically applied

Completion of the 4PN mass quadrupole moment

[Larrouturou, Blanchet, Henry & Faye 2021ab]

- All UV divergences treated by dimensional regularization and all UV poles shown to be renormalized by appropriate shifts of the particles' worldlines
- Presence at 4PN order of a non-local-in-time term associated with tail radiation mode and containing a crucial IR pole
- IR divergences (poles $\propto \frac{1}{d-3}$) appear already at 3PN order but are cancelled (as well as the finite part beyond) by poles coming from "tails-of-tails" propagating in the wave zone
- At 4PN order the IR poles are cancelled by more complicated "tails-of-memory" but there remains a crucial finite contribution specifically due to dimensional regularization
- Finally we have obtained the finite renormalized 4PN quadrupole moment of compact binaries ready to be used for 4PN/4.5PN templates

Towards 4.5PN templates

- 4.5PN "tail-of-tail-of-tail" completed [Marchand, Blanchet & Faye 2017]
- 3PN mass octupole moment and 3PN current quadrupole moment completed
  - UV divergences treated by dimensional regularization
  - IR divergences treated by Hadamard regularization equivalent to dimensional regularization at that order
- 2PN mass dodecapole and current octupole, as well as higher-order moments are already known
- Cubic interactions at 4PN order in the radiative quadrupole moment need to be completed in (computation to be done in ordinary $3d$)
  - relation between canonical and source/gauge moments
  - relation between radiative and canonical moments

Problem of cubic interactions at 4PN order

$$U_{ij}(u) = M^{(2)}_{ij}(u) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau M^{(4)}_{ij}(u-\tau)\left[2\ln\left(\frac{c\tau}{2b_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail}}$$

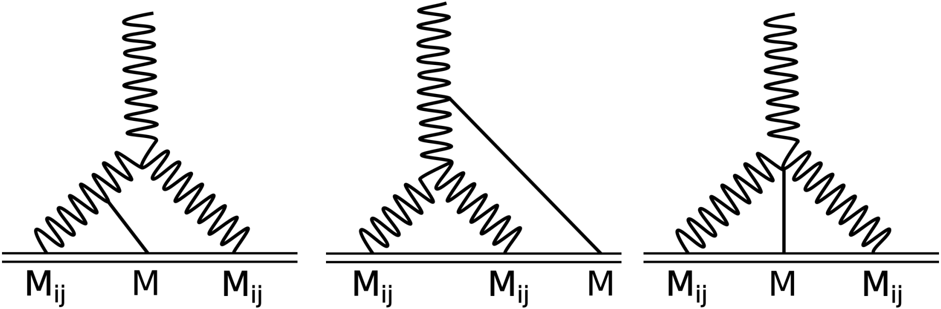
$$+ \frac{G}{c^5}\left\{\underbrace{\frac{2G}{7c^5}\int_0^{+\infty} \mathrm{d}\tau M^{(3)}_{a<i}M^{(3)}_{j>a}(u-\tau)}_{\text{2.5PN memory}} + \cdots\right\}$$

$$+ \underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau M^{(5)}_{ij}(u-\tau)\left[2\ln^2\left(\frac{c\tau}{2r_0}\right)+\frac{57}{35}\ln\left(\frac{c\tau}{2r_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tails-of-tail}}$$

$$+ \frac{G^2}{c^8}\left\{\underbrace{\sum M\int_0^{+\infty} \mathrm{d}\rho\, M^{(p)}_{a<i}(u-\rho)\int_0^{+\infty} \mathrm{d}\tau\, K(\rho,\tau)\, M^{(q)}_{j>a}(u-\rho-\tau)}_{\text{4PN tails-of-memory}}\right\}$$

$$+ \underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau M^{(6)}_{ij}(u-\tau)\left[\frac{4}{3}\ln^3\left(\frac{c\tau}{2r_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail}} + \cdots$$

Gravitational-wave tails of memory



$(M_{ij} \times M) \times M_{ij}$ $\qquad$ $(M_{ij} \times M_{ij}) \times M$ $\qquad$ $M_{ij} \times M \times M_{ij}$

Powerful integration formula [Blanchet & Damour 1986]

- The tails of memory are computed using a powerful formula to solve

$$\Box\Psi_L = \underbrace{\hat{n}_L\, S(r, t-r)}_{\text{source with given multipolarity } \ell}$$

- When the source term $S(r,u)$ tends to zero suffiently rapidly when $r \to 0$

$$\Psi_L = \int_{-\infty}^{t-r} \mathrm{d}s\, \hat{\partial}_L \left[\frac{R\left(\frac{t-r-s}{2}, s\right) - R\left(\frac{t+r-s}{2}, s\right)}{r} \right]$$

$$\text{where} \quad R(\rho, s) \equiv \rho^\ell \int_0^\rho \mathrm{d}\lambda\, \frac{(\rho-\lambda)^\ell}{\ell!} \left(\frac{2}{\lambda}\right)^{\ell-1} S(\lambda, s)$$

- Alternative form

$$\Psi_L = -\frac{2^{\ell-1}}{\ell!} \int_{-\infty}^{t-r} \mathrm{d}s \int_{\frac{t-r-s}{2}}^{\frac{t+r-s}{2}} \frac{\mathrm{d}\lambda}{\lambda^{\ell-1}}\, S(\lambda, s)\, \hat{\partial}_L \left[\frac{1}{r} \left(\frac{t-r-s}{2}\right)^\ell \left(\frac{t-r-s}{2} - \lambda\right)^\ell \right]$$

Gravitational-wave tails of memory [Trestini & Blanchet 2023]

- Computation performed using the MPM construction in radiative coordinates which avoids far zone logarithms which plague harmonic coordinates

$$U_{ij}^{\text{ToM}} = \frac{2G^2 M}{7c^8}\Bigg\{\underbrace{\int_0^{+\infty} \mathrm{d}\rho\, M^{(4)}_{a\langle i}(u-\rho)\int_0^{+\infty}\mathrm{d}\tau\, M^{(4)}_{j\rangle a}(u-\rho-\tau)\ln\left(\frac{\tau}{2r_0}\right)}_{\text{“genuine” tail-of-memory}}$$

$$+\underbrace{\int_0^{+\infty}\mathrm{d}\tau\,(M^{(3)}_{a\langle i}M^{(4)}_{j\rangle a})(u-\tau)\left[-15\ln\left(\frac{\tau}{2b_0}\right)-10\ln\left(\frac{\tau}{2r_0}\right)\right]}_{\text{tail like terms}}$$

$$+\cdots\cdots\cdots$$

$$\underbrace{-\,8M^{(2)}_{a\langle i}\int_0^{+\infty}\mathrm{d}\tau\, M^{(5)}_{j\rangle a}(u-\tau)\left[\ln\left(\frac{\tau}{2r_0}\right)+\frac{27521}{5040}\right]}_{\text{tail like terms}}\Bigg\}$$

- The 4PN genuine tail-of-memory can be retrieved by inserting 1.5PN tails into the lowest order 2.5PN memory

Tail modulation of the GW phase at the 4PN order

[Wiseman 1993; Blanchet & Schafer 1993; Blanchet, Iyer, Will & Wiseman 1996]

- Because of GW tails the GW phase ψ differs from the orbital phase ϕ by a logarithmic, tail-induced phase modulation

$$\psi = \phi - \frac{2GM\,\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right)$$

which affects the waveform at the 4PN order

- The GW frequency $\Omega = \dot{\psi}$ is shifted with respect to the orbital one $\omega = \dot{\phi}$

$$\Omega = \omega - \frac{2GM\,\dot{\omega}}{c^3}\left[\ln\left(\frac{\omega}{\omega_0}\right) + 1\right]$$

$$\boxed{\Omega = \omega\left\{1 - \frac{192}{5}\,\nu\left(\frac{Gm\omega}{c^3}\right)^{8/3}\left[\ln\left(\frac{\omega}{\omega_0}\right)+1\right] + \mathcal{O}\left(\frac{1}{c^{10}}\right)\right\}}$$

- Expressing the flux and modes in terms of the directly observable GW phase ψ and frequency Ω we find that all arbitrary constants cancel out at 4PN order

Post-adiabatic calculation of the tail integral

- The tail integral arises at the 1.5PN order

$$\propto \int_0^{+\infty} \mathrm{d}\tau \, \big[\omega(u-\tau)\big]^\alpha \, \overbrace{\mathrm{e}^{-\mathrm{i}n\,\phi(u-\tau)}}^{\text{oscillating phase}} \ln\left(\frac{\tau}{\tau_0}\right)$$

- At 4PN order we must include a 2.5PN post-adiabatic correction

$$\boxed{\xi(u) \equiv \frac{\dot{\omega}(u)}{\omega^2(u)} = \mathcal{O}\left(\frac{1}{c^5}\right)}$$

- Changing variable $\tau \longrightarrow v = \xi[\phi(u) - \phi(u-\tau)]$

$$\propto \frac{\mathrm{e}^{-\mathrm{i}n\phi(u)}}{\xi(u)} \int_0^{+\infty} \mathrm{d}v \, \underbrace{\big[\omega(u-\tau(v))\big]^{\alpha-1} \, \mathrm{e}^{\frac{\mathrm{i}n\,v}{\xi(u)}} \ln\left(\frac{\tau(v)}{\tau_0}\right)}_{\text{fast oscillating integrand in the limit } \xi(u)\to 0}$$

- The integral can be computed by replacing the integrand by its expansion when $v \to 0$ which yields the asymptotic post-adiabatic expansion

The 4.5PN GW energy flux for circular orbits

[Blanchet, Faye, Henry, Larrouturou & Trestini 2023ab]

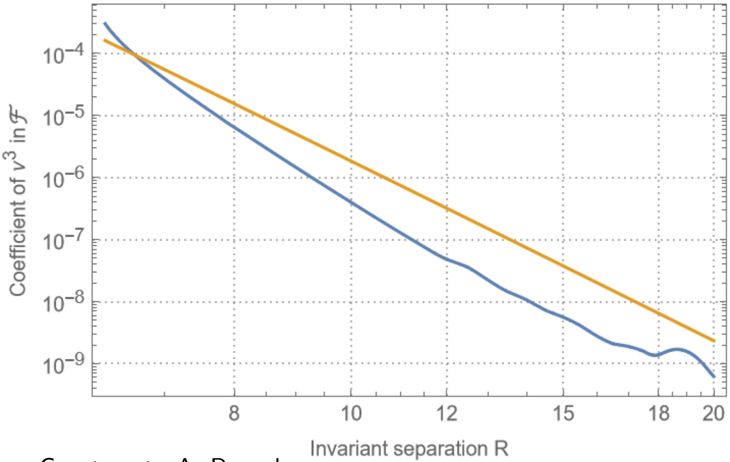
$$\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} \\
& + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathsf{E} - \frac{856}{105}\ln(16\,x) + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} \\
& + \left[-\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\mathsf{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3 + \frac{232597}{8820}\ln x \right. \\
& \quad + \left(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\mathsf{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3 + \frac{20739}{245}\ln x\right)\nu \\
& \quad \left. + \left(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\right)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4\right]x^4 \\
& + \left[\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathsf{E} - \frac{3424}{105}\ln(16\,x) + \left(\frac{2062241}{22176} + \frac{41}{12}\pi^2\right)\nu \right. \\
& \quad \left. - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\right]\pi x^{9/2} \Bigg\}
\end{aligned}$$

In the test-mass limit $\nu \to 0$, we exactly retrieve the result of linear black-hole perturbation theory [Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

Comparison with second-order GSF results

[Warburton, Pound, Wardell, Miller & Durkan 2021]

The 4.5PN flux agrees well with recent numerical second-order self-force results



Courtesy to A. Pound

The 4.5PN phase evolution of compact binaries

[Blanchet, Faye, Henry, Larrouturou & Trestini 2023ab]

Apply the energy flux-balance equation $\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}$

$$
\begin{aligned}
\psi = \psi_0 - \frac{x^{-5/2}}{32\nu}\Bigg\{ & 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right)x - 10\pi x^{3/2} \\
& + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right)x^2 + \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi x^{5/2}\ln x \\
& + \Bigg[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathrm{E} - \frac{856}{21}\ln(16\,x) \\
& \quad + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\Bigg]x^3 \\
& + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi x^{7/2} \\
& + \Bigg[\frac{2550713843998885153}{2214468081745920} - \frac{9203}{126}\gamma_\mathrm{E} - \frac{45245}{756}\pi^2 - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 - \frac{9203}{252}\ln x \\
& \quad + \left(-\frac{680712846248317}{337983528960} - \frac{488986}{1323}\gamma_\mathrm{E} + \frac{109295}{1792}\pi^2 - \frac{1245514}{1323}\ln 2 + \frac{78975}{392}\ln 3 - \frac{244493}{1323}\ln x\right)\nu \\
& \qquad + \left(\frac{7510073635}{24385536} - \frac{11275}{1152}\pi^2\right)\nu^2 + \frac{1292395}{96768}\nu^3 - \frac{5975}{768}\nu^4\Bigg]x^4 \\
& + \Bigg[-\frac{93098188434443}{150214901760} + \frac{1712}{21}\gamma_\mathrm{E} + \frac{80}{3}\pi^2 + \frac{856}{21}\ln(16x) \\
& \quad + \left(\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right)\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3\Bigg]\pi x^{9/2}\Bigg\}
\end{aligned}
$$

Number of cycles contributed by each PN order

Contribution of each PN order to the total number of accumulated cycles

Detector	LIGO/Virgo		ET		LISA	
Masses ($M_\odot$)	1.4×1.4	10×10	1.4×1.4	500×500	$10^5 \times 10^5$	$10^7 \times 10^7$
PN order	cumulative number of cycles					
Newtonian	$2\,562.599$	95.502	$744\,401.36$	37.90	$28\,095.39$	9.534
1PN	143.453	17.879	$4\,433.85$	9.60	618.31	3.386
1.5PN	-94.817	-20.797	$-1\,005.78$	-12.63	-265.70	-5.181
2PN	5.811	2.124	23.94	1.44	11.35	0.677
2.5PN	-8.105	-4.604	-17.01	-3.42	-12.47	-1.821
3PN	1.858	1.731	2.69	1.43	2.59	0.876
3.5PN	-0.627	-0.689	-0.93	-0.59	-0.91	-0.383
4PN	-0.107	-0.064	-0.12	-0.04	-0.12	-0.013
4.5PN	0.098	0.118	0.14	0.10	0.14	0.065

- The PN approximation seems to converge well for comparable masses
- This suggests that systematic errors due to the PN modeling may be dominated by statistical errors and negligible for LISA
- However, this should be confirmed by detailed investigations along the lines [Owen, Haster, Perkins, Cornish & Yunes 2023]