UNIVERSITY OF LYON-1 / UNIVERSITY OF JOHANNESBURG

QUASINORMAL MODE APPLICATIONS IN THE EPOCH OF GRAVITATIONAL-WAVE ASTRONOMY

Anna Chrysostomou
Aldo Deandrea & Alan Cornell

AC, AC, AD, É. Ligout, D. Tsimpis, Eur. Phys. J. C 83, 325 (2023)

PhD Day 2 | 27 April 2023

Outline

Einstein-Hilbert action:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, (R - 2\Lambda + \mathcal{L}_m)$$

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Einstein's grav. constant:	$\kappa = 8\pi G c^{-4}$
determinant:	$g = \det(g_{\mu\nu})$
Ricci scalar:	$R = g^{\mu\nu} R_{\mu\nu}$
cosmological constant:	$\Lambda = 3/L^2$
matter fields:	$\mathcal{L}_m = \dots$

Einstein-Hilbert action:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, (R - 2\Lambda + \mathcal{L}_m)$$

$$\Big\Downarrow \; \delta S = 0$$

Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

Einstein-Hilbert action:

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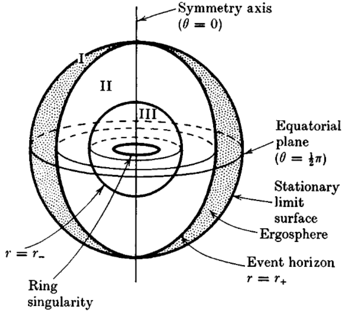
Einstein field equations for *flat space*, *in vacuum*:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \cancel{\Lambda g_{\mu\nu}} = \cancel{\kappa T_{\mu\nu}}$$



Hawking & Ells, *The Large Scale Structure of Space-time*

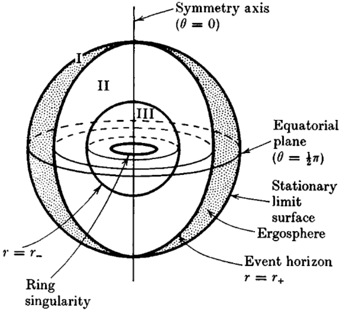
The "no-hair" conjecture

All stationary black hole solutions in GR can be completely characterised by three independent, externally observable, and classical parameters:

- *mass M,*
- *electric charge Q,*
- *angular momentum a.*

Stationary, neutral, spherically-symmetric black hole:

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$



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$$f(r) = 1 - \frac{2M}{r}$$

event horizon:	$r_H = 2M$
photon orbit:	$r_c = 3M$
length scale:	$M = Gm^{BH}c^{-2}$

Bekenstein

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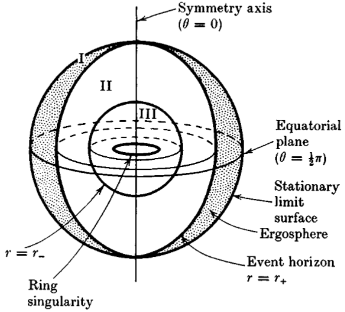

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$$f(r) = 1 - \frac{2M}{r}$$

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photon orbit: $r_c = 3M$

length scale: $M = Gm^{BH}c^{-2} = 1$

$(G = c = 1)$

Bekenstein

1957: Regge and Wheeler's grav. perturbations

The birth of black hole perturbation theory:

L REVIEW VOLUME 108, NUMBER 4 NOVEMBER

Stability of a Schwarzschild Singularity

TULLIO REGGE, *Istituto di Fisica della Università di Torino, Torino, Italy*

AND

JOHN A. WHEELER, *Palmer Physical Laboratory, Princeton University, Princeton, New Jersey*

(Received July 15, 1957)

It is shown that a Schwarzschild singularity, spherically symmetrical and endowed with mass, will undergo small vibrations about the spherical form and will therefore remain stable if subjected to a small nonspherical perturbation.

$$\begin{aligned} g_{\mu\nu}dx^{\mu}dx^{\nu} &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ g_{\mu\nu} \to g'_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu} \qquad (g_{\mu\nu} \gg h_{\mu\nu}) \end{aligned}$$

Quasinormal mode and frequency

$$\Psi(x^\mu) = \sum_{n=0}^{\infty} \sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{r} e^{-i\omega t} Y_{\ell m}(\theta,\phi) \;, \qquad \omega_{sn\ell} = \omega_R - i\omega_I$$

- ⋆ $\mathbb{R}e\{\omega\}$ = physical oscillation frequency
- ⋆ $\mathbb{I}m\{\omega\}$ = damping → dissipative, "quasi"

Quasinormal mode and frequency

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- ⋆ s: spin of perturbing field
- ⋆ m: azimuthal number for spherical harmonic decomposition in θ_i
- ⋆ ℓ: angular/multipolar number for spherical harmonic decomposition in θ, ϕ
- ⋆ n: overtone number labels QNMs by a monotonically increasing $|\mathbb{I}m\{\omega\}|$

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Due to symmetries, only 2 ODEs needed:

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Due to symmetries, only 2 ODEs needed:

⋆ *Angular behaviour encapsulated by spheroidal harmonics:*

$$\nabla^2 Y^s_{m\ell}(\theta,\phi) = -\frac{\ell(\ell+1)}{r^2} Y^s_{m\ell}(\theta,\phi)$$

⋆ *s.t. QNM computations depend on radial behaviour*

Black hole wave equation:

$$\frac{d^2}{dr_*^2}\varphi(r_*) + \left[\omega^2 - V(r)\right]\varphi(r_*) = 0\ , \qquad \frac{dr}{dr_*} = f(r)$$

$\rightarrow$ reduces to a second-order ODE in r

Black hole wave equation:

$$\frac{d^2}{dr_*^2}\varphi(r_*) + \left[\omega^2 - V(r)\right]\varphi(r_*) = 0 \,, \qquad \frac{dr}{dr_*} = f(r)$$

→ subjected to **QNM boundary conditions**

purely ingoing:	$\varphi(r_*) \sim e^{-i\omega(t+r_*)}$	$r_* \to -\infty \ (r \to r_H)$
purely outgoing:	$\varphi(r_*) \sim e^{-i\omega(t-r_*)}$	$r_* \to +\infty \ (r \to +\infty)$

Waves escape domain of study at the boundaries ⇒ dissipative

Dolan & Ottewill

Class. Quant. Grav. 26 (2009), Phys. Rev. D 84 (2011)

A new computation method for BH QNMs through a novel ansatz based on **null geodesics** + expansion of the QNF in inverse powers of $L = \ell + 1/2$

$$\Phi(r) = e^{i\omega z(r_*)} v(r) \ , \quad \omega = \sum_{k=-1}^{\infty} \omega_k L^{-k}$$

- ⋆ iterative procedure best performed in the eikonal limit
- ⋆ more efficient means of calculating detectable QNMs?

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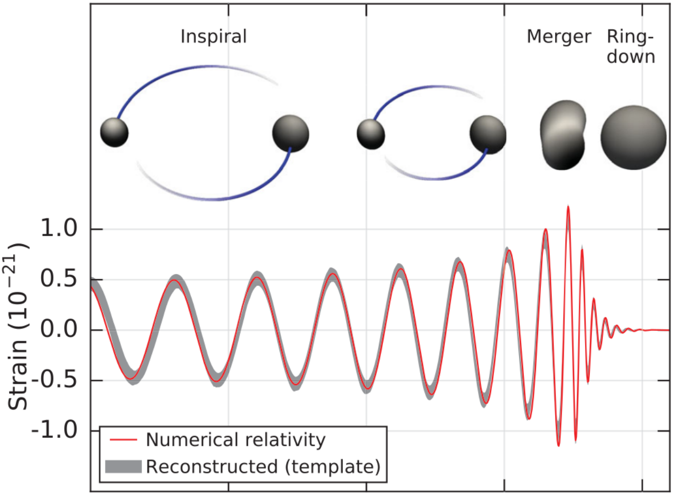
- ⋆ more efficient means of calculating detectable QNMs?
- ⋆ can extend to compute QNM wavefunctions [♦ rare find!]

Quasinormal mode: "ringdown"



B. P. Abbott *et al.*, PRL **116**, 061102 (2016)

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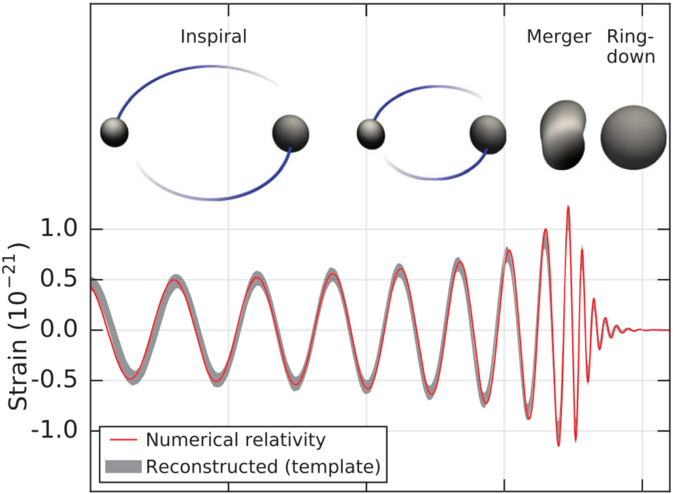

Ringdown: *a superposition of QNMs*

Higher harmonics & overtones

"the *fundamental* $(\ell, m, n) = (2, 2, 0)$ *mode* dominates ringdown"

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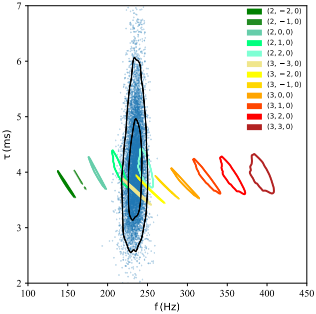

a multimodal analysis of the GW150914 data using PyRing, *see Carullo et al.*

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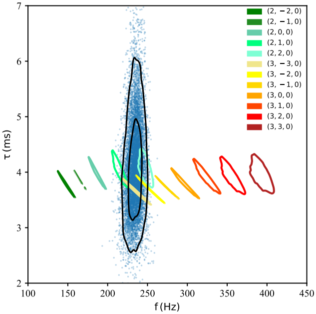

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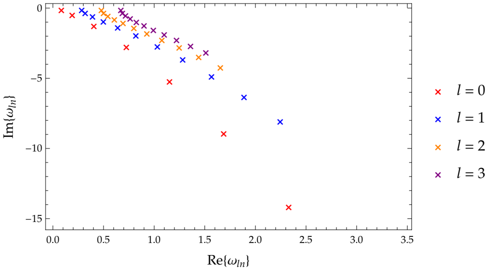

first 10 overtones for $s = 0$ QNFs of various ℓ

Controversial: a post-peak ringdown starttime?

PHYSICAL REVIEW X **9**, 041060 (2019)

Black Hole Ringdown: The Importance of Overtones

Matthew Giesler,[1,*] Maximiliano Isi,[2,3] Mark A. Scheel,[1] and Saul A. Teukolsky[1,4]

...By modelling the ringdown of accurate numerical relativity simulations, we find, in agreement with previous findings, that the ***fundamental mode alone is insufficient*** *to recover the true underlying mass and spin, unless the analysis is started very late in the ringdown. Including higher overtones associated with this $\ell = m = 2$ harmonic resolves this...*

Quasinormal excitation factors:

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In the GW context, strain is a function of the excitation coefficient
Oshita, Phys. Rev. D 104 (2021)

$$h_+ + ih_\times = \sum_{\ell mn} \mathcal{C}_{\ell n}\, Y_{\ell m}(\theta,\phi)\, \frac{\psi_{\ell n}}{r}\, e^{-i\omega_{\ell n} t}\,,$$

which is a product of a source factor (initial data) & an independent quasinormal excitation factor.

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Formally, we model the QNM contribution to the black hole response through a Green's function analysis. This requires explicit expressions for the wavefunction, evaluated at the QNF.

S. Detweiler, Proc. R. Soc. A 352 (1977)
E. W. Leaver, Phys. Rev. D 34 (1986)
N. Andersson, Phys. Rev. D 51 (1995)

Quasinormal excitation factor

at the horizon:

$$\psi_{r_H} \sim \begin{cases} e^{+i\omega r_\star} & r_\star \to -\infty \\ A^-_{\ell\omega} e^{-i\omega r_\star} + A^+_{\ell\omega} e^{+i\omega r_\star} & r_\star \to +\infty \end{cases}$$

at spatial infinity:

$$\psi_\infty \sim e^{+i\omega r_\star} \qquad r_\star \to +\infty$$

Quasinormal excitation factor

We require two linearly-independent solutions,

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$$W(\ell,\omega) = \psi_{r_H} \frac{d\psi_\infty}{dr_\star} - \psi_\infty \frac{d\psi_{r_H}}{dr_\star} = 2i\omega A^{-}_{\ell\omega}$$

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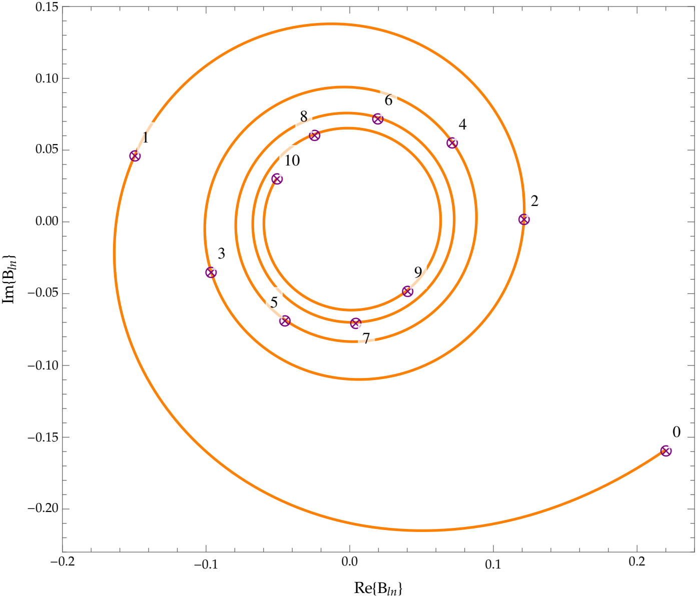
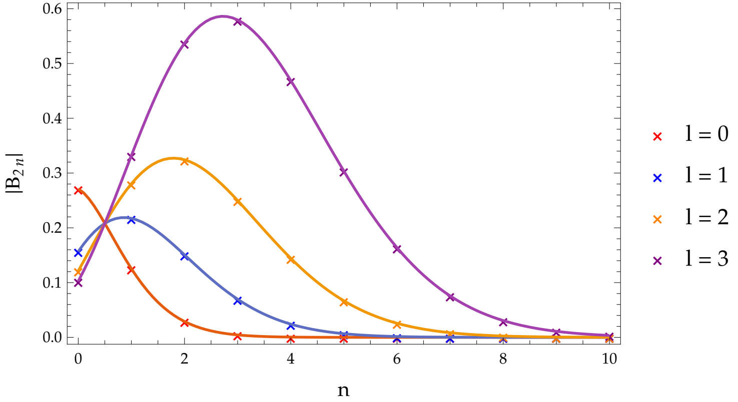
$$\boxed{\mathcal{B}_{\ell mn} \equiv \left[\frac{A^+_{\ell\omega}}{2\omega} \left(\frac{\partial A^-_{\ell\omega}}{\partial\omega} \right)^{-1} \right]_{\omega=\omega_{\ell n}}}$$



Schwarzschild QNEFs for $n = 0$ and increasing ℓ (labelled)

Quasinormal excitation factor: Schwarzschild case



Larger excitation for higher harmonics/overtones.

Note that $n = \ell = 0$ *&* $n = \ell = 2$ *are similarly excited but* $n = \ell = 1$ *is the least.*

How can we exploit available GW searches to investigate extra-dimensional scenarios?

The extra-dimensional metric

$$ds_D^2 = g_{\mu\nu}^{BH}(\mathbf{x})dx^\mu dx^\nu + g_{ij}^{D-4}(\mathbf{y})dy^i dy^j$$

$$\Psi^s_{n\ell m}(t, r, \theta, \phi, \mathbf{y}) = \sum_{n=0}^{\infty}\sum_{\ell,m} \frac{\psi_{sn\ell}(r)}{e^{i\omega t} r} Y^s_{m\ell}(\theta, \phi) Z(\mathbf{y})$$

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Laplacian of a product space is the sum of its parts

$$\left(\nabla^2_{BH} + \nabla^2_{D-4}\right)\sum \Phi(\mathbf{x})Z_k(\mathbf{y}) = 0\ ,$$

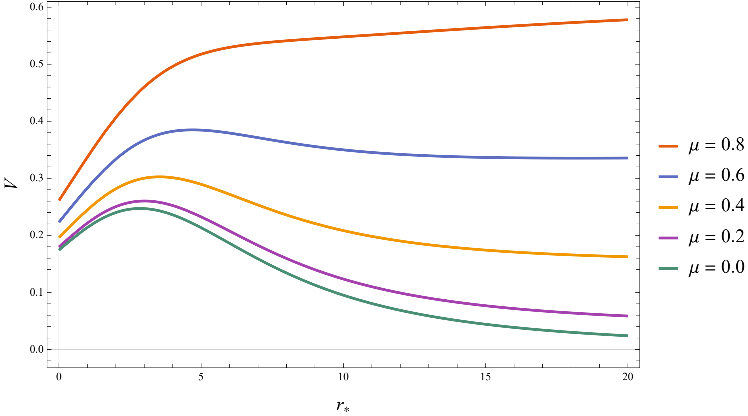
$$\nabla^2_{D-4}Z_k(\mathbf{y}) = -\mu_k^2 Z_k(\mathbf{y})$$

The wavelike equation

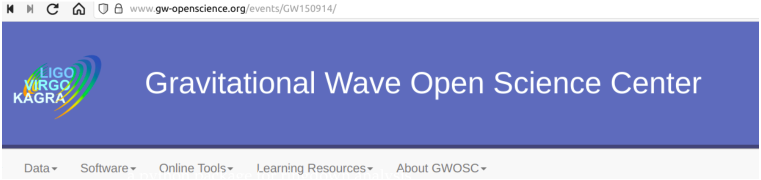
$$\frac{d^2\psi}{dr_*^2} + \left(\omega^2 - V(r)\right)\psi = 0$$

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3} + \mu^2\right)$$

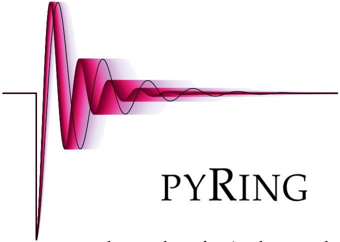
The fundamental mode: $n = 0$, $\ell = 2$



Agnostic bounds from ringdown GWs



LVK's *Tests of GR with binary black holes from* GWTC-1, GWTC-2, GWTC-3

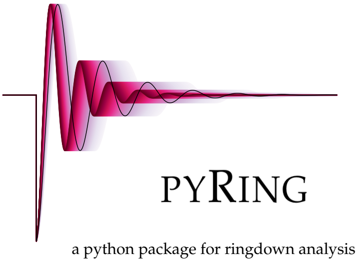


a python package for ringdown analysis

$$\delta\omega = \omega^{GR}(1+\delta\omega)$$

$$\delta\tau = \tau^{GR}(1+\delta\tau)$$

LVK's *Tests of GR with binary black holes from GWTC-1, GWTC-2, GWTC-3*



$$\delta\omega_{O3} = 0.02^{+0.07}_{-0.07}$$

$$\delta\tau_{O3} = 0.13^{+0.21}_{-0.22}$$

The fundamental mode: $n = 0$, $\ell = 2$

μ	$\mathbb{R}e\{\omega\}$	$\mathbb{I}m\{\omega\}$	$\delta\omega$	$\delta\tau$
0.0	0.4836	−0.0968	0.0000	0.0000
0.1	0.4868	−0.0957	0.0065	0.0113
0.2	0.4963	−0.0924	0.0262	0.0473
0.3	0.5124	−0.0868	0.0594	0.1149
0.4	0.5352	−0.0787	0.1066	0.2302
0.5	0.5653	−0.0676	0.1687	0.4306
0.6	0.6032	−0.0532	0.2472	0.8206
0.7	0.6500	−0.0343	0.3440	1.8181

Result: a detectability bound on the QNM probe

$$\boxed{\mu \lesssim 0.3681}$$

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From the dimensionless parameter $M\mu$,

$$\begin{aligned} M\mu &= \frac{G m^{\mathrm{BH}} m}{\hbar c} \\ \Rightarrow \quad & m = \frac{1}{m^{\mathrm{BH}}} \frac{\hbar c}{G} M\mu \\ & m \sim 10^{-\chi} 10^{-46} \mathrm{kg} \sim 10^{-(\chi+10)} \mathrm{eV/c^2} \end{aligned}$$

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$$\Rightarrow \quad m = \frac{1}{m^{\mathrm{BH}}} \frac{\hbar c}{G} M\mu$$

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$$m_{KK} \lesssim 10^{-13}\ \mathrm{eV/c^2}$$

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$\sim$ *light scalar hypotheses*

cf. Cajohare's *Axion Limits*, PDG 2022

Conclusions

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- ⋆ Detecting modifications to GR is considered to be beyond the sensitivity of modern detectors (still model-agnostic, first-order, etc.)

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- ⋆ Detecting modifications to GR is considered to be beyond the sensitivity of modern detectors (still model-agnostic, first-order, etc.)

- ⋆ Here, we have determined a QNM detectability bound on extra dimensions

  ↪ roadmap for new model-agnostic extra dimensions search

Thank you

Thank you

And a warm thanks to

science & technology
Department: Science and Technology
REPUBLIC OF SOUTH AFRICA

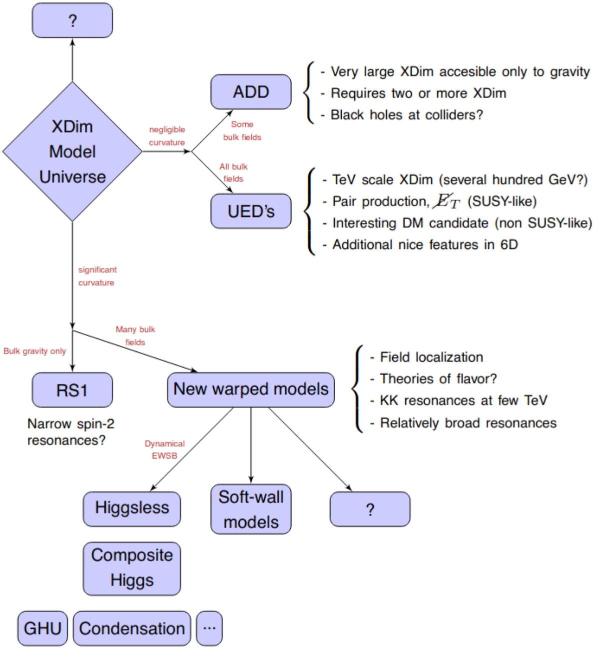
NRF
National Research Foundation

iThemba LABS
Laboratory for Accelerator Based Sciences

CERN

CAMPUS FRANCE

Backup slides

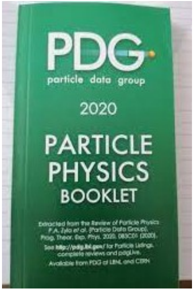


Taken from E. Pontón's 2011 TASI lectures, "TeV scale EDs"

Constraining EDs

- Limits on R from Deviations in Gravitational Force Law
- Limits on R from On-Shell Production of Gravitons: $\delta = 2$
- Mass Limits on M_{TT}
- Limits on $1/R = M_c$
- Limits on Kaluza-Klein Gravitons in Warped Extra Dimensions
- Limits on Kaluza-Klein Gluons in Warped Extra Dimensions
- Black Hole Production Limits
  – Semiclassical Black Holes
  – Quantum Black Holes

ATLAS, CMS, DELPHI, ALEPH, CDF, D0, OPAL, etc.



Negatively-curved EDs: a BSM landscape of untapped potential?

Phenomenological implications:

- natural resolution to the hierarchy problem

  $\rightarrow$ volume grows exponentially with ℓ_G/ℓ_c

  $\rightarrow$ RSI-like KK mass spectrum w/o light KK modes

- zero modes of Dirac operator emerges w/o gauge breaking
- enables homogeneity & flatness of observed universe

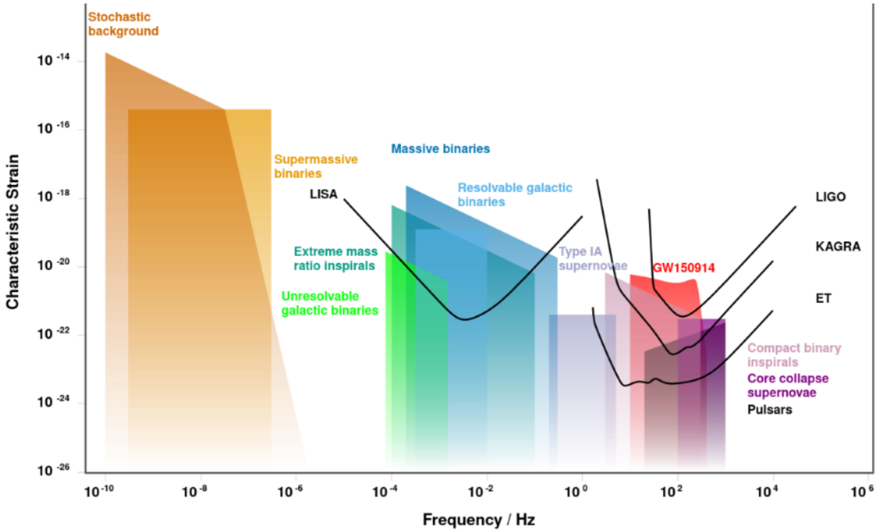
Explore beyond the CMB and CνB...

- Phase transitions: QCD ($\sim$ 100 MeV), EW ($\sim$100 GeV)

  $\hookrightarrow$ Baryogenesis + baryon asymmetry, EWSB (BSM!)
- Inflation ($\leq 10^{16}$ GeV)
- Exotic: cosmic strings, primordial black holes, Planck scale
- GR violation: > 2 polarisation states, modified dispersion relation, superluminal propagation, etc.

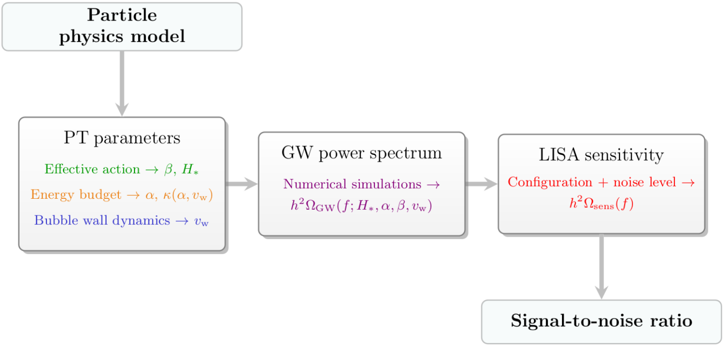
What's next?

Using established techniques to probe the GW BSM landscape...



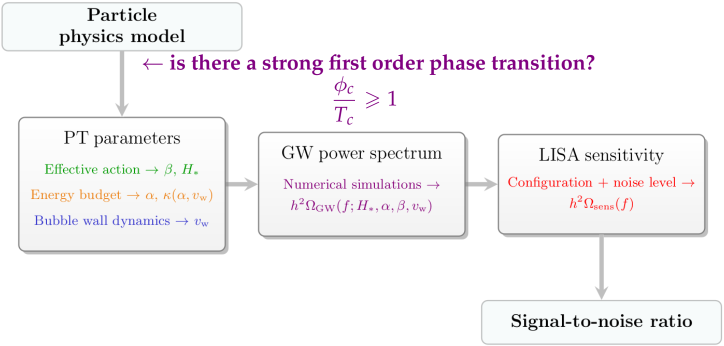
C. Moore, R. Cole, & C. Berry's GWplotter

Algorithm for finding GWs from PTs

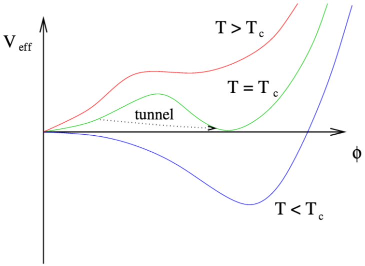


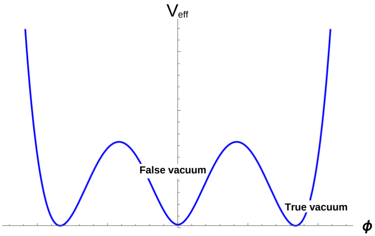
C. Caprini *et al.* JCAP 03 (2020) 024.

Algorithm for finding GWs from PTs



To determine the first order phase transition

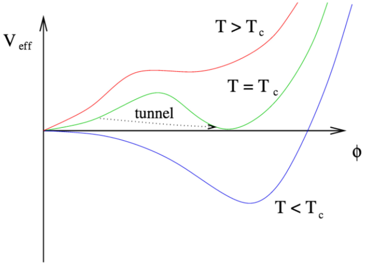


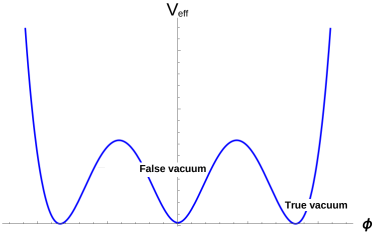


G. White, *A Pedagogical Intro to Baryogenesis*; W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

To determine the first order phase transition

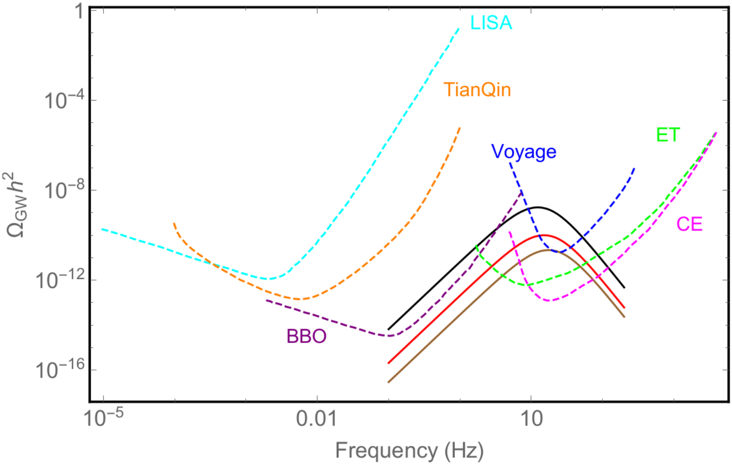
$$V_{eff} \approx V_{tree} + V_{1loop} + V_T$$





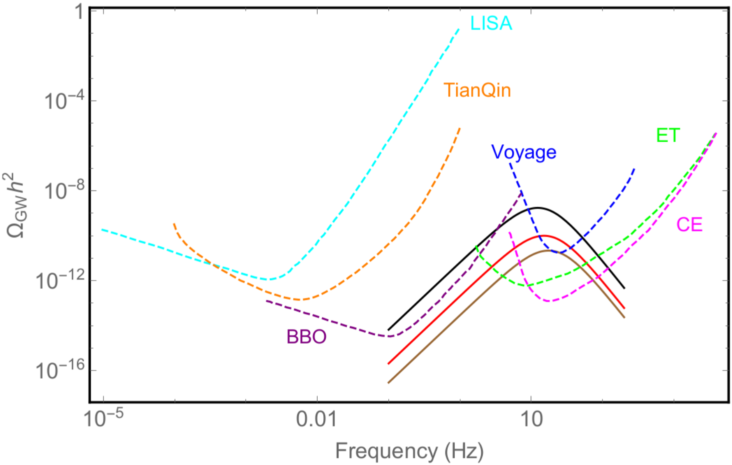
G. White, *A Pedagogical Intro to Baryogenesis*; W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

Sensitivity to new signals



W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

Stay tuned!

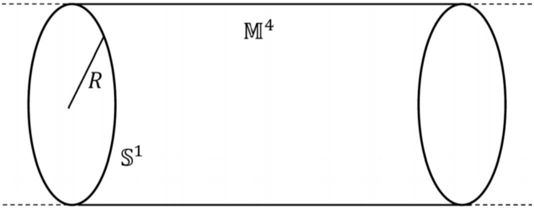


W.C. Huang, F. Sannino, and Z.W. Wang, Phys. Rev. D 102 (2020) 095025

Compactification: infinite $(3+1)$ dims; finite x_5
periodic BCs: $x_5 \to x_5 + 2\pi R$

KK tower of states:

$$\Phi(x^\mu, x^5) = \sum_{n=0}^{\infty} \Phi^{(n)}(x^\mu)\, e^{inx^5/R}\ ,\ m_n = \sqrt{m_0^2 + \left(\frac{n}{R}\right)^2}$$



In the Kaluza-Klein 5D universe

$$V_{eff} \approx V_{tree} + V_{1loop} + V_T$$

$$V_{eff} \approx V_{tree} + V_{1loop} + V_T$$

For $\beta = 1/T,\ L_5 = 2\pi R$:

$$\begin{aligned} V_T &= -\frac{3}{4\pi^2}\zeta(5)\frac{1}{L_5^4} \\ &\quad -\frac{3}{4\pi^2}\zeta(5)\frac{L_5}{\beta^2} - \frac{\Gamma(5/2)}{\pi^{5/2}L_5^4}\, 2\sum_{m,n=1}^{\infty}\left[\left(\frac{\beta m}{L_5}\right)^2 + n^2\right]^{-5/2} \end{aligned}$$

$$V_T \sim -\frac{3}{4\pi^2}\zeta(5)\frac{1}{L_5^4} \qquad L_5 \ll \beta$$

$$V_T \sim -\frac{3}{4\pi^2}\zeta(5)\frac{1}{\beta^5} \qquad L_5 \gg \beta$$

Dolan & Ottewill (2009)

A new computation method for BH QNMs through a novel ansatz based on **null geodesics** + expansion of the QNF in inverse powers of $L = \ell + 1/2$

$$\Phi(r) = e^{i\omega z(x)} v(r) \ , \quad \omega = \sum_{k=-1}^{\infty} \omega_k L^{-k}$$

We explore the method for Schwarzschild, RN, and SdS in 4D:

- more efficient means of calculating detectable BH QNMs?
- explore interplay of θ, λ in large-ℓ limit

The DO multipolar expansion method

Components of the ansatz

$$v(r) = \exp\left\{\sum_{k=0}^{\infty} S_k(r) L^{-k}\right\}, \quad z(x) = \int^x \rho(r)dx = \int^x b_c k_c(r) dx$$

$$k_c(r)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2}$$

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$$r_c = \frac{2f(r)}{\partial_r f(r)}\bigg|_{r=r_c}, \quad b_c = \sqrt{\frac{r^2}{f(r)}}\,\bigg|_{r=r_c}, \quad k_c(r)^2 = \frac{1}{b^2} - \frac{f(r)}{r^2}$$

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We generalise the consequent ODE

$$f(r)\frac{d}{dr}\left(f(r)\frac{dv}{dr}\right) + 2i\omega\rho(r)\frac{dv}{dr} + \left[i\omega f(r)\frac{d\rho}{dr} + \left(1-\rho(r)^2\right)\omega^2 - V(r)\right] v(r) = 0$$

We solve iteratively for ω_k and $S'_k(r)$ and sub into ω_k

QNF expansions for the Schwarzschild BH

$$r_c = 3\ ,\ b_c = \sqrt{27} \quad \Rightarrow \quad \rho(r) = \left(1 - \frac{3}{r}\right)\sqrt{1 + \frac{6}{r}}$$

$$\begin{aligned}
\omega(L,\mu) = &+\frac{1}{3}L - \frac{i}{6}L^0 + \left[\frac{3\mu^2}{2} + \frac{7}{648}\right]L^{-1} \\
&+ \left[\frac{5i\mu^2}{4} - \frac{137i}{23328}\right]L^{-2} + \left[\frac{9\mu^4}{8} - \frac{379\mu^2}{432} + \frac{2615}{3779136}\right]L^{-3} \\
&+ \left[\frac{27i\mu^4}{16} - \frac{2677i\mu^2}{5184} + \frac{590983i}{1088391168}\right]L^{-4} \\
&+ \left[\frac{63\mu^6}{16} - \frac{427\mu^4}{576} + \frac{362587\mu^2}{1259712} - \frac{42573661}{117546246144}\right]L^{-5} \\
&+ \left[\frac{333i\mu^6}{32} + \frac{6563i\mu^4}{6912} + \frac{100404965i\mu^2}{725594112} + \frac{11084613257i}{25389989167104}\right]L^{-6}\ .
\end{aligned}$$

The fundamental mode: $n = 0$, $\ell = 2$

μ	ω (WKB)	ω (PT)	ω (DO)
0.0	$0.4836 - 0.0968i$	$0.4874 - 0.0979i$	$0.4836 - 0.0968i$
0.1	$0.4868 - 0.0957i$	$0.4909 - 0.0968i$	$0.4868 - 0.0957i$
0.2	$0.4963 - 0.0924i$	$0.5015 - 0.0936i$	$0.4963 - 0.0924i$
0.3	$0.5123 - 0.0868i$	$0.5192 - 0.0881i$	$0.5124 - 0.0868i$
0.4	$0.5351 - 0.0787i$	$0.5443 - 0.0800i$	$0.5352 - 0.0787i$
0.5	$0.5649 - 0.0676i$	$0.5770 - 0.0690i$	$0.5653 - 0.0676i$
0.6	$0.6022 - 0.0528i$	$0.6181 - 0.0541i$	$0.6032 - 0.0532i$
0.7	$0.1396 + 0.2763i$	$0.6695 - 0.0312i$	$0.6500 - 0.0343i$

In agreement with massive scalar QNFs of S. Dolan, Phys. Rev. D 76 (2007) 084001

Suppose we place a 4D Schwarzschild black hole within a 7D spacetime, perturbed by a 7D scalar test field of mass μ:

$$\text{KG:}\quad \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Psi\right)-\mu^2\Psi=0\ ,$$

$$g_{\mu\nu}dx^\mu dx^\nu = g_{ab}(x)dx^a dx^b + g_{ij}(y)dx^i dx^j\ ,$$

$$g_{\mu\nu}=\begin{bmatrix} -f(r) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f(r)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r_2^2+r_3^2N^2y_1^2 & r_3^2Ny_1 \\ 0 & 0 & 0 & 0 & 0 & r_3^2Ny_1 & r_3^2 \end{bmatrix},$$

where $f(r)=1-2M/r$

QNMs: Deriving the radial equation

Variable-separable QNM solution:

$$\Psi^s_{n\ell m\mu}(t,r,\theta,\phi,y_1,y_2,y_3) = \sum_{n=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} R^s_{n\ell\mu}(r)Y^s_{m\ell}(\theta,\phi)Z_\mu(y_1,y_2,y_3)e^{i\omega t} .$$

Laplacian of a product space is the sum of its parts

$$\left(\nabla^2_{BH} + \nabla^2_{nil}\right)\sum \Phi(x)Z_k(y) = 0 ,$$

- $\nabla^2 Y^s_{m\ell}(\theta,\phi) = \frac{-\ell(\ell+1)}{r^2}Y^s_{m\ell}(\theta,\phi)$
- $\nabla^2 Z_k(y) = -\mu_k^2 Z_k(y)$

$$\mu^2_{k,j,m} = \frac{4\pi^2 k^2}{(r_3)^2}\left[1 + \frac{(2m+1)r_3}{2\pi|k|}|\mathtt{f}|\right]$$

Gravitational perturbations

Table I. Stabilities of generalised static black holes. In this table, "d" represents the spacetime dimension, $n+2$. The results for tensor perturbations apply only for maximally symmetric black holes, while those for vector and scalar perturbations are valid for black holes with generic Einstein horizons, except in the case with $K=1, Q=0, \lambda>0$ and $d=6$.

		Tensor		Vector		Scalar	
		$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$	$Q=0$	$Q\neq 0$
$K=1$	$\lambda=0$	OK	OK	OK	OK	OK	$d=4,5$ OK $d\geq 6$?
	$\lambda>0$	OK	OK	OK	OK	$d\leq 6$ OK $d\geq 7$?	$d=4,5$ OK $d\geq 6$?
	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?
$K=0$	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?
$K=-1$	$\lambda<0$	OK	OK	OK	OK	$d=4$ OK $d\geq 5$?	$d=4$ OK $d\geq 5$?

$$\mathcal{R}_{ED} = (d-3)K\gamma_{ij}$$