Tunneling effect and istantons solutions

Student : Lorenzo Cane Advisor: Prof. Marco Billo'

UNIVERSITÀ DEGLI STUDI DI TORINO
Corso di Laurea in Fisica

19 Luglio 2023

Index

I. Introduction
- Symmetries and tunneling effect

II. Path integrals

III. Particle in a periodoc potential (PPP)
- instantonic solution
- Expansion around the instanton solution
- Instantons dilute gas
- Kernels and hyperbolic Bessel functions

IV. Instanton in QFT

Symmetries and tunneling effect

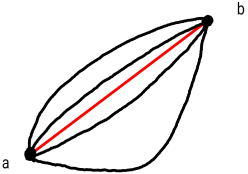
- Properties of a quantum system with finite degrees of freedom:
  - The ground state of a quantum system can never be degenerate (unless there is spin degeneracy).
  - The ground state of a quantum system has the same symmetry properties as the system's Hamiltonian.
- They appear to be falsified by systems with separated potential minima.

  $\rightarrow$ Finite-action solutions that connect the different minima are needed.

  $\rightarrow$ Degeneracy breaking and *tunneling effect*.

Path integrals
Kernel

- The formalism of path integrals leads to a Lagrangian formulation of quantum mechanics. In this formalism, the propagator takes the form



$$K(q_b, t_b; q_a, t_a) = K(q_b, q_a; \beta) = \int \mathcal{D}[q(t)]\, e^{i/\hbar\, S[q(t)]}$$

- Some Kernel properties:

$$\psi(q_b; t_b) = \int K(q_b, t_b; q_a, t_a)\, \psi(q_a; t_a)\, dq_a$$

$$K(b, a) = \int_{q_N} \dots \int_{q_2} \int_{q_1} K(b, N) \dots K(1, a)\, dq_1\, dq_2 \dots dq_N$$

- We will work in a 4-dimensional Euclidean space, reached through a *Wick rotation.*:

$$\mathbb{M}^{1,3} \to \mathbb{R}^4 \qquad t \to -i\tau$$

- Where we define:

$$S[q(t)] \to iS_E[q(\tau)] = \int d\tau \left[\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) \right] = \int d\tau \ \mathcal{L}_E$$

$$K(q_b, q_a; \beta) \to K_E(q_b, q_a; \beta)$$

- So the euclidean kernel is such that:

$$K_E(q_b, q_a; \beta) \overset{\beta \to \infty}{\simeq} \psi_0^*(q_b)\psi_0(q_a)e^{-\beta E_0/\hbar} + O(e^{-\beta \Delta E/\hbar})$$

- Every path can be write as : $q(\tau) = q_{cl}(\tau) + \eta(\tau)$.
  the (Euclidean) action to the second order around the minimum:

$$S_E[q(\tau)] = S_{cl} + \frac{1}{2}\int d\tau\ \eta(\tau)\hat{F}(q_{cl})\eta(\tau) + O\left(\eta(\tau)^3\right)$$

- And writng $\eta(\tau)$ with autofunction of $\hat{F}$: $\eta(\tau) = \sum_{n=0}^{\infty} C_n\varphi_n(\tau)$

$$K_E(q_b, q_a; \beta) = e^{-S_{cl}/\hbar}\int dC_0 \prod_{n=1}^{\infty}\int dC_n \,\mathrm{Det}\left(\underbrace{-\frac{d^2}{d\tau^2} + \frac{d^2V[q]}{d\tau^2}}_{\hat{F}}\right)^{-1/2}$$

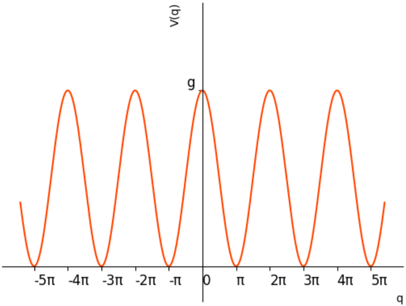
Particle in a periodoc potential (PPP)

- We study a potential that satisfies the condition

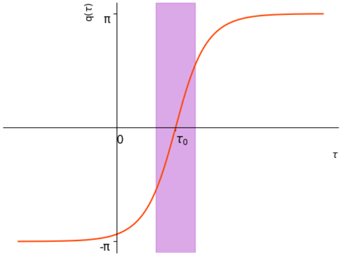
$$V(q) = V(q + 2\pi) \quad \forall q$$

$$\downarrow$$

$$V(q) = g^2(1 + \cos q)$$



- We search for the classical solution q_{cl} that interpolates between two consecutive minima, which is called an *instanton*.



- Energy conservation

$$\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 - V(q) = 0$$

$$\frac{dq}{g\sqrt{2(1+\cos q)}} = d\tau \rightsquigarrow \frac{dq}{\cos\frac{q}{2}} = 2g d\tau$$

- We choose an arbitrary timeτ_0 in which $q_{cl}(\tau_0) = 0$

$$q(\tau) = \pm 2 \arcsin \left\{ \tanh[g(\tau - \tau_0)] \right\} \ \textit{(Anti-)instanton}$$

Particle in a periodic potential (PPP)

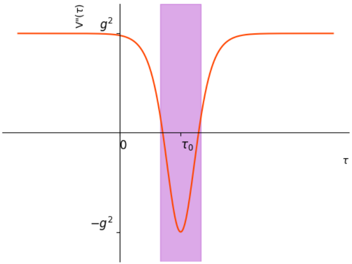
Instantonic solution

- The speed of such a transition can be evaluated:

$$V''(q_{cl}) = -g^2 + 2g^2 \tanh^2(g\Delta\tau)$$

- From this we find out the second-order expansion of the potential around a $q = \pi$ $(V(\pi) = 0)$

$$V(q_{cl}) \simeq \frac{1}{2} g^2 (q - \pi)^2$$



- Energy conservation again :

$$\frac{dq_{cl}}{d\tau} \simeq \sqrt{g^2(q_{cl} - \pi)^2} \Rightarrow \ln \frac{(q_{cl}(\tau) - \pi)}{(q_{cl}(\tau_0) - \pi)} = g(\tau - \tau_0)$$

- Consider$\Delta q(\tau) = q_{cl}(\tau) - \pi$:

$$\Delta q(\tau) = \Delta q(\tau_0)\ e^{-g\Delta\tau}$$

- The core typical dimension is $\Delta\tau = g^{-1}$
- Such a time interval is negligible for $\beta \to \infty$

Particle in a periodic potential (PPP)

Expansion around the instanton solution

- The classical solution alone does not reproduce the known characteristics of the system.
- As mentioned before, the saddle-point approximation is used:

$$K_E(-\pi,\pi;\beta) = e^{-S_{cl}/\hbar} \int dC_0 \int dC_n \,\mathrm{Det}\left(-\frac{d^2}{dt^2} + \frac{d^2 V(q)}{d\tau^2}\right)^{-1/2}$$

- The integral in C_0 is divergent, this is related to the arbitrariness of τ_0:

$$K_E(-\pi,\pi;\beta) = e^{-S_{cl}/\hbar}\sqrt{S_{cl}} \int d\tau_0 \int dC_n \,\mathrm{Det}'\left(-\frac{d^2}{d\tau^2} + \frac{d^2 V(q)}{d\tau^2}\right)^{-1/2}$$

- Exploiting the similarity between the studied potential and the potential of a harmonic oscillator:

$$\mathcal{R}^{-2} = \frac{\mathrm{Det}'\left(-\frac{d^2}{d\tau^2} + \frac{d^2 V[q]}{d\tau^2}\right)}{\mathrm{Det}'\left(-\frac{d^2}{d\tau^2} + \omega^2\right)}$$

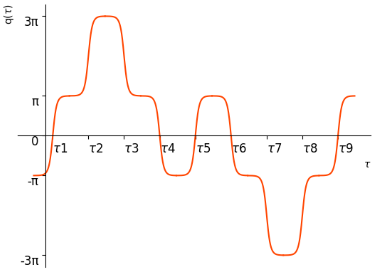
- We obtain the kernel:

$$K_E(-\pi, \pi; \beta) = e^{-S_{cl}/\hbar} \mathcal{R} \beta \sqrt{S_{cl}} \sqrt{\frac{\omega}{\pi \hbar}}\, e^{-\beta\omega/2}$$

Particle in a periodic potential (PPP)

Instantons dilute gas

- Consider all possible series of n_1 instantons and n_2 widely separated anti-instantons (with n_1 and n_2 being arbitrary).
- To maintain the correct boundary conditions, we must require $n_1 - n_2 = 1$
- There is no constraint on the order of the (anti-)instantons.



Particle in a periodic potential (PPP)

Instantons dilute gas

- Exploiting the properties of kernels and exponentials:

$$K_E(-\pi,\pi;\beta) = \sqrt{\frac{\omega}{\pi\hbar}}\, e^{-\beta\omega/2} \sum_{n_1,n_2} \frac{\overbrace{\left(e^{-S_{cl}/\hbar}\sqrt{S_{cl}}\ \mathcal{R}\beta\right)}^{x}{}^{n_1+n_2}}{n_1!\ n_2!}\ \delta_{n_1-n_2,1}\ .$$

- I decouple the summation ($\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i\theta(n_1-n_2-1)} = \delta_{n_1-n_2,1}$):

$$K_E(-\pi,\pi;\beta) = \sqrt{\frac{\omega}{\pi\hbar}} e^{-\beta\frac{\beta\omega}{2}} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta} \sum_{n_1} \frac{\left(xe^{-i\theta}\right)^{n_1}}{n_1!} \sum_{n_2} \frac{\left(xe^{i\theta}\right)^{n_2}}{n_2!}$$

$$K_E(-\pi,\pi;\beta) = \sqrt{\frac{\omega}{\pi\hbar}} e^{-\frac{\beta\omega}{2}} \int_0^{2\pi} \frac{d\theta}{2\pi}\ e^{i\theta}\ e^{2e^{-S_{cl}/\hbar}\sqrt{S_{cl}}\ \mathcal{R}\beta\cos\theta}$$

- With non-adjacent minima ($q_b - q_a = 2N\pi$):

$$K_E\left(-\pi,(2N-1)\pi;\beta\right) = \sqrt{\frac{\omega}{\pi\hbar}} e^{-\frac{\beta\omega}{2}} \int_0^{2\pi} \frac{d\theta}{2\pi}\ e^{iN\theta}\ e^{2e^{-S_{cl}/\hbar}\sqrt{S_{cl}}\ \mathcal{R}\beta\cos\theta}$$

- From here, one derives Bloch's theorem and the energy of the first band

$$\langle(2N-1)\pi|\phi_\theta\rangle = e^{iN\theta}\langle-\pi|\phi_\theta\rangle$$

$$E_\theta = \frac{1}{2}\hbar\omega - \hbar\left(e^{-S_{cl}/\hbar}\sqrt{S_{cl}}\ \mathcal{R}\cos\theta\right)$$

Kernels and hyperbolic Bessel functions

- The kernel can be written as

$$K_E(-\pi, (2N-1)\pi; \beta) = \sqrt{\frac{\omega}{\pi\hbar}} e^{-\frac{\beta\omega}{2}} \int_0^{2\pi} \frac{d\theta}{2\pi} \cos(N\theta)\, e^{2x\cos\theta} =$$
$$= \sqrt{\frac{\omega}{\pi\hbar}} e^{-\frac{\beta\omega}{2}} \, I_N(2x)$$

- The first-order approximation for $\beta \to \infty$ ($x \to \infty$) does not depend on N.

$$I_N(2x) \overset{2x\to\infty}{\simeq} \frac{e^{2x}}{\sqrt{2\pi\, 2x}} (1 - O(2x)) \ .$$

Instanton in QFT

- Let's consider a field $\varphi(x,t)$, a potential $U(\varphi)$ and Q as the topological space of solutions with finite energy.
- Q might have disconnected components, which would make impossible connections between different minima.
- Static solutions with finite energy are found: *solitons*.
- Solitons and instantons have different similarities.

$$U(\varphi) = g^2(1+\cos\varphi) \rightsquigarrow \varphi(x) = 2\arcsin\{\tanh[g(x-x_0)]\}$$

$$V(q) = g^2(1+\cos q) \rightsquigarrow q(\tau) = 2\arcsin\{\tanh[g(\tau-\tau_0)]\}$$

Bibliografia

Richard Phillips Feynman, Albert R. Hibbs, and Daniel F. Styer.
Quantum Mechanics and Path Integrals.
Dover Publications, 2010.

Hilmar Forkel.
A Primer on Instantons in QCD.
August 2000.

R Rajaraman.
Solitons and instantons.
Elsevier Science, London, England, April 1982.

Z. Ambroziński.
Tunneling in cosine potential with periodic boundary conditions.
Acta Physica Polonica B, 44(6):1261, 2013.

- The images used here are generated with codes written by me and freely downloadable:
  `https://github.com/LorenzoCane/Instanton-and-tunnelling.git`

```
#***********************************************
#Control panel
# This code will draw: - DW Potential multiple-instantons : V (q) = \lambda / 4 (q^2 - a^2)^2 (real and imaginary time)
#                                                           V_sec = 2*lam*a^22*(1-3/2*(np.cosh(a*(lam/2)^(1/2)*(t-tau0)))^(-2))
#                        - SG multiple-instantons (adjacent minima)
#                        - anti - instanton (generic minima)
#See the other files for variable meaning



Letter_mode = True #True: the ticks on x and y axis are generic constant, no number will appear

lam = 40
a =1
omega= a*sqrt(2*lam)
tau0 = 2
N= 5 # # instantons + #anti-instantons for DW

g = 1
tau0_PP = 3
N_PP= 7 # # instantons + #anti-instantons
custom = 'IIAAIAAII' #custom order of (anti-)instantons for periodic potential. I=instanton , A=Anti-instantons
#remember that #I-#A= distance , please modify (1) and (2) if you need (automatic in future releases)


FunctionColor = 'orangered' #color of the plot
SecondFunctionColor ='b'
```

Thank you for your attention!