On the trace decomposition of tensors via the Brauer algebra

Yegor Goncharov and Thomas Helpin

(based on [arXiv:2212.14496] with D. Bulgakova)

All is implemented in the [xBrauer] package which is linked to [xAct]

INSTITUT DENIS POISSON ORLÉANS • TOURS Mathématiques & Physique Théorique

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Outline

Introduction: irreducible trace decompositions

Irreducible trace decompositions

- Trace operation:
  - ▷ a pair of tensor indices can be contracted by the metric:

$$T_{a_1 \ldots a_n} \quad \xrightarrow{\mathrm{tr}_{ij}} \quad g^{a_i a_j} T_{a_1 \ldots a_i \ldots a_j \ldots a_n} \equiv T_{a_1 \ldots}{}^{b}{}_{\ldots b \ldots a_n}$$

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  - a tensor is 2-*traceless* if

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    and if it is *not traceless*.

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  - By analogy, define f-traceless tensors for $f = 3, 4, \dots$.

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Given T_n a tensor with n indices

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- **Problem$_1$:** construct the f-traceless projection:

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- **Problem$_2$:** construct the **trace decomposition** [H. Weyl'46]

$$T_n = \overset{(1)}{T}_n \oplus \overset{(2)}{T}_n \oplus \ldots$$

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- **Problem$_3$:** construct the **irreducible central decomposition**:

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$$\overset{(f)}{P}_n = \sum_{\lambda} P_n^{\lambda} \,, \quad P_n^{\lambda} P_n^{\mu} = \delta_{\lambda\mu} \, P_n^{\lambda}$$

Irreducible trace decompositions

- Examples:
  - ▷ Multipole decomposition of the electrostatic potential :

$$\triangle\varphi(x) = 0 \quad \Rightarrow \quad T_{a_1\ldots a_n}(x) = \partial_{a_1}\ldots\partial_{a_n}\varphi(x) \text{ are symmetric and traceless}$$

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  - ▷ Fronsdal fields: spin-s particles on Minkowski background via symmetric tensor fields $\phi_{a_1 \dots a_s}(x)$:

$$\underbrace{\phi^{b}{}_{b}{}^{c}{}_{c\,a_1 \dots a_{s-4}}(x)}_{\text{2-traceless}} = 0$$

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  - ▷ Colour decomposition of scattering amplitudes in the S-matrix bootstrap approach :

$$\langle p_{\mathrm{a}}, p_{\mathrm{b}}|\,\widehat{S}\,|p_{\mathrm{c}}, p_{\mathrm{d}}\rangle \sim \quad S_0(s)\underbrace{\frac{1}{d}\delta_{\mathrm{ab}}\delta_{\mathrm{cd}}}_{\text{trace}} + S_{+}(s)\underbrace{\left(\frac{\delta_{\mathrm{ac}}\delta_{\mathrm{bd}} + \delta_{\mathrm{ad}}\delta_{\mathrm{bc}}}{2} - \frac{1}{d}\delta_{\mathrm{ab}}\delta_{\mathrm{cd}}\right)}_{\text{traceless symmetriser}}$$

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  - Decomposition of the Riemann tensor in metric-affine gravity.

Decomposition of Riemann tensors

Decomposition of the metric Riemann tensor

- Riemann tensor $R_{[ab][cd]}$ in Riemannian geometry (M, g) :

Trace decomposition : $R_{[ab][cd]} = \underbrace{W_{[ab][cd]}}_{\textit{1-traceless}} + \underbrace{E_{[ab][cd]}}_{\textit{2-traceless}} + \underbrace{S_{[ab][cd]}}_{\textit{full trace}}.$

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$$\textit{SO(d) - decomposition :}\quad R^{\boxplus}_{[ab][cd]} = \underbrace{W^{\boxplus}_{[ab][cd]}}_{\textit{1-traceless}} + \underbrace{E^{\square\square}_{[ab][cd]}}_{\textit{2-traceless}} + \underbrace{S^{\varnothing}_{[ab][cd]}}_{\textit{full trace}} .$$

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The explicit expressions for this decomposition are known :

$$W^{\boxplus}_{[ab][cd]} = R^{\boxplus}_{[ab][cd]} - E^{\square\square}_{[ab][cd]} - S^{\varnothing}_{[ab][cd]}$$

$$E^{\square\square}_{[ab][cd]} = \frac{1}{d-2}\left(g_{db}\underline{R}_{ca} - g_{da}\underline{R}_{cb} + g_{ca}\underline{R}_{db} - g_{cb}\underline{R}_{da}\right)$$

$$S^{\varnothing}_{[ab][cd]} = \frac{R}{d(d-1)}\left(g_{ca}g_{db} - g_{cb}g_{da}\right) .$$

with $\underline{R}_{ab} = R_{ab} - \frac{g_{ab}}{d}R$, the traceless part of Ricci tensor.

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but ...

$$\mathcal{R}_{ab[cd]} = \mathcal{R}^{\yng(3,1)}_{ab\,[cd]} + \mathcal{R}^{\yng(2,2)}_{ab\,[cd]} + \mathcal{R}^{\yng(2,1,1)}_{ab\,[cd]} + \mathcal{R}^{\yng(1,1,1,1)}_{ab\,[cd]}\,, \qquad \mathcal{R}^{\,\mu}_{ab\,[cd]} = Z^{\,\mu}(\mathcal{R}_{ab\,[cd]})$$

$Z^{\yng(4)} \to$ total symmetrisation $\qquad Z^{\yng(1,1,1,1)} \to$ total anti-symmetrisation

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SO(d) decomposition : $\yng(2,2)$: GL(d) irreps $\supset$ $\yng(2,2)$: SO(d) irreps

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The explicit expressions for this decomposition are not available in the literature

On the complexity of the trace decomposition

$$T_{a_1\,a_2\,\ldots\,a_n} = \overset{(1)}{T}_{a_1\,a_2\,\ldots\,a_n} + \overset{(2)}{T}_{a_1\,a_2\,\ldots\,a_n} + \ldots + \overline{T}_{a_1\,a_2\,\ldots\,a_n}.$$

[H. Weyl, Classical Groups 1946]

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- Brute force strategy for 1-traceless part :

1. Write down the most general Ansatz for $\overset{(1)}{T}_{i_1\,i_2\,\dots\,i_n}$:

$$\overset{(1)}{T}_{a_1\,a_2\,\dots\,a_n} = T_{a_1\,a_2\,\dots\,a_n} - c_1\, g_{a_1 a_2}\, T^{c}{}_{c\,a_3\,\dots\,a_n} - c_2\, g_{a_1 a_3}\, T^{c}{}_{a_2 c\, a_4\,\dots\,a_n} - \ldots$$

2. Solve the algebraic equations: $\overset{(1)}{T}{}^{c}{}_{c a_3\,\dots\,a_n} = 0$, $\overset{(1)}{T}{}^{c}{}_{a_2 c a_4\,\dots\,a_n} = 0, \ldots.$

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- Examples 1-traceless tensors :
  - $\overset{(1)}{T}_{a_1\,a_2} = T_{a_1\,a_2} - \frac{1}{d}T^c{}_c$ (easy)
  - $\overset{(1)}{T}_{a_1\,a_2\,a_3} = T_{a_1 a_2 a_3} - \frac{g_{a_1 a_2}}{(d-1)(d+2)}\Big((\,d+1\,)T^c{}_{c a_3} - T^c{}_{a_3 c} - T_{a_3}{}^c{}_c\Big) + \ldots$ (hard)
  - $\overset{(1)}{T}_{a_1\,a_2\,a_3\,a_4} = T_{a_1\,a_2\,a_3\,a_4} - \ldots$ (very hard)

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- Examples 1-traceless projectors :

  - $\overset{(1)}{P}_{a_1\,a_2}{}^{b_1\,b_2} = \delta_{a_1}{}^{b_1}\delta_{a_2}{}^{b_2} - \frac{1}{d}\,g_{a_1 a_2}g^{b_1 b_2}$
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$\rightarrow$ Computations are very long both by hand and with a computer.
No explicit formulas except for particular cases (*e.g.* symmetric tensors).

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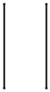
$\delta_{a_1}{}^{b_1}\delta_{a_2}{}^{b_2} \mapsto$  $g_{a_1a_2}g^{b_1b_2} \mapsto$  $\mathcal{B}_2(d)$

[R. Brauer, 1936]

On the complexity of the trace decomposition

$$T_{a_1\,a_2\,\dots\,a_n} = \overset{(1)}{T}_{a_1\,a_2\,\dots\,a_n} + \overset{(2)}{T}_{a_1\,a_2\,\dots\,a_n} + \,\dots + \overline{T}_{a_1\,a_2\,\dots\,a_n}.$$

[H. Weyl, Classical Groups 1946]

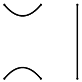
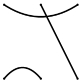
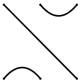
- Examples 1-traceless projectors :

  ▷ $\overset{(1)}{P}_{a_1\,a_2}{}^{b_1\,b_2} = \delta_{a_1}{}^{b_1}\delta_{a_2}{}^{b_2} - \frac{1}{d}\,g_{a_1a_2}g^{b_1b_2}$

  ▷ $\overset{(1)}{P}_{a_1\,a_2\,a_3}{}^{b_1\,b_2\,b_3} = \delta_{a_1}{}^{b_1}\delta_{a_2}{}^{b_2}\delta_{a_3}{}^{b_3} - \frac{1}{(d-1)(d+2)}\left(g_{a_1a_2}g^{b_1b_2}\delta_{a_3}{}^{b_3}(d+1) - g_{a_1a_2}g^{b_1b_3}\delta_{a_3}{}^{b_2} - g_{a_1a_2}g^{b_2b_3}\delta_{a_3}{}^{b_1}\right) + \dots$

  ▷ $\overset{(1)}{P}_{a_1\,a_2\,a_3\,a_4}{}^{b_1\,b_2\,b_3\,b_4} = \delta_{a_1}{}^{b_1}\delta_{a_2}{}^{b_2}\delta_{a_3}{}^{b_3}\delta_{a_4}{}^{b_4} - \dots$

  $\rightarrow$ We need algebraic tools : the diagrammatic Brauer algebra $\mathcal{B}_n(d)$

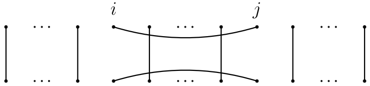
$g_{a_1a_2}g^{b_1b_2}\delta_{a_3}{}^{b_3} \mapsto$  $g_{a_1a_2}g^{b_1b_3}\delta_{a_3}{}^{b_2} \mapsto$  $g_{a_1a_2}g^{b_2b_3}\delta_{a_3}{}^{b_1} \mapsto$  $\mathcal{B}_3(d)$

[R. Brauer, 1936]

Traceless projection and the Brauer algebra

The element A_n : 1^{st} building block of the construction

$$A_n = \sum_{1 \leqslant i < j \leqslant n}$$



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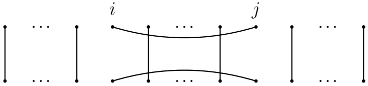

$$A_2 =$$

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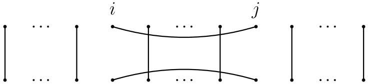
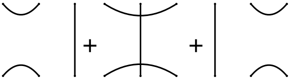


$$A_3 =$$



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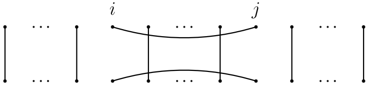
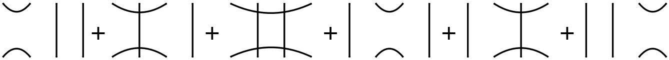


$$A_4 =$$



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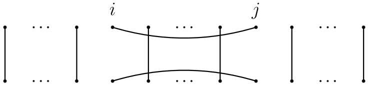

Lemma:

1. $\mathrm{Ker}(A_n) =$ *space of 1-traceless tensors.*
2. *The action of A_n on $V^{\otimes n}$ is diagonalisable, in particular* *

$$A_n(v^{\mu\backslash\lambda}) = \alpha_{\mu\backslash\lambda}\, v^{\mu\backslash\lambda}, \text{ with } \quad \alpha_{\mu\backslash\lambda} = \frac{|\mu| - |\lambda|}{2}(\mathrm{d}-1) + \mathrm{c}(\mu) - \mathrm{c}(\lambda).$$

3. *The eigenvalues of A_n are non-negative integers :* $\alpha_{\mu\backslash\lambda} \in \mathbb{N}_0$

μ : GL(d) irreps $\qquad$ λ : SO(d) irreps

* **[M. Nazarov, 1996]**

[D. Bulgakova, Y. G, T. H, 2022]

The element A_n : 1^{st} building block of the construction

$$\mathcal{R}_{ab[cd]} = \mathcal{W}^{\yng(3,1)}_{ab[cd]} + \mathcal{W}^{\yng(2,2)}_{ab[cd]} + \mathcal{W}^{\yng(2,1,1)}_{ab[cd]} + \mathcal{W}^{\yng(1,1,1,1)}_{ab[cd]} + \mathcal{E}^{\yng(2)}_{ab[cd]} + \mathcal{E}^{\yng(1,1)}_{ab[cd]} + \mathcal{S}^{\varnothing}_{[ab][cd]}.$$

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1. $A_4(\mathcal{W}^{\lambda}_{ab[cd]}) = 0.$

2. $A_4(Z^{\yng(2,2)}(\mathcal{E}^{\yng(2)}_{ab[cd]})) = \alpha\ Z^{\yng(2,2)}(\mathcal{E}^{\yng(2)}_{ab[cd]})$ with $\alpha = d - 2$

$A_4(Z^{\yng(3,1)}(\mathcal{E}^{\yng(1,1)}_{ab[cd]})) = \alpha\ Z^{\yng(3,1)}(\mathcal{E}^{\yng(1,1)}_{ab[cd]})$ with $\alpha = d + 2$

Traceless projectors

Theorem: 1-traceless projectors are given by

$$\overset{(1)}{P}_n = \prod_{\alpha \in \mathrm{spec}^*(A_n)} \left(\mathbb{1} - \frac{1}{\alpha} A_n\right)$$

[D. Bulgakova, Y. G, T. H, 2022]

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Then, $\overset{(1)}{P}_n(T) = \overset{(1)}{T}$

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$$\overset{(1)}{P}{}_n^{\lambda} = Z^{\lambda} \prod_{\alpha \,\in\, \mathrm{spec}_{\lambda}^*(A_n)} \left(\mathbb{1} - \frac{1}{\alpha} A_n\right)$$

Z^{λ} = Central Young projectors $\in \mathbb{C}\mathfrak{S}_n$

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? ?

From traceless to multi-traceless projection

Arc induction : 2^{nd} building block of the construction

Idea : Construct 2-traceless projectors from 1-traceless projectors

$$\overset{(1)}{P}{}^{\lambda}_{n-2} \longrightarrow \overset{(2)}{P}{}^{\lambda}_{n} \longrightarrow \overset{(3)}{P}{}^{\lambda}_{n+2} \ \ldots$$

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Tool : The arc induction map $\mathfrak{A} : \mathcal{B}_{n-2}(d) \to \mathcal{B}_n(d)$

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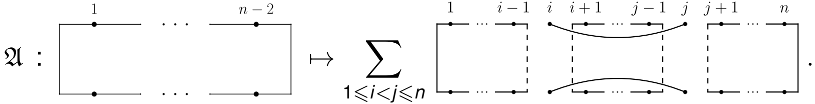

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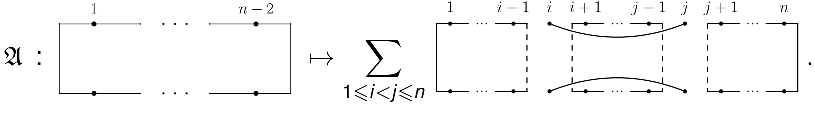

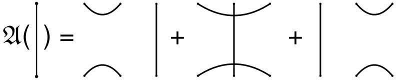

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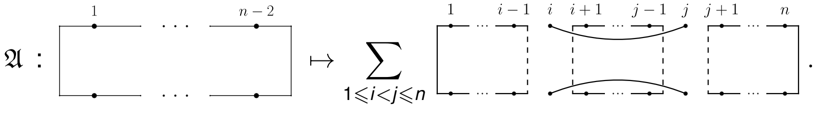

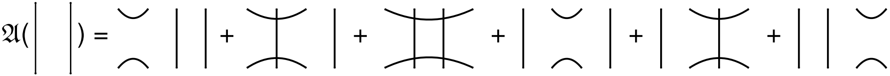

Arc induction : 2^{nd} building block of the construction

Idea : *Construct 2-traceless projectors from 1-traceless projectors*

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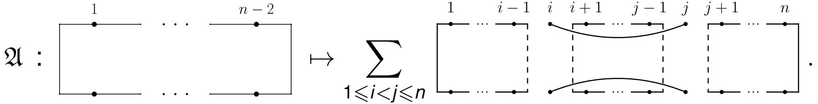
Tool : *The arc induction map* $\mathfrak{A} : \mathcal{B}_{n-2}(d) \to \mathcal{B}_n(d)$



$$\mathfrak{A}(\,\mathbb{1}_{n-2}\,) = A_n$$

Arc induction : 2^{nd} building block of the construction

Idea : Construct 2-traceless projectors from 1-traceless projectors

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Tool : The arc induction map $\mathfrak{A} : \mathcal{B}_{n-2}(d) \to \mathcal{B}_n(d)$

Lemma:

$$\mathfrak{A}(\overset{(f-1)}{P}{}^{\lambda}_{n-2}) = A_n \overset{(f)}{P}{}^{\lambda}_{n} .$$

As a consequence,

1. $\mathfrak{A}(\overset{(f-1)}{P}{}^{\lambda}_{n-2})(v^{\mu\backslash\lambda}) = \alpha_{\mu\backslash\lambda}\, v^{\mu\backslash\lambda} ,$
2. $\mathfrak{A}(\overset{(f-1)}{P}{}^{\lambda}_{n-2})(v^{\mu\backslash\beta}) = 0, \qquad \forall\beta \neq \lambda .$

μ : *GL(d) irreps* λ : *SO(d) irreps*

[Y. G, T. H, (in preparation)]

Theorem: f-traceless projectors are given by

$$\overset{(f)}{P}{}_n^{\lambda} = \sum_{\mu\,\in \mathrm{cl}_n(\lambda)} \overset{(f)}{P}{}_n^{\mu\backslash\lambda}\,, \quad \text{with}$$

$$\overset{(f)}{P}{}_n^{\mu\backslash\lambda} = \frac{\mathfrak{A}(\overset{(f-1)}{P}{}_{n-2}^{\lambda})}{\alpha_{\mu\backslash\lambda}} \prod_{\substack{\beta\,\in\,\mathrm{spec}^{\lambda}(A_n)\,,\\ \beta\neq\alpha_{\mu\backslash\lambda}}} \left(\frac{\beta - A_n}{\beta - \alpha_{\mu\backslash\lambda}}\right).$$

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[Y. G, T. H, (in preparation)]

The explicit expressions for the SO(d) decomposition

$$\mathcal{R}^{\yng(3,1)}_{ab[cd]} = \mathcal{W}^{\yng(3,1)}_{ab[cd]} + \mathcal{E}^{\yng(3,1)\backslash\yng(2)}_{ab[cd]} + \mathcal{E}^{\yng(3,1)\backslash\yng(1,1)}_{ab[cd]}$$

$$\mathcal{W}^{\yng(3,1)}_{ab[cd]} = \mathcal{R}^{\yng(3,1)}_{ab[cd]} - \mathcal{E}^{\yng(3,1)\backslash\yng(2)}_{ab[cd]} - \mathcal{E}^{\yng(3,1)\backslash\yng(1,1)}_{ab[cd]}$$

$$\mathcal{E}^{\yng(3,1)\backslash\yng(2)}_{ab[cd]} = \frac{1}{2\,d}\left(g_{bc}\left(\overset{(1)}{\underline{R}}_{(ad)} - \overset{(2)}{\underline{R}}_{(ad)}\right) - g_{bd}\left(\overset{(1)}{\underline{R}}_{(ac)} - \overset{(2)}{\underline{R}}_{(ac)}\right) + (a \leftrightarrow b)\right)$$

$$\mathcal{E}^{\yng(3,1)\backslash\yng(1,1)}_{ab[cd]} = \frac{1}{4\,(d+2)}\left(g_{bd}\left(\overset{(1)}{\underline{R}}_{[ac]} + \overset{(2)}{\underline{R}}_{[ac]} - \overset{(3)}{R}_{[ac]}\right) - g_{bc}\left(\overset{(1)}{\underline{R}}_{[ad]} + \overset{(2)}{\underline{R}}_{[ad]} - \overset{(3)}{R}_{[ad]}\right)\right.$$
$$\left. +(a \leftrightarrow b) - g_{ab}\left(\overset{(1)}{\underline{R}}_{[cd]} + \overset{(2)}{\underline{R}}_{[cd]} - \overset{(3)}{R}_{[cd]}\right)\right).$$

$$\overset{(1)}{\underline{R}}_{ab} = \overset{(1)}{R}_{ab} - \frac{g_{ab}}{d}R, \quad \textit{(traceless part of Ricci tensor)},$$

$$\overset{(2)}{\underline{R}}_{ab} = \overset{(2)}{R}_{ab} - \frac{g_{ab}}{d}R, \quad \textit{(traceless part of co-Ricci tensor)}$$

$$\overset{(3)}{R}_{[ab]} = \mathcal{R}^{c}{}_{c[ab]}, \quad \textit{(Homothetic tensor)}$$

The explicit expressions for the SO(d) decomposition

$$\mathcal{R}^{\boxplus}_{ab[cd]} = \mathcal{W}^{\boxplus}_{ab[cd]} + \mathcal{E}^{\boxplus \setminus \square\square}_{ab[cd]} + \mathcal{S}^{\varnothing}_{ab[cd]}$$

$$\mathcal{W}^{\boxplus}_{ab[cd]} = \mathcal{R}^{\boxplus}_{ab[cd]} - \mathcal{E}^{\boxplus \setminus \square\square}_{ab[cd]} - \mathcal{S}^{\varnothing}_{ab[cd]}$$

$$\mathcal{E}^{\boxplus \setminus \square\square}_{ab[cd]} = \frac{1}{2\,(d-2)} \left(g_{bd} \left(\overset{(1)}{\underline{R}}_{(ac)} + \overset{(2)}{\underline{R}}_{(ac)} \right) - g_{bc} \left(\overset{(1)}{\underline{R}}_{(ad)} + \overset{(2)}{\underline{R}}_{(ad)} \right) + (a \leftrightarrow b) \right)$$

$$\mathcal{S}^{\varnothing}_{ab[cd]} = \frac{R}{d(d-1)} \left(g_{ad} g_{bc} - g_{ac} g_{bd} \right)$$

$$\overset{(1)}{\underline{R}}_{ab} = \overset{(1)}{R}_{ab} - \frac{g_{ab}}{d} R, \quad \textit{(traceless part of Ricci tensor),}$$

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$$\mathcal{R}^{\yng(2,1,1)}_{ab[cd]} = \mathcal{W}^{\yng(2,1,1)}_{ab[cd]} + \mathcal{E}^{\yng(2,1,1)\backslash\yng(1,1)}_{ab[cd]}$$

$$\mathcal{W}^{\yng(2,1,1)}_{ab[cd]} = \mathcal{R}^{\yng(2,1,1)}_{ab[cd]} - \mathcal{E}^{\yng(2,1,1)\backslash\yng(1,1)}_{ab[cd]}$$

$$\mathcal{E}^{\yng(2,1,1)\backslash\yng(1,1)}_{ab[cd]} = \frac{1}{4\,(d-2)}\left(g_{bc}\left(\overset{(1)}{\underline{R}}_{[ad]} - 3\,\overset{(2)}{\underline{R}}_{[ad]} + \overset{(3)}{R}_{[ad]}\right) - g_{bd}\left(\overset{(1)}{\underline{R}}_{[ac]} - 3\overset{(2)}{\underline{R}}_{[ac]} + \overset{(3)}{R}_{[ac]}\right)\right.$$
$$\left. + \left(a \leftrightarrow b\,;\ (1) \leftrightarrow (2)\right) + g_{ab}\left(\overset{(1)}{\underline{R}}_{[cd]} + \overset{(2)}{\underline{R}}_{[cd]} + \overset{(3)}{R}_{[cd]}\right)\right).$$

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Conclusions

1. Trace decomposition is a computationally hard problem

2. One can resolve the problem with the aid of the representation theory of Brauer algebras $\mathcal{B}_n(d)$:

   - 1-traceless projectors
   - multi-traceless projectors via induction from the traceless ones

3. Application to metric affine gravity :

   - Algebraically transparent construction of quadratic curvature Lagrangians
   - Possible classification of such theories

Thank you for your attention

More remarks

1. Z^μ projects onto the direct sum of equivalent $GL(d)$ irreps.

   P_n^λ projects onto the direct sum of equivalent $O(d)$ irreps.

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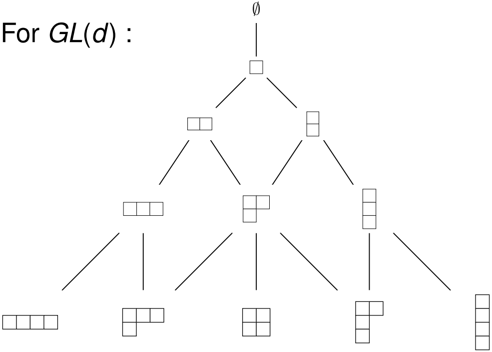
For $GL(d)$:



$$Y_T = \prod_{\mu \in T} Z_{\mu}$$

Semi-normal Young units

[A.M. Vershik, A.Y. Okunkov, (2005)]

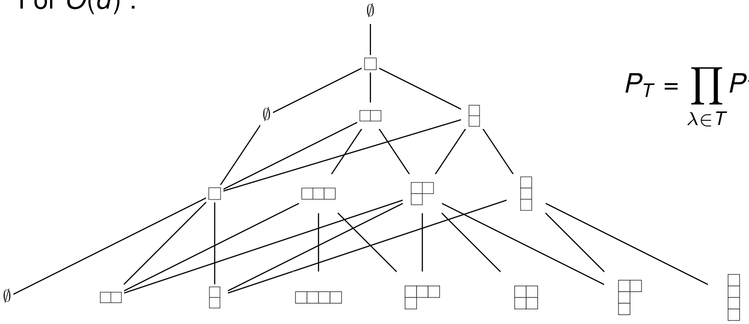
More remarks

1. Z^μ projects onto the direct sum of equivalent $GL(d)$ irreps.

   P_n^λ projects onto the direct sum of equivalent $O(d)$ irreps.

2. Projection onto a specific irrep $\rightarrow$ follow the path on the Bratteli diagram.

For $O(d)$:



$$P_T = \prod_{\lambda \in T} P^\lambda$$

[S. Doty, A. Lauve, G. Seelinger, (2016)]