Parametrization and localization of Einstein's initial and asymptotic data sets

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- construction of Einstein's initial data sets
- constraint equations: nonlinear elliptic, underdetermined, degenerate
- asymptotic properties on decay and blow-up
- gluing techniques, parametrization, localization

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Two main objectives in this talk

- **Scattering and classification for gravitational bouncing**
  with Bruno Le Floch (LPTHE, Sorbonne)
  Gabriele Veneziano (CERN, Geneva)

- **Localized seed-to-solution method for isolated systems**
  with Bruno Le Floch (LPTHE, Sorbonne)
  The-Cang Nguyen (Paris)

Junctions between spacetimes

Regime of interest

- complex dynamics near singularities — Belinsky, Khalatnikov, Lifshitz, Damour, etc.
- quiescent regime, monotone behavior — oscillation-free

  spatial derivatives negligible, observers cannot communicate
- Einstein-matter system — scalar field, stiff or compressible fluid

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Junction conditions

asymptotics near a singularity hypersurface

- bouncing behavior — contracting/expanding singularity hypersurfaces
- beyond Israel junction condition (Penrose, cut and paste) — regularity hypersurfaces
- Objective: parametrize *all* meaningful junctions — physically, mathematically

  literature: special junctions, symmetric spacetimes

1. SCATTERING MAPS for GRAVITATIONAL BOUNCING

Junctions between spacetimes

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Proposed framework

- work with *general* spacetimes, asymptotic version of the constraints
- (past, future) singularity scattering data/maps

  $$\mathbf{S} : (g^-, K^-, \phi_0^-, \phi_1^-) \mapsto (g^+, K^+, \phi_0^+, \phi_1^+)$$
- classification/parametrization — degrees of freedom
- **S**-cyclic spacetime

Vicinity of a spacelike singularity

ADM formulation

- Gaussian foliation by spacelike hypersurfaces $\mathcal{M}^{(4)} = \bigcup_\tau \mathcal{H}_\tau$

$g^{(4)} = -d\tau^2 + g(\tau)$ $g(\tau) = g_{ij}(\tau)dx^i dx^j$ τ near zero

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- Einstein's evolution equations — induced metric g, extrinsic curvature K

$$\partial_\tau g_{ij} = -2\, K_{ij} \qquad \partial_\tau K^i_j = \mathrm{Tr}(K) K^i_j + R^i_j - 8\pi\, M^i_j$$

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$$M^i_j(\phi) = \tfrac{1}{2}\rho\, g^i_j + T^i_j - \tfrac{1}{2}\mathrm{Tr}(T) g^i_j$$

- Einstein's constraints — Hamiltonian, momentum, nonlinear elliptic

$$R + |K|^2 - \mathrm{Tr}(K^2) = 16\pi\rho \qquad \nabla_i K^i_j - \nabla_j(\mathrm{Tr} K) = 8\pi J_j$$

- wave equation $\Box_{g^{(4)}} \phi = 0$ for a scalar field ϕ — expressions for ρ, J_i

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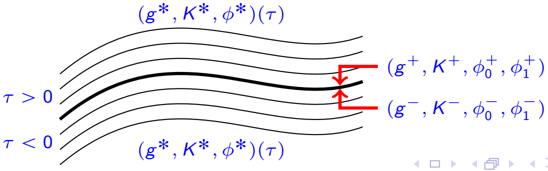
Behavior near a singularity hypersurface

approximate solution



A typical asymptotic behavior: the Kasner profiles

$$g^*_{\text{Kasner}}(\tau, x) = (-\tau)^{2p_1(x)}(dx^1)^2 + (-\tau)^{2p_2(x)}(dx^2)^2 + (-\tau)^{2p_3(x)}(dx^3)^2$$

$$K^*_{\text{Kasner}}(\tau, x) = \frac{-1}{\tau}\text{diag}(p_1, p_2, p_3)(x)$$

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- In this case, the singularity data are the
  - Euclidean metric g^-
  - tensor K^- with constant eigenvectors
  - $K^- \equiv \text{diag}(p_1, p_2, p_3)$ in some coordinates
  - exponents p_1, p_2, p_3 possibly depending upon the spatial variable x
  - matter data (ϕ_0^-, ϕ_1^-)
- This is an "asymptotic profile", in a sense we define next.

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Our standpoint

- a systematic study of the asymptotic data
- parametrize and analyze an asymptotic version of the Einstein constraints
- formulation and parametrization of junction conditions

A singularity hypersurface is a given 3-manifold $\mathcal{H}$.

Definition

(Past) **asymptotic profile** associated with some given data $(g^-, K^-, \phi_0^-, \phi_1^-)$ is the ancient geometric flow defined on $\mathcal{H}$ by

$$\tau \in (-\infty, 0) \mapsto \big(g^*, K^*, \phi^*\big)(\tau)$$

$$g^*(\tau) = |\tau|^{2K^-} g^- \qquad K^*(\tau) = \frac{-1}{\tau} K^- \qquad \phi^*(\tau) = \phi_0^- \log|\tau| + \phi_1^-$$

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(Past) **singularity initial data set** on a 3-manifold $\mathcal{H}$

two symmetric 2-tensor fields (g^-, K^-)

two scalar fields (ϕ_0^-, ϕ_1^-)

Riemannian metric	g^-
CMC symmetric $(1,1)$-tensor	$\mathbf{Tr}(K^-) = 1$
Hamiltonian constraint	$1 - |K^-|^2 = 8\pi\,(\phi_0^-)^2$
momentum constraints	$\mathbf{div}_{g^-}(K^-) = 8\pi\,\phi_0^-\,d\phi_1^-$

which we refer to as the **Einstein's asymptotic constraints.**

Notation $\mathbf{I}(\mathcal{H})$: space of all singularity data $(g^-, K^-, \phi_0^-, \phi_1^-)$

Scattering maps and gluing

Past-to-future singularity scattering map on a manifold $\mathcal{H}$

$$\mathbf{S} : \mathbf{I}(\mathcal{H}) \to \mathbf{I}(\mathcal{H}) \qquad (g^-, K^-, \phi_0^-, \phi_1^-) \mapsto (g^+, K^+, \phi_0^+, \phi_1^+)$$

- *diffeomorphism-covariant* — coordinate invariant
- *pointwise or ultra-local map* — pointwise values only
  $\mathbf{S}(g^-, K^-, \phi_0^-, \phi_1^-)(p)$ depends only on $(g^-, K^-, \phi_0^-, \phi_1^-)(p)$
- *quiescent regime* $K^- > 0$ and $K^+ > 0$

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Singular junction condition

- class of manifolds $\mathcal{M}^4$ with Lorentzian metric $g^{(4)}$ and scalar field ϕ
- $g^{(4)}$ and ϕ bounded outside a singularity locus $\mathcal{L} \subset \mathcal{M}^4$
- assume the existence of rescaled limits

$$(g^\pm, K^\pm) = \lim_{\substack{\tau \to 0 \\ \tau \gtrless 0}} \left(|\tau|^{2\tau K} g,\ -\tau K\right)$$

$$(\phi_0^\pm, \phi_1^\pm) = \lim_{\substack{\tau \to 0 \\ \tau \gtrless 0}} \left(\tau \partial_\tau \phi,\ \phi - \tau \log|\tau| \partial_\tau \phi\right)$$

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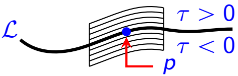

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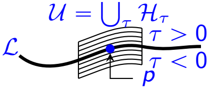
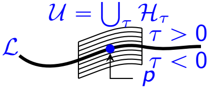
relation between the past / future singularity data

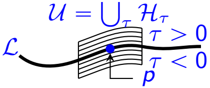
$$(g^+, K^+, \phi_0^+, \phi_1^+) = \mathbf{S}(g^-, K^-, \phi_0^-, \phi_1^-)$$



$\mathcal{U} = \bigcup_\tau \mathcal{H}_\tau$
$\tau > 0$
$\mathcal{L}$
$\tau < 0$
p



- a three-manifold $\mathcal{H}_0$ and a quiescence-preserving scattering map **S**
- past singularity data $(g^-, K^-, \phi_0^-, \phi_1^-)$ defined on $\mathcal{H}_0$



Local gluing at singularities — B. Le Floch, PLF, G. Veneziano

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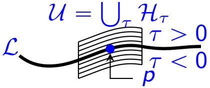
Then:

- a $\mathbf{S}$-spacetime $(\mathcal{M}^{(4)}, g^{(4)})$ with singularity locus $\mathcal{H}_0$
- a local Gaussian foliation $\mathcal{M}^{(4)} = \bigcup_{\tau \in [\tau_{-1}, \tau_1]} \mathcal{H}_\tau$ with time function τ
- the flow $\tau \mapsto (g(\tau), K(\tau), \phi(\tau))$ satisfies the Einstein equations coupled to a scalar field ϕ away from $\tau = 0$
- the junction $(g^+, K^+, \phi_0^+, \phi_1^+) = \mathbf{S}(g^-, K^-, \phi_0^-, \phi_1^-)$ holds on $\mathcal{H}_0$.



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If $\mathcal{H}$ is compact:

- shrinking volume of the slices $\mathrm{Vol}_{g(\tau)}(\mathcal{H}_\tau) \to 0$
- crushing singularity: mean curvature blowup $\lim_{\tau \to 0} \tau H(\tau) = -1$ on $\mathcal{H}_\tau$
- curvature singularity spacetime scalar (and Weyl) curvature

$$\lim_{\tau \to 0^\pm} \tau^2 R^{(4)}(\tau) = -8\pi (\phi_0^\pm)^2 \text{ on } \mathcal{H}_\tau$$

Proposed strategy

locality property for any $p \in \mathcal{H}$

$\mathbf{S}(g^-, K^-, \phi_0^-, \phi_1^-)(p)$ depends upon $(g^-, K^-, \phi_0^-, \phi_1^-)(p)$ and possibly derivatives at the point p, only

A singularity scattering map $\mathbf{S}$ is said to be

- **a ultra-local map** if — pointwise values only

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- **a rigidly conformal map** if g^+ and g^- differ by a conformal factor

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Conditions satisfied at the junction

we rely on the asymptotic constraints

CMC symmetric $(1,1)$-tensor K^-	$\mathbf{Tr}(K^-) = 1$
Hamiltonian constraint	$1 - \|K^-\|^2 = 8\pi\, (\phi_0^-)^2$
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first the subclass of rigidly conformal maps

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Classification and flexible framework

- uncovered *all possible* classes of junction

geometrically / physically meaningful

conformal/non-conformal spacelike/null/timelike

scalar field stiff fluid compressible fluid

a complete classification

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a complete classification

- discovered three universal laws constrain macroscopic aspects of spacetime junction

regardless of their origin from different microscopic corrections

a guide to uncover specific structures

Main classification results

Rigidly conformal bounces — B. Le Floch, PLF, G. Veneziano

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- **Isotropic rigidly conformal bounce $\mathbf{S}^{\text{iso, conf}}_{\lambda,\varphi}$**

$$g^+ = \lambda^2 g^- \qquad K^+ = \delta/3 \qquad \phi_0^+ = 1/\sqrt{12\pi} \qquad \phi_1^+ = \varphi$$

parametrized by a conformal factor $\lambda = \lambda(\phi_0^-, \phi_1^-, \det K^-) > 0$ and a constant φ

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$$g^+ = c^2 \mu^2 g^- \qquad K^+ = \mu^{-3}(K^- - \delta/3) + \delta/3$$

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$$\mu(\phi_0, \phi_1) = \big(1 + 12\pi(\phi_0)^2 f(\phi_1)\big)^{1/6} \qquad F(\phi_1) = \int_0^{\phi_1} (1 + f(\varphi))^{-1/2} d\varphi$$

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General classification

Only two classes of ultra-local spacelike, singularity scattering maps

- **Isotropic bounce $\mathbf{S}^{\text{iso}}_{\lambda,\varphi}$**
- **Non-isotropic bounce $\mathbf{S}^{\text{ani}}_{\Phi,c}$**

where now λ is a two-tensor, Φ a "canonical transformation", c a constant.

More conveniently stated as three laws, as follows.

- **First law: scaling of Kasner exponents**

  Our classification uncovers three universal laws obeyed by any ultra-local bounce. First, Kasner exponents are scaled as

  There exists a (dissipation) constant $\gamma \in \mathbb{R}$ such that

$$|g^+|^{1/2} \mathring{K}^+ = -\gamma \, |g^-|^{1/2} \mathring{K}^-_{\text{before}}$$

spatial metric g in synchronous gauge, volume factor $|g|^{1/2}$

traceless part $\mathring{K}$ of the extrinsic curvature (as a $(1,1)$ tensor)

- **First law: scaling of Kasner exponents**

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- **Third law: directional metric scaling**

  $$g^+ = \exp\big(\sigma_0 + \sigma_1 K + \sigma_2 K^2\big) g^-$$

  nonlinear scaling in each proper direction of K

  $\gamma = 0$: isotropic scattering, no restriction $\sigma_0, \sigma_1, \sigma_2$

  $\gamma \neq 0$: non-isotropic scattering, explicit formulas in terms of Φ, γ

References

- Joint with **Bruno Le Floch and Gabriele Veneziano**
  - *Universal scattering laws for quiescent bouncing cosmology*
    Physical Review D (2021)
  - *Cyclic spacetimes through singularity scattering maps.*
    The laws of quiescent bounces J. High Energy Physics (2022)
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    Class. Quantum Gravity (2022)
- Joint with **Bruno Le Floch**
  - *On the global evolution of self-gravitating matter. Nonlinear interactions in Gowdy symmetry*
    Arch. Rational Mech. Analysis (2019)
  - *Compensated compactness and corrector stress tensor for the Einstein equations in T2 symmetry* Portugaliae Math (2020)
  - *Scattering maps for interfaces in self-gravitating matter flows*
    Preprint (2023)

ArXiv:2005.11324, ArXiv:2106.09666, ArXiv:1912.12981, etc.

Some illustrations...

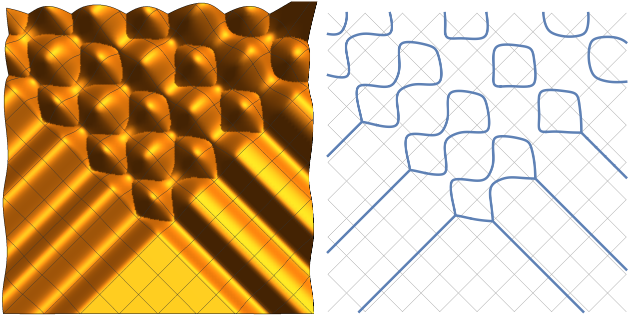

Figure: **Cyclic spacetime arising from colliding plane gravitational waves**

- Left: area A of plane-symmetry orbits = height of spacetime "bubbles"
- Right: singular locus $A = 0$ across which we apply the junction relation

$$(g^+, k^+, \phi_0^+, \phi_1^+) = (e^{2(k^- - 1/3)} g^-, k^-, \phi_0^-, \phi_1^- + \phi_0^-)$$

- For this example of junction, the global evolution problem is well-posed in a class of "cyclic spacetimes".

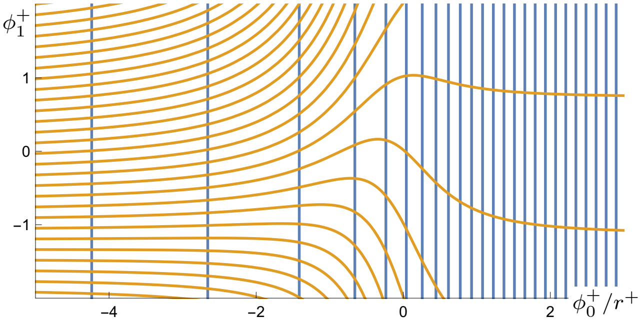

Figure: **Image of equally-spaced constant-**(ϕ_0^-/r^-) (vertical lines) **and constant-**ϕ_1^- (curved lines)

- under the matter map Φ of the Pre Big Bang scenario

$$\beta^+ = -\beta^-, \; u_+ = u_-$$

- It preserves $d(\phi_0^\pm/r^\pm)d\phi_1^\pm$ so each region has the same area.

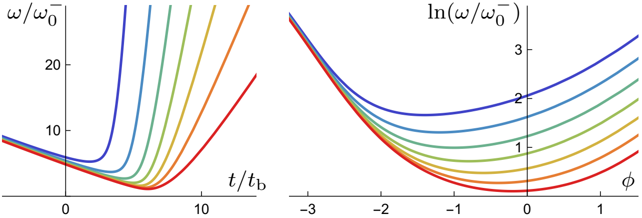

Figure: **Bianchi I symmetric modified matter bounces**

- Lagrangian $\mathcal{L} = \frac{1}{2}\dot\phi^2 - 2|\dot\phi|e^{-\phi^2/2}/t_\mathrm{b} + e^{-\phi^2}/t_\mathrm{b}^2$ for fixed t_0^-, ϕ_0^-, and ω_0^- (normalized to 1).
- Each color corresponds to one value of ϕ_1^-
- which affects the $t \to +\infty$ asymptotics for $\omega \simeq \omega_0^+(t - t_0^+)$ and $\phi \simeq \phi_0^+ \ln\omega + (\phi_1^+ - \phi_0^+ \ln\omega_0^+)$ manifest in the two plots.

Existence of initial data sets

"prescribed curvature problem"

- manifold $(\mathbf{M}, g, k)$ with finitely many asymptotic ends
- unknowns: Riemannian metric g and symmetric $(0,2)$-tensor field k

  extrinsic curvature in the dynamical picture
- matter content: scalar field $H_\star : \mathbf{M} \to \mathbb{R}_+$ vector field $M_\star$
- Einstein's **Hamiltonian and momentum constraints**

$$R_g + (\mathrm{Tr}_g k)^2 - |k|_g^2 = H_\star \qquad \mathrm{Div}_g\big(k - (\mathrm{Tr}_g k)g\big) = M_\star$$

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Notation

It is convenient to introduce the $(2,0)$-tensor h by $h := \big(k - \mathrm{Tr}_g(k)g\big)^{\sharp\sharp}$

$$\mathcal{H}(g,h) := R_g + \frac{1}{2}(\mathrm{Tr}_g h)^2 - |h|_g^2 \qquad \mathcal{M}(g,h) := \mathrm{Div}_g h$$

$$\mathcal{G}(g,h) := \big(\mathcal{H}, \mathcal{M}\big)(g,h) = (H_\star, M_\star)$$

In the dynamical picture, $\mathcal{G}(g,h)$ is a spacetime vector.

Vast and rich literature

- Conformal method Lichnerowicz (1960s), Choquet-Bruhat, Chrusciel, Corvino, Delay, Dilts, Galloway, Gicquaud, Holst, Isenberg, Maxwell, Mazzeo, Miao, Pollack, Schoen, etc.
- Variational method Corvino, Corvino-Schoen, Chrusciel-Delay, Carlotto-Schoen, etc.

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Major achievements

- existence of initial data, explicit constructions, physically relevant solutions
- general relativity, Riemannian geometry

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Major achievements

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- numerous classes of solutions: compact, various types of asymptotic ends
- including gluing techniques, combine two different solutions together

- A. Carlotto, *The general relativistic constraint equations,* Living Reviews in Relativity (2021).

Localization in initial data sets

Shielding gravity at infinity

- asymptotically Euclidean initial data sets
- phenomena of anti-gravity (or shielding) — Carlotto and Schoen; Chruściel and Delay
- solutions that are localized at infinity
  - The Positive Mass Theorem implies restrictions on gluing at infinity.
  - identically Euclidian near infinity except in a cone
- Other recent developments
  - S. Aretakis, S. Czimek, I. Rodnianski: characteristic gluing problem
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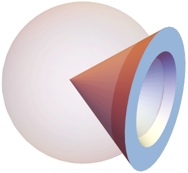
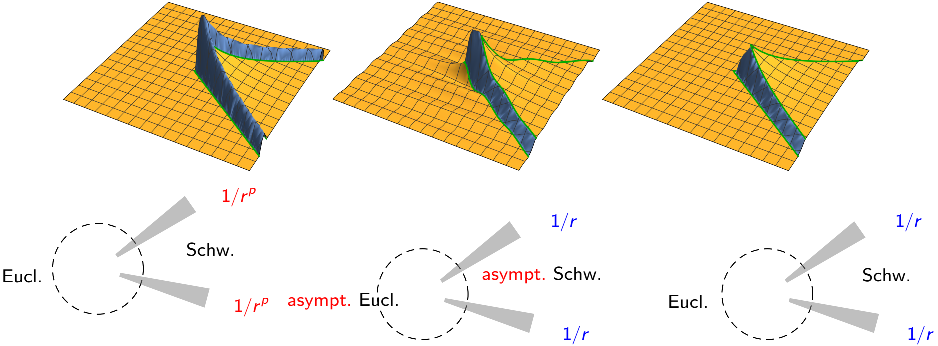
Localization with (super-)harmonic control

- Improve upon Carlotto-Schoen's theory
  - solutions with *sub-harmonic* control r^p with $p \in \left(\frac{n-2}{2}, n-2\right)$
  - conjecture: gluing should be possible at harmonic level
- Localization results with *harmonic* and *super-harmonic* control
  - PLF & The-Cang Nguyen, 2020: *The seed-to-solution method for the Einstein constraint equations*
  - Bruno Le Floch & PLF, 2023: *The* **localized** *seed-to-solution method for the construction of Einstein's initial data sets*





Carlotto, Schoen, Chrusciel, Delay (2017) | PLF-TC Nguyen (2020) | B. Le Floch-PLF (2023)

sub-harmonic localization | harmonic asymptotic localization | harmonic localization

5. LOCALIZATION AT SUPER-HARMONIC RATES

Theorem. The seed-to-solution parametrization **(PLF & T-C Nguyen)**
and a localized version by Bruno Le Floch & PLF

Given any seed data set $(\mathbf{M}, g_1, h_1)$ on a 3-manifold (with a single end, say):
a Riemannian metric g_1 and a symmetric two-tensor h_1
satisfying (suitable smallness conditions and)

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$$1/2 < p_M < +\infty$$

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- **Super-harmonic decay:** $p_M > 1$ $\qquad p = \min(p_G + 1, p_M, 2)$

$$g = g_1 + \frac{\tilde{m}}{r} + \mathcal{O}(r^{-p}) \qquad h = h_1 + \mathcal{O}(r^{-2})$$

Mass modulator $\tilde{m} = \tilde{m}(g_1, h_1) = \texttt{const.} \int_{\mathbf{M}} \mathcal{H}(g_1, h_1)\, dV_{g_1} + \mathcal{O}(\mathcal{G}(g_1, h_1)^2)$

Exact localization at sub-harmonic rates

Carlotto and Schoen

– Vacuum constraint Einstein equations

Decompose asymptotic infinity into three angular regions

- $\mathscr{C}_a$: cone with (possibly arbitrarily small) angle $a \in (0, 2\pi)$
- $\mathscr{C}^c_{a+\varepsilon}$: complement of the same cone with (slightly) larger angle $a + \varepsilon$
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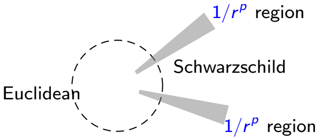
– *Sub-harmonic control in* $\mathscr{T}_{a,\varepsilon}$, *that is,* r^{-p} *with* $p \in (1/2, 1)$

– Question raised by Carlotto and Schoen

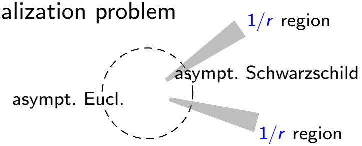
construct solutions (with prescribed asymptotic)
enjoying the $1/r$ harmonic decay in all angular directions



Asymptotic localization at super-harmonic rates

slightly *relax the localization condition*

- asymptotic at a super-harmonic rate to prescribed metrics
- physically as natural as the exact localization problem



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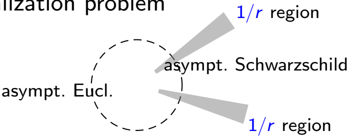

Theorem. The asymptotic localization problem (PLF–Nguyen)

- Vacuum Einstein equations on a manifold **M** with a single asymptotic end
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$$\mathscr{C}_a \cup \mathscr{C}^c_{a+\varepsilon} \cup \mathscr{T}_{a,\varepsilon} \subset \mathbb{R}^3$$

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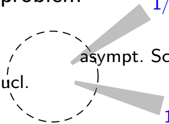

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By considering (for instance) the Euclidean metric g_{Eucl} and the Schwarzschild metric $g_{\mathrm{Sch}} = (1 + 2m_{\mathrm{Sch}}/r)\, g_{\mathrm{Eucl}}$ (with mass $m_{\mathrm{Sch}} > 0$),
there exists a solution to the vacuum Einstein equations $\mathcal{G}(g, h) = 0$:

$$g = g_{\mathrm{Eucl}} + \mathcal{O}(r^{-q}) \qquad \text{in } \mathscr{C}^c_{a+\varepsilon}$$

$$g = g_{\mathrm{Sch}} + \mathcal{O}(r^{-q}) \qquad \text{in } \mathscr{C}_a \qquad q \in (1,2)$$

$$g = g_{\mathrm{Eucl}} + \mathcal{O}(r^{-1}) \qquad \text{in } \mathscr{T}_{a,\varepsilon}$$

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`Only a SKETCH of our results`

$(\mathbf{M}, \Omega, g_0, h_0)$: cone-like, asymptotically Euclidian, reference set

(g_1, h_1): seed data set localized to Ω

- Strongly tame: $(p_G, q_G) \geqslant (\frac{n-2}{2}, \frac{n}{2})$ and $(p_M, q_M) > (n-2, n-1)$
- Strongly effective $(p_\star, q_\star) > (n-2, n-1)$

Theorem. Localized gluing beyond harmonic decay — Bruno LF & PLF, 2023

For each (p_G, q_G, p_M, q_M)-localized seed data set $(g_1, h_1, H_\star, M_\star)$ there exists a modulated seed data set

$$(\widetilde{g}_1, \widetilde{h}_1) = (g_1, h_1) + \text{sup}_{a=1,2,\ldots} \text{ localized harmonic terms at each end}$$

$$\|g - \widetilde{g}_1\|^{N,\alpha}_{g_0,\Omega,p_\star,P} + \|h - \widetilde{h}_1\|^{N,\alpha}_{g_0,\Omega,q_\star,P} \lesssim \mathcal{E}(g_1, h_1) + \sum_{a=1,2,\ldots} \left(|m^\star_a| + |P^\star_a|\right)$$

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$$\mathcal{E}(g_1, h_1) = \left\|\mathcal{H}(g_1, h_1) - H_\star\right\|^{N-2,\alpha}_{\Omega,g_0,p+2,P-2} + \left\|\mathcal{M}(g_1, h_1) - M_\star\right\|^{N-1,\alpha}_{\Omega,g_0,q+1,P-1}$$

the scalars $m_a^\star = m_a^\star(g_1, h_1, H_\star)$ and vectors $P_a^\star = P_a^\star(g_1, h_1, M_\star)$ being

$$m_a^\star := \frac{1}{16\pi} \int_{\mathbf{M}} \left(H_\star - \mathcal{H}(g_1, h_1)\right) \kappa_a \lambda^2 \, dV_{g_1}$$

$$P_a^\star := \frac{1}{8\pi} \int_{\mathbf{M}} \left(M_\star - \mathcal{M}(g_1, h_1)\right) \kappa_a \lambda^2 \, dV_{g_1}$$

κ_a : partition of unity for the asymptotic ends

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$$(\widetilde{g}_1, \widetilde{h}_1) = (g_1, h_1) + \sup_{a=1,2,\dots} \text{localized harmonic terms at each end}$$

$$\|g - \widetilde{g}_1\|^{N,\alpha}_{g_0,\Omega,p_\star,P} + \|h - \widetilde{h}_1\|^{N,\alpha}_{g_0,\Omega,q_\star,P} \lesssim \mathcal{E}(g_1, h_1) + \sum_{a=1,2,\dots} \left(|m^\star_a| + |P^\star_a|\right)$$

$$\mathcal{E}(g_1, h_1) = \left\|\mathcal{H}(g_1, h_1) - H_\star\right\|^{N-2,\alpha}_{\Omega,g_0,p+2,P-2} + \left\|\mathcal{M}(g_1, h_1) - M_\star\right\|^{N-1,\alpha}_{\Omega,g_0,q+1,P-1}$$

the scalars $m^\star_a = m^\star_a(g_1, h_1, H_\star)$ and vectors $P^\star_a = P^\star_a(g_1, h_1, M_\star)$ being

$$m^\star_a := \frac{1}{16\pi}\int_{\mathbf{M}} \left(H_\star - \mathcal{H}(g_1, h_1)\right) \kappa_a \lambda^2 \, dV_{g_1}$$

$$P^\star_a := \frac{1}{8\pi}\int_{\mathbf{M}} \left(M_\star - \mathcal{M}(g_1, h_1)\right) \kappa_a \lambda^2 \, dV_{g_1}$$

κ_a : partition of unity for the asymptotic ends

ADM energy-momentum $(\widetilde{m}_a, \widetilde{P}_a)$ of the localized harmonic contributions

$$\sup_{a=1,2,\dots} \left|\widetilde{m}_a - m^\star_a\right| + \left|\widetilde{P}_a - P^\star_a\right| \lesssim \mathcal{E}(g_1, h_1)$$

At each asymptotic end $\lambda \simeq$ distance to $\partial\Omega$

$$\lambda^P \left(\mathbf{r}^{n-2}\,|g - \widetilde{g}_1| + \mathbf{r}^{n-1}\left(|\partial(g - \widetilde{g}_1)| + |h - \widetilde{h}_1|\right) + \mathbf{r}^n\,|\partial(h - \widetilde{h}_1)|\right) \to 0$$