Supersymmetric string vacua with flat backgrounds

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Based on collaborations with Ilarion Melnikov and Ruben Minasian

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2. Perturbative analysis of F-theory on FHSV
3. Flat background in IIA and M thoery

Discrete torsion as flat B-field background

Genaralities of 2d CFTS

- Pertubative string theory is formulated on 2d CFTs, whose Hilbert space carries a module structure over Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1}$$

- Orbifold CFTS: gauging a non-anomalous symmetry G (assume finite and Abelian for simplicity).
- The Hilbert space of orbifold CFT is direct sum of projected states P in each sector:

$$\mathcal{H}^{orb} = \oplus \mathcal{H}_g^G \tag{2}$$

different sectors are lablelled by $g \in G$. The partition function is:

$$Z_{orb} = \frac{1}{|G|}\sum_{g,h\in G} Z_{g,h} = \frac{1}{|G|}\sum_{g,h\in G} Tr_{\mathcal{H}_g}(\rho(h)\mathbf{q}^{L_0-\frac{c}{24}}\bar{\mathbf{q}}^{\bar{L}_0-\frac{c}{24}}) \tag{3}$$

Example: Compact boson with $c = 1$

Gauging the $\mathbb{Z}_2$ symmetry generated by σ: $X \to X + \pi r$ (the half circle shift) gives a orbifold CFT.

- The $\mathbb{Z}_2$ action depends on the momentum number: $\sigma\mathcal{H}_m = (-1)^m\mathcal{H}_m$. Projected states have even momentum modes
- Twisted sector ($X = X + \pi r$) gives twisted Hilbert space (after projection) is $\mathcal{H}_\sigma = \oplus\mathcal{H}_{2m,n+\frac{1}{2}}$.
- This is a compact boson CFT with radius $\frac{r}{2}$

Discrete Torsion Vafa 1986

A natural generalization will be giving a weight $\epsilon(g,h)$ to every (g,h) sector, the partition function then becomes:

$$Z_{orb} = \frac{1}{|G|} = \sum_{g,h} \epsilon(g,h) Z_{g,h} \tag{4}$$

After this generalization, modularity imposes non-trivial constraints on $\epsilon(g,h)$. Under a very mild condition (factorization property on a genus 2 surface) these constraints are:

$$\epsilon(gh,l) = \epsilon(g,l)\epsilon(h,l), \epsilon(g,h)\epsilon(h,g) = \epsilon(g,g) = 1 \tag{5}$$

Vafa also proposed a family of natural solutions to it, parameterized by second group cohomology $H^2(G, U(1))$:

$$\forall \omega \in H^2(G, U(1)) \rightarrow \epsilon(g,h) = \frac{\omega(g,h)}{\omega(h,g)} \tag{6}$$

This set of solutions $\epsilon(g,h)$ are called discrete torsion.

Example: B-field on $S^1 \times S^1$

Bosonic CFT with target space $S^1 \times S^1$

- Bosonic CFT with target space $S^1 \times S^1$ can be viewed as gauging $\mathbb{Z} \times \mathbb{Z}$ symmetry of a bosonic CFT with target space $\mathbb{R} \times \mathbb{R}$ ($S^1 = \mathbb{R}/\mathbb{Z}$).
- Discrete torsion given by $H^2(\mathbb{Z} \times \mathbb{Z}, U(1)) = \frac{H^2(T^2,\mathbb{R})}{H^2(T^2,\mathbb{Z})}$ can be identified with the B-field on $S^1 \times S^1$.

5d Seifert manifold X_5

For simplicity we will only discuss one type of 5d Seifert manifold, $\mathbb{Z}_2$ quotient of T^5. Here T^5 is described by:

$$(x_1, x_2, x_3, x_4, x_5), x_i \simeq x_i + 1, i = 1, 2, 3, 4, 5 \tag{7}$$

and the $\mathbb{Z}_2$ action is:

$$\sigma : (x_1, x_2, x_3, x_4, x_5) = (-x_1, -x_2, -x_3, -x_4, x_5 + \frac{1}{2}) \tag{8}$$

Half circle shift on 5-th direction makes X_5 is a smooth 5d manifold. We call the 5-th circle as the Seifert circle.

Two ways of viewing X_5

- A: X_5 could be viewed as T^4 fibered over the Seifert circle and it picks up a $\mathbb{Z}_2$ involution when moves around the circle once.
- B: X_5 could be viewed as the Seifert circle fibered over orbifold $T^4/\mathbb{Z}_2$ and the radius shrinks half its size over 16 fixed points.

From above description, it is clear that X_5 is a flat 5d manifold that preserves half super symmetry and the Seifert circle is important to make the geometry smooth. So how does T-duality goes with this circle?

T-duality of Type IIA string on X_5

CFT on X_5

Focus on the Seifert circle, the Seifert $\mathbb{Z}_2$ acts on the relevant quantum number as:

$$\sigma(m_5, n_5) = (-1)^{m_5}(m_5, n_5) \tag{9}$$

So after T-dual of Seifert circle $\mathbb{Z}_2$ should act as:

$$\sigma^T(m_5, n_5) = (-1)^{n_5}(m_5, n_5) \tag{10}$$

i.e. the action depends on the winding number rather than momentum number. Unlike the Seifert $\mathbb{Z}_2$ which is a half circle shift, the geometric understanding of its T-dual is not clear.

The picture from spacetime point of view

Consider X_5 as Seifert circle fibered over $T^4/\mathbb{Z}_2$.

- Compactify on the Seifert circle gives $T^4/\mathbb{Z}_2$ with holonomy $\int_{\pi_i} g_{\mu 5} = \frac{1}{2}$ due to half circle shift on the fiber (π_i is the non trivial loop around i-th fixed point caused by the $\mathbb{Z}_2$ action).
- T-dual the Seifert circle exchanges $g_{\mu 5}$ and $B_{\mu 5}$.
- Starts with $T^4/\mathbb{Z}_2$ with $(\int_{\pi_i} g_{\mu 5} = \frac{1}{2}, \int_{\pi_i} B_{\mu 5} = 0)$. T-dual gives $(\int_{\pi_i} g_{\mu 5} = 0, \int_{\pi_i} B_{\mu 5} = \frac{1}{2})$.
- Lifts back gives IIB on $T^4/\mathbb{Z}_2 \times S^1$ with $\int_{\pi_i \times S^1} B = \frac{1}{2}$

As in IIA X_5 is smooth, the addional massless degrees of freedom from singualrities in IIB should be freezed by the flat B-field background.

Flat B-field as discrete torsion

What is the CFT associated with the flat B-field background $\int_{\pi_i \times S^1} B = \frac{1}{2}$?
The answer is discrete torsion.

- CFT with target space $T^4/\mathbb{Z}_2 \times S^1$ coould be viewed as gauging the symmetry $\mathbb{Z}_2 \times \mathbb{Z}$ of a $T^4 \times \mathbb{R}$ CFT.
- as $H^2(\mathbb{Z}_2 \times \mathbb{Z}, U(1)) = \mathbb{Z}_2$, there is a non trivial discrete torsion could be turned on. The discrete torsion is given by:

$$\epsilon((a, n_1), (b, n_2)) = \exp \pi i(an_2 - bn_1) \tag{11}$$

  for $(a, n_1), (b, n_2) \in \mathbb{Z}_2 \times \mathbb{Z}$.
- It is easy to check with this discrete torsion turned on, the spectrum is the same as expected from IIA side, e.g. in the untwisted $\mathbb{Z}_2$ sector, the action becomes:

$$\sigma' |m_5, n_5\rangle = (-1)^{n_5} |m_5, n_5\rangle \tag{12}$$

With non-trivial discrete torsion turned on, the twisted sectors associated with spacetime singular locus become massive.

Frozen singularities

With the non-trivial B field around each singularity $\int_{\pi_i \times S^1} B = \frac{1}{2}$, we see that 16 A_1 singularities of $T^4/\mathbb{Z}_2$ doesn't enter into the low energy spectrum.
Spacetime singularities doesn't enter into low energy spectrum is referred to frozen of singularities and typically a non-perturbative mechanism. John H. Schwarz et al 1995, Jan de Boer et al 2001 discussed one frozen mechanism in IIA by turing on non-trivial C_1 holonomy $\int_{\pi_i} C_1 = \frac{1}{2}$ on $T^4/\mathbb{Z}_2$. The one we discussed above via B-field is a perturbative one and is connected to this mechanism via dualities.

- Put M theory on $X_5 \times S^1$
- First compactify on the Seifert circle, it gives IIA on $T^4/\mathbb{Z}_2 \times S^1$ with non-trivial C_1 holonomy mentioned above. T-dual S^1 gives IIB on $T^4/\mathbb{Z}_2 \times S^1$ with holonomy $\int_{\pi \times S^1} C_2 = \frac{1}{2}$ around singularities.
- First compactify on S^1 gives IIA on Seifert. T-dual the Seifert circle gives IIB on $T^4/\mathbb{Z}_2 \times S^1$ with B-field holonomy $\int_{\pi \times S^1} B = \frac{1}{2}$ around singularities.
- The relation now becomes clear, $\int_{\pi \times S^1} B = \frac{1}{2}$ and $\int_{\pi \times S^1} C_2 = \frac{1}{2}$ are related to each other with $SL(2, \mathbb{Z})$ duality in IIB.

Perturbative analysis of F-theory on FHSV

$SL(2,\mathbb{Z})$ and F theory

Type IIB string theory is believed to have $SL(2,\mathbb{Z})$ duality group under which the NS-NS 2-form B_2, RR 2-form C_2 and the axiodilaton $\tau = C_0 + ie^{-\phi}$ transform as:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{13}$$

F theory is type IIB string theory with 7-branes ($D7$ and its $SL(2,\mathbb{Z})$ duality transformation) source the axiodilation τ. Hence τ varies in spacetime and can be represented as modulus of an elliptic curve over spacetime. It is non-perturbative in general.

However, if all the monodromy involved is just the center of $SL(2,\mathbb{Z})$, i.e. $diag(-1,-1)$. Then

$$\tau \to \frac{-\tau}{-\tau} = \tau \tag{14}$$

and string coupling can be small universally along spacetime and a perturbative description of it seems possible.

Sen's example

- View $T^4/\mathbb{Z}_2$ (a special locus of the moduli) as an elliptic fibration:

$$T^2 \hookrightarrow T^4/\mathbb{Z}_2 \to T^2/\mathbb{Z}_2 = \mathbb{P}^1 \tag{15}$$

- The monodromy involved is the center $diag(-1,-1)$, and the perturbative description given by Sen is gauging the worldsheet CFT by $(-)^{F_L}\Omega R$, where R is the involution on T^2, Ω is the worldsheet parity and F_L measures the spacetime fermion number comes from left moving sector.

FHSV manifold

FHSV manifold S. Ferrara et al 1995 is a smooth elliptic CY_3 fold given by taking a special $\mathbb{Z}_2$ quotient of $K3 \times \mathbb{T}^2$

- The $\mathbb{Z}_2$ generator σ acts on $\mathbb{T}^2$ is an involution, i.e. $\sigma \cdot z = -z$.
- σ action of $K3$ is fixed point free, i.e. $K3/\mathbb{Z}_2$ is a smooth Kaehler manifold. There is only one such possibility and the resulting $K3/\mathbb{Z}_2$ ia an Enriques surface. σ acts on the middle integral cohomology $\Gamma_{3,19}$ in the following way:

$$\sigma : \Gamma_{3,19} = \Gamma_{1,1} \oplus \Gamma^a_{1,9} \oplus \Gamma^b_{1,9} \rightarrow (-\Gamma_{1,1}) \oplus \Gamma^b_{1,9} \oplus \Gamma^a_{1,9}$$

- As for holomorphic structure $(\Omega^{2,0}, \omega^{1,1})$

$$\sigma : (\Omega^{2,0}, \omega^{1,1}) \rightarrow (-\Omega^{2,0}, \omega^{1,1})$$

- Enriques action only exists on a speical locus of K3 moduli space.

Some features of FHSV

- The action on the elliptic fiber is involution, i.e. τ can be fixed uniformly and a perturbative treatment should be possible
- Enriques surface E is smooth (no orientifold needed) and doesn't admit a spin structure.

What is worldsheet theory for this specific background?

Find the worldsheet action x of $diag(-1,-1) \in SL(2,\mathbb{Z})$ and worldsheet action σ of the spacetime Enriques involution.

FHSV manifold

Some hints:

- NS-NS, R-R sectors give spacetime bonsonic field, hence the worldsheet action should satisfy $\sigma^2 = x^2 = id$.
- NS-R, R-NS sectors give spacetime fermions, so the action of σ and x should satisfy:

$$\sigma^2 = x^2 = -id$$

- The Enriques action σ on the worldsheet should give the spacetime action, e.g. it acts on the chiral primary operators in the same way as it acts on $H^*(K3, \mathbb{R})$.
- The action x:

field	g	ϕ	B_2	C_0	C_2	C_4^+
x	$+$	$+$	$-$	$+$	$-$	$+$

(16)

- FHSV geometry suggests that when move around the nontirvial loop on E, B_2 goes to $-B_2$, which suggests the worldsheet action x should contain worldsheet parity Ω:

$$x : \int_{\Sigma_{2d}} \phi^* B \to -\int_{\Sigma_{2d}} \phi^* B$$

- $x\sigma$ is the $\mathbb{Z}_2$ action on the worldsheet.

The action x:

$$\begin{array}{ccccccc} \text{field} & g & \phi & B_2 & C_0 & C_2 & C_4^+ \\ x & + & + & - & + & - & + \end{array} \tag{17}$$

and it should involve worldsheet parity Ω leaves only one natural choice:

$$x = (-)^{F_L}\Omega$$

where F_L is the spacetime fermion number comes from left moving sector.

As for σ, recall some basic properties of the $K3$ SCFT ((4, 4) SCFT with $\hat{c} = 4$, i.e. $k_{su(2)_R} = 1$):
In the NS-NS sector, the relevant chiral primary operators (states) are

- Identity operator $h = l = \bar{h} = \bar{l} = 0$
- 20 quartets $\mathcal{O}_i^{\pm\pm}$ with $2h = l = 2\bar{h} = \bar{l} = 1$, they are in 1-1 correspondence with 20 harmonic $(1, 1)$-forms on $K3$. They furthur split into 1 self-dual quartet (Kaehler class) and 19 anti-selfdual quartets.
- The action of σ on the chiral primary operators can be read out from the Enriques action on $H^2(K3, \mathbb{R})$.
- NS-R, R-NS, R-R sectors are obtained by spectral flow operators.
- Picking out a $U(1)_R \subset SU(2)_R$ in both left and right moving part, then the $U(1)_R$ charges (q_L, q_R) satisfies:
  –NS-NS, R-R sectors: $q_L - q_R = 2\mathbb{Z}$
  –NS-R,R-NS sectors: $q_L - q_R = 2\mathbb{Z} + 1$

The σ is $exp\{i\frac{\pi}{2}J_0^{U(1)} + i\frac{\pi}{2}\bar{J}_0^{U(1)}\} \cdot g_E$, where g_E is of order 2 and acts on the $(4, 4)$ superconformal characters. It has all the required properties, i.e. $\sigma^2 = (-)^{q_L - q_R}$.

The worldsheet theory related to Enriques surface is given by orbifolding $x\sigma$. Standard worldsheet techniques can be used to give the massless spectrum:

- Half of spacetime supersymmetries projected out, i.e. Spacetime supersymmetry is 6d $(1, 0)$.
- Vacuum gives a $(1, 0)$ gravity multiplet and a $(1, 0)$ hypermultiplet.
- Self dual quartet gives a $(1, 0)$ hypermultiplet.
- 19 antiself dual quartet gives 9 tensor multiplets and 10 hypermultiplets

matches the F theory analysis.

Supersymmetric vacua on Pin^- manifold

Further put this theory on S^1 and T-dual it to IIA, it gives a supersymmetric vacua on internal manifold $\frac{K3\times S^1}{\mathbb{Z}_2}$ (Ω needs to be replaced by ΩR_{S^1} to suit the asymmetric GSO projection in IIA). $\mathbb{Z}_2$ acts as:

$$i : (x, \phi) \to (\sigma \cdot x, -\phi)$$

It is an smooth unorientable Pin^- manifold and the above construction gives a IIA supersymmetric vacua on smooth Pin^- manifold.

$Pin^{\pm}$ structure

- The topological condition for a Pin^+ structure is:

$$w_2(TM \oplus det(TM)^{\oplus 3}) = 0 \to w_2(TM) = 0 \tag{18}$$

- The topological condition for a Pin^- structure is:

$$w_2(TM \oplus det(TM)) = 0 \to w_2(TM) + w_1^2(TM) = 0 \tag{19}$$

Flat background in IIA and M theory: K theory versus Integral cohomology

More about FHSV

Perhaps the most striking feature of FHSV manifold is that it is mirror to itself, and there is no worldsheet instanton corrections to the Kaehler moduli space in IIA. One way to see it is noticing the holonomy is $\mathbb{Z}_2 \ltimes SU(2)$, the hyperkaehler sturcutre is only "softly" broken.

One evidence for the self-mirror property is given by the fact:

$$K^0_{tor}(FHSV) = \mathbb{Z}_2^{\oplus 4} = K^1_{tor}(FHSV) \tag{20}$$

and $K^{0(1)}_{tor}(FHSV)$ classfies torsionful flat RR background of IIA(B) (hence 0 at differential form level) on FHSV Gregory Moore et al 1999 (and torsionful D-branes in IIB(A)). In IIA, the flat RR background generators are $[C_1], [C_1^+], [C_1^-], [C_3]$.

More about FHSV

More interstingly, Ilka Brunner al 2002 observed that with certain flat $[C_1]$ turned on, not all the torsionful D-branes (given by $K^1_{tor}(FHSV)$)are allowed to exist in the spectrum. The argument goes as follows;

- In a specific point of the moduli space with gravity decoupled, IIA comapctify on FHSV gives a 4d $N = 2$ supersymmetric gauge theory with gauge group $SU(2)$.
- IIA on FHSV receives no worldsheet instanton corrections, implies this 4d $N = 2$ $SU(2)$ gauge theory is a finite $N = 2$ theory: $SU(2)$ with $N_f = 4$.
- By analyzing the massless spectrum in this limit, Ilka Brunner al 2002 found only when certain torsionful D branes are excluded in the spectrum can the effective theory be finite. (Otherwise it gives $N_f = 8$).

Self consistency of string theory requires the absence of some torsionful D-branes.

The observation from M theory perspective

Instead of K-theory, the C_3 field in M theory seems to be classfied by (equivariant) integral cohomolgy.

From M-theory perspective, the presence of torsion flat $[C_1]$ gives the M theory geometry Y_7 is a non-trivial circle fibration:

$$S^1 \hookrightarrow Y_7 \to FHSV \tag{21}$$

with $e(S^1) = [C_1] \in H^2_{tor}(FHSV)$.

The K theory class $[C_3] + [C_1]$ is connected to K theory class $[C_1]$ when lifts to M-theory, which both sits on the left handside of the flat M theory C_3 field moduli:

$$0 \to \frac{H^3(Y_7;\mathbb{R})}{H^3(Y_7,\mathbb{Z})_{\mathbb{R}}} \to H^3(Y_7;U(1)) \to H^4_{tor}(Y_7;\mathbb{Z})(=0) \to 0 \tag{22}$$

M-theory analysis suggests the IIA moduli branch labelled $[C_1]$ and $[C_1] + [C_3]$ are connected. Combine this fact with the perfect pairing between torsion D-branes and torsion RR fluxes:

$$K^0_{tor}(FHSV) \times K^1_{tor}(FHSV) \to U(1) \tag{23}$$

suggests the allowed torison D-branes $\alpha \in K^1_{tor}(FHSV)$ should satisfy

$$[C_3] \times \alpha = 0 \in U(1) \tag{24}$$

matches with the analysis done in Ilka Brunner et al 2002

Thanks!