Axions as Thermal DM: A ChPT Approach

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[1] L. Di Luzio, J. Martín Camalich, G. Martinelli, JAO, G. Piazza, PRD108,035025(2023)

[2] R. Gao, Z.-H. Guo, JAO, H.-Q. Zhou, JHEP 04, 022 (2023)

Axions++ 2023, September 25-28th 2023, LAPTh, Annecy, France,

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At GGI 2023: QCD axion couplings "beyond standard" chiral perturbation theory

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Outline

1. Introduction
2. $a\pi \to \pi\pi$ scattering
3. Current algebra and the $\rho(770)$
4. Calculation of $a\pi \to \pi\pi$
5. Axion EFT Lagrangian
6. Masses, mixing and $\gamma\gamma$ couplings
7. Conclusions

1.- Introduction

Why "beyond standard"?

In the first part: Thermal production of axions

The reaction $a\pi \to \pi\pi$ scattering is dominant for $T_D \lesssim T_c$, $T_c \approx 155$ MeV, $m_a \sim 1$ eV

- The Chiral Perturbation Theory (ChPT) series in many instances is poorly convergent for $\sqrt{s} \gtrsim 0.4$ GeV
- We implement a *unitarization method*

[1] L. Di Luzio, J. Martín Camalich, G. Martinelli, JAO, G. Piazza, PRD108,035025(2023)

In the second part: a, π, η, η', K

- No removal of the $G\tilde{G}$ term. Isosinglet axial current $\bar{q}\gamma_5 q$
- $U(3)$ ChPT.
- δ counting: It combines chiral plus large N_C countings

[2] R. Gao, Z.-H. Guo, JAO, H.-Q. Zhou, JHEP 04, 022 (2023)

Scattering amplitudes in the low-energy EFT (ChPT) satisfy

- Lorentz invariance
- (Perturbative) Unitarity
- Analyticity
- Crossing

Schnitzer, H.J. *Current algebra and unitarity.* PRL,24,1384**(1970)**; PRD,2,1621**(1970)**

- The tree approximation to the scattering amplitudExplicit account of isosinglet axial transformations.es typically badly violates unitarity.
- The Lagrangians are nonlinear and nonrenormalizable, which makes difficult to compute higher-order corrections. (Proliferation of higher-order counterterms).
- The resultant renormalized perturbation series would probably diverge

Pole of the $f_0(500)$ or σ resonance: $|s_\sigma| \simeq 0.25$ GeV^2

3.- Current algebra and the $\rho(770)$

The main feature of medium energy $\pi\pi$ scattering is the $\rho(770)$ $J^{PC} = 1^{--}$

$\pi\pi$ scattering lengths and volume Weinberg,PRL17,616(1966) in current algebra

Soon afterwards L.S.Brown,R.L.Goble, PRL20,346(1968)

"In this note ... **we may extend this low-energy result through a useful range of physical energies ... with general requirements such as unitarity and correct analytic structure**"

It is similar to how renormalization group equation is used in quantum field theory to improve perturbative calculations

Basics of Unitarization

- **Partial-wave amplitudes (PWA)**

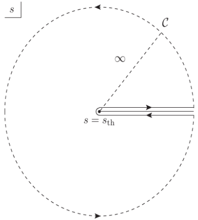
$$T_{IJ}(s) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta P_J(\cos\theta)\, T_I(s,t,u)$$

- **Unitarity**

$$T_{IJ}(s) = e^{i\delta_{IJ}} \sin\delta_{IJ}\, \frac{k}{8\pi\sqrt{s}}$$

$$\mathrm{Im}\, T_{IJ}(s)^{-1} = -\frac{k}{8\pi\sqrt{s}}$$

- **Analyticity** is used to isolate the unitarity cut



$$h(s) = -\frac{s}{8\pi^2}\int_{s_{th}}^{\infty} ds' \frac{k(s')/\sqrt{s'}}{(s'-s)s}$$
$$= \frac{\sigma(s)}{16\pi^2}\log\frac{\sigma(s)+1}{\sigma(s)-1}$$

$$\sigma(s) = \sqrt{1-\frac{4m^2}{s}}$$

- Master unitarity formula

$$T_{IJ}(s) = \left[M_{IJ}(s) + h(s)\right]^{-1}$$

Different unitarization methods differ in the calculation of $M_{IJ}(s)$

Extended Effective-Range Expansion
Brown and Goble 1968

$$\frac{1}{8\pi} M_{IJ}(s) = -\frac{1}{a_1 k^2} + \frac{r_1}{2}$$

- $a_1 = \frac{1}{12\pi f_\pi^2}$ Weinberg (1966)

r_1 is fixed by reproducing the $\rho(770)$ mass

$$\frac{1/a_1}{m_\rho^2/4 - m_\pi^2} + \frac{r_1}{2} + \mathrm{Re}[h(m_\rho^2)] = 0$$

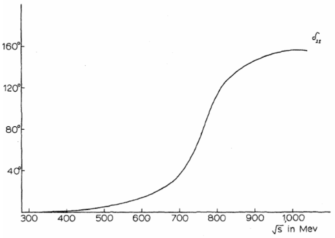
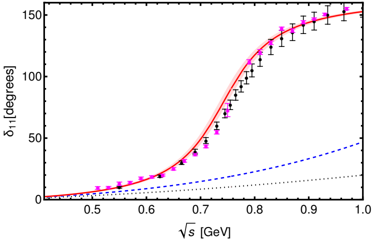


FIG. 2. *P*-wave pion-pion shift as a function of the total center-of-mass energy.

Brown,Goble(1968)



$a\pi \to \pi\pi$ PRD108,035025(2023)
LO (dotted), NLO (dashed) ChPT

The ρ width was predicted as 130 MeV (experimentally 150 MeV)

Final-state interactions (FSI)

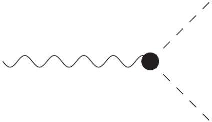
Soon after Weinberg's calculation (1966)
Immediately after Brown-Goble (1968) unitarization

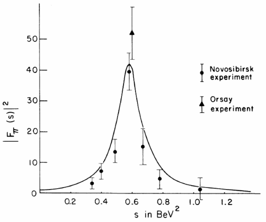
Gounaris,Sakurai work out the FSI to $\rho \to e^+e^-$
Gounaris,Sakurai,PRL21,244(1968) *Finite-width corrections to the vector-meson-dominance prediction for $\rho \to e^+e^-$*

Vector form factor of the π: $F_\pi(s)$
$F(0) = 1$

$$\tau_{11}(s) = \frac{T_{11}(s)}{k^2}$$

$$F_\pi(s) = \frac{\tau_{11}(s)}{\tau_{11}(0)} = \frac{-1/a_1}{-1/a_1 + r_1 k^2/2 + k^2 h(s)}$$





Omnés function

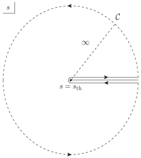
$\Omega(s) = \dfrac{\tau_{11}(s)}{\tau_{11}(0)}$ is an Omnés function

Watson FSI theorem:

$$\mathrm{Im} F_\pi(s) = \frac{\sigma(s)}{16\pi} F_\pi(s) T_{11}(s)^*$$

$F_\pi(s)$ has the same phase as $T_{11}(s)$

$$\Omega(s) = \exp\left(\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_{11}(s')}{(s'-s)s'} \right)$$



Solution to the integral equation required by unitarity and analyticity

$$F_\pi(s) = 1 + \frac{s}{16\pi^2} \int_{s_{\rm th}}^{\infty} \frac{ds' \sigma(s') F_\pi(s') T_{11}(s')^*}{(s'-s)s'}$$

4.- LO+NLO ChPT result for $a\pi \to \pi\pi$

Di Luzio,Martinelli,Piazza PRL126,241801(2021); Di Luzio,Piazza,JHEP41(2022) Error corrected in [1] PRD108,035025(2023)

$$\mathcal{M}^{\rm LO}_{a\pi^0\to\pi^+\pi^-} = \frac{3C_{a\pi}}{2f_\pi f_a}(m_\pi^2 - s)\ , \quad C_{a\pi} = \frac{1}{3}\left(\frac{m_d - m_u}{m_d + m_u} + c_d^0 - c_u^0\right)$$

$$\begin{aligned}
\mathcal{M}^{\rm NLO}_{a\pi^0\to\pi^+\pi^-} = \frac{C_{a\pi}}{192\pi^2 f_\pi^3 f_a}\Bigg\{ & 15m_\pi^2(u+t) - 11u^2 - 8ut - 11t^2 \\
& - 6\overline{\ell_1}\left(m_\pi^2 - s\right)\left(2m_\pi^2 - s\right) - 6\overline{\ell_2}\left(-3m_\pi^2(u+t) + 4m_\pi^4 + u^2 + t^2\right) \\
& + 18\overline{\ell_4}m_\pi^2(m_\pi^2 - s) + 3\Bigg[3s\left(m_\pi^2 - s\right)\sigma(s)\ln\left(\frac{\sigma(s)-1}{\sigma(s)+1}\right) \\
& + \left(m_\pi^2(t-4u) + 3m_\pi^4 + t(u-t)\right)\sigma(t)\ln\left(\frac{\sigma(t)-1}{\sigma(t)+1}\right) \\
& + \left(m_\pi^2(u-4t) + 3m_\pi^4 + u(t-u)\right)\sigma(u)\ln\left(\frac{\sigma(u)-1}{\sigma(u)+1}\right)\Bigg]\Bigg\} \\
& - \frac{4\ell_7 m_\pi^2 m_d\left(s - 2m_\pi^2\right)m_u\left(m_d - m_u\right)}{f_\pi^3 f_a\left(m_d + m_u\right)^3}
\end{aligned}$$

One only needs $a\pi^0 \to \pi^+\pi^-$ and $a\pi^0 \to \pi^0\pi^0$ because of crossing symmetry

Inverse Amplitude Method for $a\pi \to \pi\pi$

We follow the Inverse Amplitude Method (IAM)

H. Lehmann, Phys.Lett.41,529(1972)

T.N. Truong, PRL161,25626(1988)

for calculating $a\pi \to \pi\pi$ in [1]

This is a FSI problem $A_{IJ}(s)$

$$\mathrm{Im} A_{IJ}(s) = \frac{\sigma(s)}{16\pi} A_{IJ}(s) T_{IJ}(s)^*$$

$$\mathrm{Im} \frac{1}{A_{IJ}(s)} = -\frac{\sigma}{16\pi} \frac{T_{IJ}(s)}{A_{IJ}(s)}$$

The inverse $1/A_{IJ}(s)$ is more perturbative

- Associated poles to resonances become zeros of $1/A_{IJ}$
- $|T_{IJ}(s)| \leq 8\pi\sqrt{s}/k$ because of unitarity
  $\mathrm{Im} A_{IJ}(s)^{-1}$ remains small even if $|A_{IJ}(s)|$ is large

From ChPT up to NLO Di Luzio,Martinelli,Piazza PRL126,241801(2021); Di Luzio,Piazza,JHEP41(2022)

[Error corrected in [1] PRD108,035025(2023)]

$$A_{IJ}(s) = A_{IJ}^{(2)}(s) + A_{IJ}^{(4)}(s) + \mathcal{O}(p^6)$$

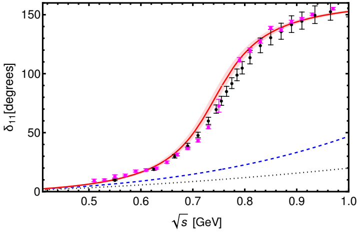
Expansion of the inverse of the amplitude

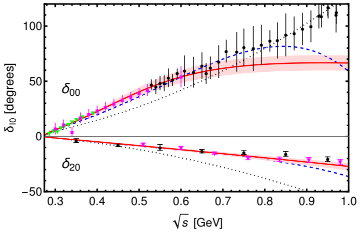
$$\frac{1}{A_{IJ}(s)} = \frac{1}{A_{IJ}^{(2)}(s)} - \frac{A_{IJ}^{(4)}(s)}{A_{IJ}^{(2)}(s)^2}$$

Take the inverse

$$A_{IJ}(s) = \frac{A_{IJ}^{(2)}(s)^2}{A_{IJ}^{(2)}(s) - A_{IJ}^{(4)}(s)}$$

- This formula does not build in explicitly Watson FSI theorem
- Unitarity check
- It is nicely fulfilled
  δ_{IJ}, I: Isospin, J: Angular momentum





For δ_{00} the rise for $\sqrt{s} \gtrsim 0.8$ GeV is due to the $K\bar{K}$ threshold and the $f_0(980)$
Albaladejo,JAO, PRD86,034003(2012)

Generation of resonances: $\rho(770)$ and $f_0(500)$ (aka σ)

Recently Notari, Rompineve, Villadoro, PRL131,011004(2023) [Villadoro's talk this workshop] derived the rule

$$T_{a\pi^i\to\pi^j\pi^k} = \theta_{a\pi}\cdot T_{\pi^0\pi^i\to\pi^j\pi^k} + \underbrace{\mathcal{O}\left(\frac{m_\pi^2}{s}\right)}_{\leq 10\%}$$

$\theta_{a\pi}$ is the a-π^0 LO mixing angle

- The axial isosinglet current $\bar{q}\gamma^\mu\gamma_5 q$ is suppressed by one power of quark mass at least
- The axial isovector current can be eliminated by a proper axial $SU(2)$ transformation

Loophole for $a\pi^0\to\pi^0\pi^0$ & $\pi^0\pi^0\to\pi^0\pi^0$

At LO

$$T^{(2)}_{a\pi^0\to\pi^0\pi^0} = 0$$

$$T^{(2)}_{\pi^0\pi^0\to\pi^0\pi^0} = \frac{m_\pi^2}{f_\pi^2}$$

$$T^{(2)}_{a\pi^0\to\pi^0\pi^0} \neq \theta_{a\pi}\, T_{\pi^0\pi^0\to\pi^0\pi^0} + \mathcal{O}\left(\frac{m_\pi^2}{s}\right)$$

Up to NLO:

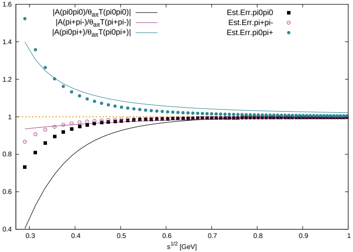
$$\underbrace{T^{(2)}_{a^0\pi^0\to\pi^0\pi^0}}_{0} + \underbrace{T^{(4)}_{a\pi^0\to\pi^0\pi^0}}_{\mathcal{O}\left(\frac{s^2}{(4\pi f_\pi)^2 f_a^2}\right)} \neq \theta_{a\pi}\left(\underbrace{T^{(2)}_{\pi^0\pi^0\to\pi^0\pi^0}}_{\frac{m_\pi^2}{f_\pi^2}} + \underbrace{T^{(4)}_{\pi^0\pi^0\to\pi^0\pi^0}}_{\mathcal{O}\left(\frac{s^2}{(4\pi f_\pi)^2 f_\pi^2}\right)}\right) + \mathcal{O}\left(\frac{m_\pi^2}{s}\right)$$

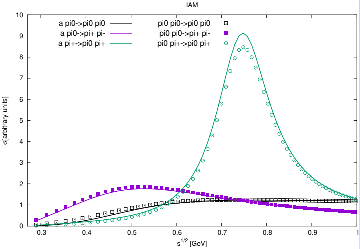
Extra suppression scale $s/(4\pi f_\pi)^2$ of the NLO contribution

$$\mathcal{O}\left(\frac{m_\pi^2(4\pi f_\pi)^2}{s^2}\right)$$

In Lattice QCD with larger pion masses this discrepancy would be significant

Phenomenologically for the hot-dark matter bound on m_a is not important





- Expected error $\pm\mathcal{O}(m_\pi^2/s)$ for the ratio $|A_{a\pi\to\pi\pi}/\theta_{a\pi}\,T_{\pi\pi\to\pi\pi}|$

- The rule is less accurate than expected for $a\pi^0 \to \pi^0\pi^0$ up to $\sqrt{s} \lesssim 0.6$ GeV Large Value!

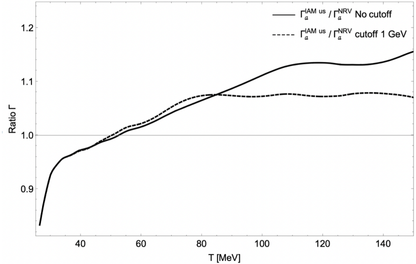
- This difference can be neglected phenomenologically speaking [left panel for the cross section]

Axion-thermalization rate

For decoupling temperatures $T_D \lesssim T_c \approx 150$ MeV or $0.1 < m_a < 1$ eV

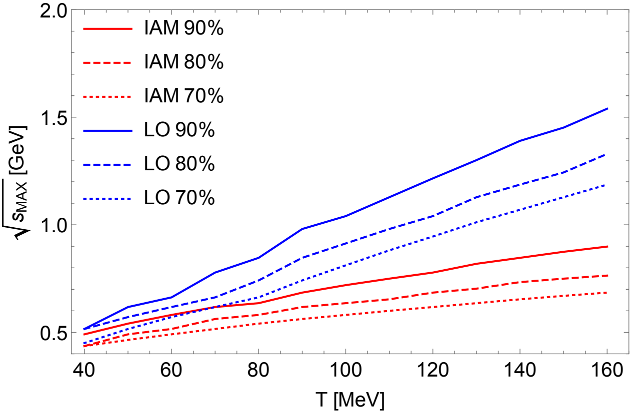
Main processes are $a\pi \leftrightarrow \pi\pi$

$$\Gamma = \frac{1}{n_{eq}} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4} \sum |T_{a\pi\to\pi\pi}|^2$$
$$\times (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4) f_1 f_2 (1 + f_3)(1 + f_3)$$
$$f_i = 1/(e^{E_i/T} - 1)$$



Good agreement with Notari, Rompineve, Villadoro, PRL131,011004(2023) better than $\sim 6\%$ (3% in the amplitude)

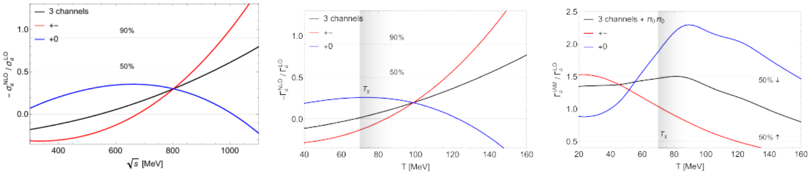
Range of energies needed is within the realm of the IAM



For Γ, most of the contribution ($\leq 90\%$) requires $\sqrt{s} < 1$ GeV

Theoretical error: Difference in Γ between the full integration and the one cut with $\sqrt{s_{\rm MAX}} = 1$ GeV

Chiral expansion limits



- Left: $-\Delta\sigma_{NLO}/\sigma_{LO}$. Maximum in $\pi^+\pi^0 \sim 35\%$ at around $\sqrt{s} = 600\,\text{MeV}$. Large correction at threshold for $\pi^+\pi^-$, $\sim 30\%$
- Middle: $-\Delta\Gamma_{NLO}/\Gamma_{LO}$. Maximum for $\pi^+\pi^0$ reached at $T \sim 70$ MeV
- Right: Γ_{IAM}/Γ_{LO}. 50% corrections are already reached at $T = 20\,\text{MeV}$ for $\pi^+\pi^-$ and at $T \sim 60\,\text{MeV}$ for $\pi^+\pi^0$. For the 3 channels summed up 50% is reached at $T \sim 70$ MeV

Similar values as $T_{max} \sim 60$ MeV deduced in Di Luzio,Martinelli,Piazza PRL(2022)

Decoupling temperature

Freeze-out condition Hannestad,Mirizzi,Raffelt,JCAP07,002(2005)

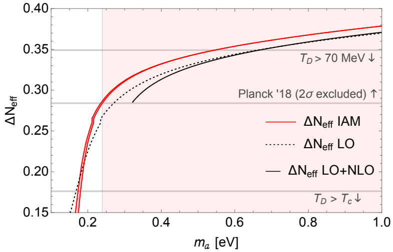
$$\Gamma(T_D) = H(T_D) = GT^2\sqrt{4\pi^3 g_*(T_D)/45}$$

g_*: # of effective relativistic dof Saikawa,Shirai,JCAP05,035(2018)

Constraint

$$\Delta N_{eff} = N_{eff} - N_\nu \approx \frac{4}{7}\left(\frac{43}{4g_S(T_D)}\right)^{4/3}$$

g_S: # of entropy dof
$N_\nu \approx 3.044$ in the SM



Planck data (2018)
$N_{eff} = 2.99^{+0.34}_{-0.33}$ 95% C.L.

$m_a < 0.24$ eV
$f_a > 2.5 \times 10^7$ GeV

Future:
CMB-S4 $\pm 0.3 \to \pm 0.06$

The same value as in Notari, Rompineve, Villadoro, PRL131,011004(2023) from an improved treatment of cosmological evolution

5.- Inclusion of the axion in the EFT

Basic axion Lagrangian for energies $\ll f_a$ with QCD

$$\mathcal{L}_a^{QCD} = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{1}{2} m_{a,0}^2 a^2 + \frac{\alpha_s}{8\pi}\frac{a}{f_a} G\tilde{G} - \bar{q}_L M_q q_R - \bar{q}_R M_q q_L$$
$$+ \frac{\partial_\mu a}{2 f_a}\bar{q} c_q^0 \gamma^\mu \gamma_5 q + \frac{1}{4} g_{a\gamma}^0 a F\tilde{F} + \ldots$$

- "Standard way": Remove the $aG\tilde{G}$ term by using an axial transformation [Fujikawa's function method]

$$q \to \exp(i\gamma_5 \frac{a}{2f_a} Q_a) q \ , \ \mathrm{Tr} Q_a = 1$$

$$M_q \to M_a = e^{i\frac{a}{2f_a}Q_a} M_q e^{i\frac{a}{2f_a}Q_a} \ , \quad c_q^0 \to c_q = c_q^0 - Q_a$$

$$g_{a\gamma} = g_{a\gamma}^0 - \frac{3\alpha}{2\pi f_a}\mathrm{Tr}(Q_a Q_{EM}^2)$$

One uses standard techniques in ChPT to handle with M_a and the isovector axial current $\propto \mathrm{Tr}[c_q^0 \sigma^i]$

Complications arise due to the isosinglet axial vector quark current $\mathrm{Tr}[c_q^0]$

LO axion-ChPT Lagrangian

See e.g. Di Luzio,Piazza,JHEP12,041(2022)

$$\mathcal{L}_a^{\chi(LO)} = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{1}{2}m_{a,0}^2 a^2 + \frac{f_\pi^2}{4}\mathrm{Tr}\left[(D^\mu U)^\dagger D_\mu U + U\chi_a^\dagger + \chi_a U^\dagger\right]$$
$$+ \frac{\partial^\mu a}{2f_a}\sum_{i=1}^{3}\mathrm{Tr}[c_q\sigma^i]\, J_{A,\mu}^i\big|_{LO}$$

$$\chi_a = 2B_0 M_a\ ,\ \ U = exp\left(i\pi^a\sigma^a/f_\pi\right)$$
$$\pi^a\sigma^a = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$$

$$m_{a,QCD}^2 = \frac{m_u/m_d}{1+(m_u/m_d)^2}\frac{m_\pi^2 f_\pi^2}{f_a^2} \approx 5.7\left(\frac{10^{12}\ \mathrm{GeV}}{f_a}\right)\ \mu eV\ .$$

Keeping $aG\tilde{G}$ term

$U(3)$ ChPT Leutwyler,PLB374,163(1996);Herrera-Siklody *et al*, NPB497,345(1997);Kaiser,Leutwyler,EPJC17,623(2000)

$U(3)_L \otimes U(3)_R$

$$q'_R = V_R q_R \ , \ \ q'_L = V_L q_L \ , \ \ V_R \in U(3)_L, \ V_R \in U(3)_R$$

We require $\mathcal{L}(q', \partial q', \ldots)$ to keep the same form as $\mathcal{L}(q, \partial q, \ldots)$ to be equivalent

Anomaly: Fermionic determinant

$$\det D' = e^{i\alpha\nu} \det D \ , \ \ e^{i\alpha} = \det V_R V_L^\dagger \ , \ \ \nu = \frac{\alpha_s}{8\pi^2} \int dx G\tilde{G}$$

This implies a **Theta** term $\mathcal{L}_{QCD} + \theta\alpha_s G\tilde{G}/8\pi^2$

$$M'_q = V_R M_q V_L^\dagger \ , \ \ \theta' = \theta - \alpha$$

Low-energy EFT: $N_C \to \infty$ the η_0 becomes a Goldstone boson

$$U = \exp(i\sqrt{2}\Phi/F)$$

$$\Phi = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & \pi^+ & K^+ \\ \pi^- & \frac{-1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 & K^0 \\ K^- & \overline{K}^0 & \frac{-2}{\sqrt{6}}\eta_8 + \frac{1}{\sqrt{3}}\eta_0 \end{pmatrix}$$

$$\mathrm{Det}\, U = \exp\left(i\sqrt{6}\eta_0/F\right)$$

Transformation

$$U(\Phi) \to V_R U(\Phi) V_L^\dagger$$

$$-i \log \mathrm{Det}\, U \to -i \log \mathrm{Det}\, U + \alpha$$

Invariance of the combination under $U_L(3) \otimes U_R(3)$

$$X \equiv \log \mathrm{Det}\, U + ia/f_a$$

and any function $f(X)$, in particular LEC's.

Large-N_C QCD and δ countings

N_C is the number of colors of QCD

Take the limit $N_C \to \infty$, then $\alpha_s \sim 1/N_C$

$$j = \bar{q}\Gamma q \;,\; \omega = G\tilde{G}$$
$$G_{n_j n_\omega} = \langle 0 | T j_1(x_1) \cdots j_n(x_n) \omega(y_1) \cdots \omega(y_{n_\omega}) | 0 \rangle \;,$$

δ counting

$G_{n_j n_\omega} \sim \sum_\ell \mathcal{O}(N_C^{2-\ell-n_\omega})$, ℓ is # of quark loops

$\delta \sim p^2 \sim 1/N_C$

For $N_C \to \infty$ the η_0 becomes the ninth Goldstone boson

It appears through: $U(\Phi)$ and $X = -i \log \mathrm{Det}\, U + a/f_a$

Gao,Guo,Oller,Zhou,JHEP04,022(2023) applies it to

Axion mixing, mass, and coupling to $\gamma\gamma$

$G_{n_j n_\omega} \sim \sum_\ell \mathcal{O}(N_C^{2-\ell-n_\omega})$

$$\mathcal{L}^{\rm LO} \sim \delta^0$$

$$\mathcal{L}^{\rm LO} = \underbrace{\frac{F^2}{4}\langle u_\mu u^\mu\rangle}_{N_C^{2-1}=N_C\,;\,F^2\sim N_C;\mathcal{O}(p^2)} + \frac{F^2}{4}\langle\chi_+\rangle + \underbrace{\frac{F^2}{12}M_0^2 X^2}_{\{N_C^{2-2}=1\,;\,F^2\sim N_C\}\to M_0^2\sim N_C^{-1}}$$

$$\mathcal{L}^{\rm NLO} \sim \delta$$

$$\mathcal{L}^{\rm NLO} = L_5\langle u^\mu u_\mu \chi_+\rangle + \frac{L_8}{2}\langle\chi_+\chi_+ + \chi_-\chi_-\rangle - \frac{F^2\Lambda_1}{12}D^\mu X D_\mu X - \frac{F^2\Lambda_2}{12}X\langle\chi_-\rangle$$

L_5 , $L_8 \sim N_C$; M_0^2 , Λ_1 , $\Lambda_2 \sim 1/N_C$

WZW terms

Change of the generating functional $Z[v, a, s, p]$ under chiral transformations Bardeen,PR184,1848(1969)

For the coupling to $\gamma\gamma$

$$\mathcal{L}^{\mathrm{LO}}_{WZW} = \mathcal{O}(p^4, N_C) = \mathcal{O}(\delta)$$
$$\mathcal{L}^{\mathrm{LO}}_{WZW} = -\frac{N_C\sqrt{2}e^2}{8\pi^2 F}\varepsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma \langle Q^2\Phi\rangle$$

which is counted as $O(p^4, N_C)$

$$\mathcal{L}^{\mathrm{NLO}}_{WZW} = \mathcal{O}(\delta^2)$$
$$\mathcal{L}^{\mathrm{NLO}}_{WZW} = i t_1 \varepsilon_{\mu\nu\rho\sigma} \underbrace{\langle f_+^{\mu\nu} f_+^{\rho\sigma}\chi_-\rangle}_{\mathcal{O}(p^6, N_C)} + k_3 \varepsilon_{\mu\nu\rho\sigma} \underbrace{\langle f_+^{\mu\nu} f_+^{\rho\sigma}\rangle X}_{\mathcal{O}(p^4, N_C^0)}$$

A^μ photon field, $f_+^{\mu\nu} = uF_L^{\mu\nu}u^\dagger + u^\dagger F_R^{\mu\nu} u$

$F^\mu_{L,R} = -eA^\mu Q$, $\;F^{\mu\nu}_{L,R} = \partial^\mu F^\nu_{L,R} - \partial^\nu F^\mu_{L,R} - i[F^\mu_{L,R}, F^\nu_{L,R}]$

6.- LO mixing and mass

- Isospin Limit for $\eta_8 - \eta_0$ mixing.

Exact diagonalization

$$\begin{pmatrix} \mathring{\eta} \\ \mathring{\eta}' \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix}$$

$$m^2_{\mathring{\eta}} = \frac{M_0^2}{2} + m_K^2 - \frac{\sqrt{M_0^4 - \frac{4M_0^2\Delta^2}{3} + 4\Delta^4}}{2}$$

$$m^2_{\mathring{\eta}'} = \frac{M_0^2}{2} + m_K^2 + \frac{\sqrt{M_0^4 - \frac{4M_0^2\Delta^2}{3} + 4\Delta^4}}{2}$$

$$\sin\theta = -\left(\sqrt{1 + \frac{\left(3M_0^2 - 2\Delta^2 + \sqrt{9M_0^4 - 12M_0^2\Delta^2 + 36\Delta^4}\right)^2}{32\Delta^4}}\right)^{-1}$$

$$\Delta^2 = m_K^2 - m_\pi^2$$

Perturbative treatment

- IB and F/f_a are kept as global factors only
- Mixing of axion field goes like F/f_a
- Isospin breaking (IB) mixing of π^0 with $\mathring{\eta}$, $\mathring{\eta}'$, and a

- Higher orders from NLO are treated perturbatively

LO diagonal fields: $\overline{\pi}^0$, $\overline{\eta}$, $\overline{\eta}'$, $\overline{a}$

$$m_{\overline{a}}^2 = m_{a,0}^2 + \frac{M_0^2 F^2}{6 f_a^2}\left[1 + \frac{c_\theta^2 M_0^2 (2m_{a,0}^2 - m_{\overline{\eta}'}^2)}{(m_{a,0}^2 - m_{\overline{\eta}'}^2)^2} + \frac{s_\theta^2 M_0^2 (2m_{a,0}^2 - m_{\overline{\eta}}^2)}{(m_{a,0}^2 - m_{\overline{\eta}}^2)^2}\right] + O(\epsilon)$$

QCD axion ($m_{a,0} = 0$)

$$m_{\overline{a}}^2 = \frac{F^2}{f_a^2}\frac{M_0^2 m_{\overline{\pi}}^2 (2m_{\overline{K}}^2 - m_{\overline{\pi}}^2)}{8M_0^2 m_{\overline{K}}^2 - 2m_{\overline{\pi}}^2(M_0^2 - 6m_{\overline{K}}^2) - 6m_{\overline{\pi}}^4}$$

In the limit m_π^2/M_0^2 and $m_\pi^2/m_K^2 \to 0$ one obtains the "standard" LO $SU(2)$ ChPT formula

NLO diagonal fields: $\hat{\pi}^0$, $\hat{\eta}$, $\hat{\eta}'$, $\hat{a}$

$$\begin{pmatrix} \hat{\pi}^0 \\ \hat{\eta} \\ \hat{\eta}' \\ \hat{a} \end{pmatrix} = \begin{pmatrix} 1 - x_{11} & -x_{12} - y_{12} & -x_{13} - y_{13} & -x_{14} - y_{14} \\ -x_{12} + y_{12} & 1 - x_{22} & -x_{23} - y_{23} & -x_{24} - y_{24} \\ -x_{13} + y_{13} & -x_{23} + y_{23} & 1 - x_{33} & -x_{34} - y_{34} \\ -x_{14} + y_{14} & -x_{24} + y_{24} & -x_{34} + y_{34} & 1 - x_{44} \end{pmatrix} \begin{pmatrix} \overline{\pi}^0 \\ \overline{\eta} \\ \overline{\eta}' \\ \overline{a} \end{pmatrix}$$

1) Matrix transformation→ diagonal kinetic term
2) Renormalize the fields to have canonical normalization
3) Final orthogonal transformation→ diagonal mass matrix

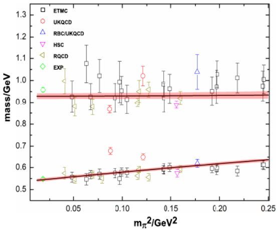
This is equivalent to the two-angle mixing formula

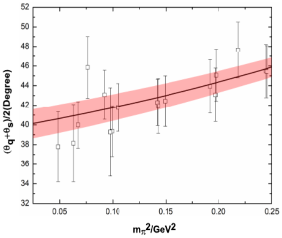
Leutwyler,NPBProcSupp64,223(1998)

$$\begin{pmatrix} \hat{\eta} \\ \hat{\eta}' \end{pmatrix} = \frac{1}{F} \begin{pmatrix} F_q \cos\theta_q & -F_s \sin\theta_s \\ F_q \sin\theta_q & F_s \cos\theta_s \end{pmatrix} \begin{pmatrix} \eta_q \\ \eta_s \end{pmatrix}$$

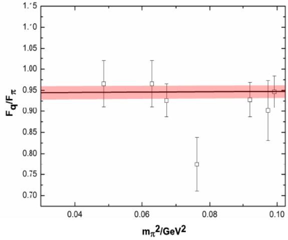
η_s and η_q: quark-flavor basis $s\bar{s}$ and $(u\bar{u} + d\bar{d})/\sqrt{2}$

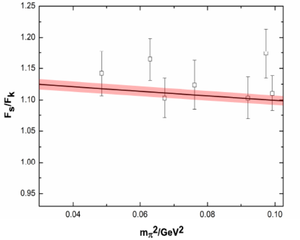
Fit to lattice QCD data



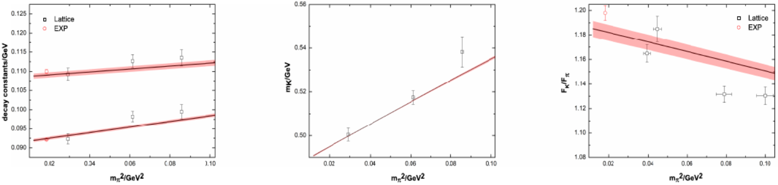


η'(top) and η(bottom) masses; ETMC,UKQCD,RBC/UKQCD,HSC,ETC





Two-mixing angle F_q and F_s; ETM



F_π, F_K and m_K; RBC, UKQCD, Durr *et al.* 1208.4412

Fitted parameters in the NLO fit

Parameters	NLO Fit
$F(\mathrm{MeV})$	$91.05^{+0.42}_{-0.44}$
$10^3 \times L_5$	$1.68^{+0.05}_{-0.06}$
$10^3 \times L_8$	$0.88^{+0.04}_{-0.04}$
Λ_1	$-0.17^{+0.05}_{-0.05}$
Λ_2	$0.06^{+0.08}_{-0.09}$
$\chi^2/(d.o.f)$	$219.9/(111-5)$

Mixing

$$\begin{pmatrix} \hat{\pi}^0 \\ \hat{\eta} \\ \hat{\eta}' \\ \hat{a} \end{pmatrix} = M^{\rm LO+NLO} \begin{pmatrix} \pi^0 \\ \eta_8 \\ \eta_0 \\ a \end{pmatrix}$$

$$M^{\rm LO+NLO} =$$

$$\begin{pmatrix} 1+(0.015\pm0.001) & 0.017+(-0.010\pm0.001) & 0.009+(-0.007\pm0.001) & \frac{12.1+(0.48\pm0.08)}{f_a} \\ -0.019+(0.007\pm0.001) & 0.94+(0.21\pm0.01) & 0.33+(-0.22\pm0.03) & \frac{34.3+(0.9\pm0.2)}{f_a} \\ -0.003+(-0.003\pm0.000) & -0.33+(-0.18\pm0.03) & 0.94+(0.13\pm0.02) & \frac{25.9+(-0.5\pm0.1)}{f_a} \\ \frac{-12.1+(-0.20\pm0.03)}{f_a} & \frac{-23.8+(1.6^{+0.8}_{-0.8})}{f_a} & \frac{-35.7+(-5.7^{+1.6}_{-1.7})}{f_a} & 1+\frac{27.6\pm1.0}{f_a^2} \end{pmatrix}$$

The axion mixing with the pseudoscalars is well behaved

For the other cases NLO contributions are still large

Masses

$$m_{\hat{\pi}} = \left[134.90 + (0.10 \pm 0.07)\right]\ \mathrm{MeV}$$
$$m_{\hat{K}} = \left[489.2 + (5.0^{+3.4}_{-3.5})\right]\ \mathrm{MeV}$$
$$m_{\hat{\eta}} = \left[490.2 + (60.9^{+10.2}_{-10.0})\right]\ \mathrm{MeV}$$
$$m_{\hat{\eta}'} = \left[954.3 + (-28.4^{+11.9}_{-12.6})\right]\ \mathrm{MeV}$$
$$m_{\hat{a}} = \left[5.96 + (0.12 \pm 0.02)\right]\ \mu\mathrm{eV}\ \frac{10^{12}\mathrm{GeV}}{f_a}$$

Clear improvement in the η mass (12% NLO effect)

Small correction (2%) for the axion

$\gamma\gamma$ couplings

$$\mathcal{L}^{\rm LO}_{WZW} = -\frac{3\sqrt{2}e^2}{8\pi^2 F}\varepsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma \left\{ \frac{1+x_{11}}{3\sqrt{2}}\hat{\pi}^0 \right.$$
$$+\left[\frac{c_\theta - 2\sqrt{2}s_\theta}{3\sqrt{6}}(1+x_{22}) + \frac{s_\theta + 2\sqrt{2}c_\theta}{3\sqrt{6}}(x_{23}-y_{23})\right]\hat{\eta}+$$
$$+\left[\frac{s_\theta + 2\sqrt{2}c_\theta}{3\sqrt{6}}(1+x_{33}) + \frac{c_\theta - 2\sqrt{2}s_\theta}{3\sqrt{6}}(x_{23}+y_{23})\right]\hat{\eta}'+$$
$$+\left[\frac{(c_\theta - 2\sqrt{2}s_\theta)v_{24} + (s_\theta + 2\sqrt{2}c_\theta)v_{34}}{3\sqrt{6}}\right.$$
$$\left.\left. + \frac{(c_\theta - 2\sqrt{2}s_\theta)(x_{24}+y_{24}) + (s_\theta + 2\sqrt{2}c_\theta)(x_{34}+y_{34})}{3\sqrt{6}}\right]\hat{a}\right\}$$

$$\mathcal{L}^{\rm NLO}_{WZW} = \frac{e^2}{F}\varepsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \partial^\rho A^\sigma \left\{ \frac{32 m_\pi^2 t_1}{3}\hat{\pi}^0 \right.$$
$$+\frac{32}{9\sqrt{3}}\left\{-9\sqrt{2}s_\theta k_3 + t_1\left[c_\theta(7m_\pi^2 - 4m_K^2) - 2\sqrt{2}s_\theta(m_K^2 + 2m_\pi^2)\right]\right\}\hat{\eta}$$
$$+\frac{32}{9\sqrt{3}}\left\{9\sqrt{2}c_\theta k_3 + t_1\left[2\sqrt{2}c_\theta(2m_\pi^2 + m_K^2) + s_\theta(7m_\pi^2 - 4m_K^2)\right]\right\}\hat{\eta}'$$
$$+\frac{32}{27}\left\{9k_3\left(\frac{F}{f_a} - \sqrt{6}s_\theta v_{24} + \sqrt{6}c_\theta v_{34}\right)\right.$$
$$+t_1\left[-\sqrt{3}s_\theta(4\sqrt{2}v_{24}m_\pi^2 - 7v_{34}m_\pi^2 + 2\sqrt{2}v_{24}m_K^2 + 4v_{34}m_K^2)\right.$$
$$\left.\left.\left.+\sqrt{3}c_\theta(4\sqrt{2}v_{34}m_\pi^2 + 7v_{24}m_\pi^2 + 2\sqrt{2}v_{34}m_K^2 - 4v_{34}m_K^2)\right]\right\}\hat{a}\right\}$$

$a \to \gamma\gamma$

$\mathcal{L}^{\rm LO}_{WZW}$ purely mixing contributions (up to NLO included)

$\mathcal{L}^{\rm NLO}_{WZW}$ direct (k_3) and mixing (up to LO included) contributions

We use the experimental decay widths

$$F^{\rm Exp}_{\pi^0\gamma\gamma} = 0.274 \pm 0.002 \text{ GeV}^{-1}$$
$$F^{\rm Exp}_{\eta\gamma\gamma} = 0.274 \pm 0.006 \text{ GeV}^{-1}$$
$$F^{\rm Exp}_{\eta'\gamma\gamma} = 0.344 \pm 0.008 \text{ GeV}^{-1}$$

to fix t_1 and k_3

$$F_{\pi^0\gamma\gamma} = 0.276 \pm 0.001 \quad \text{GeV}^{-1}$$
$$F_{\eta\gamma\gamma} = 0.276 \pm 0.009 \quad \text{GeV}^{-1}$$
$$F_{\eta'\gamma\gamma} = 0.343 \pm 0.012 \quad \text{GeV}^{-1}$$

Perfectly well reproduced with

$$t_1 = -(4.4 \pm 2.3) \times 10^{-4} \mathrm{GeV}^{-2}\,, \qquad k_3 = (1.25 \pm 0.23) \times 10^{-4}$$

$$F_{a\gamma\gamma} = -\frac{[20.1 + (0.5 \pm 0.1)] \times 10^{-3}}{f_a}$$

$$g_{a\gamma\gamma} = 4\pi\alpha_{em} F_{a\gamma\gamma} = -\frac{\alpha_{em}}{2\pi f_a}(1.63 \pm 0.01)$$

$g_{a\gamma\gamma} = 1.92 \pm 0.04$ SU(2) ChPT Grilli di Cortona *et al* JHEP01,034(2016)

$g_{a\gamma\gamma} = 2.05 \pm 0.03$ SU(3) ChPT Z.-Y.Lu *et al* JHEP05,001(2020)

No IB $\pi^0 - a$ mixing included in our result

This explains the difference $\sim 15\%$

Simple order of magnitude estimate:

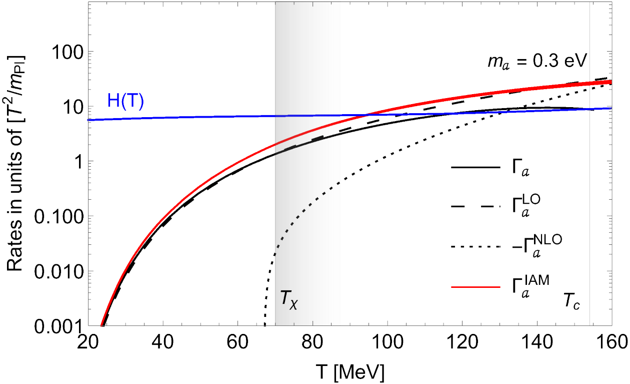
$$\sim \underbrace{\frac{12.1}{34.3 + 25.9}}_{mixing} \underbrace{\frac{F_{\pi^0\gamma\gamma}}{F_{\eta\gamma\gamma} + F_{\eta'\gamma\gamma}}}_{coupling} 1.63 = 0.15$$

7.- Conclusions

- Unitarization is convenient to reach higher energies and more precision for ChPT $a\pi \to \pi\pi$
- It is based on unitarity and analyticity
- Inverse Amplitude Method is applied, generation of the $\rho(770)$, σ or $f_0(500)$
- Novel application of $U(3)$ ChPT to axion physics: a, π, K, η, η' considered
- δ counting Combined chiral and large N_C power counting
- Calculations up to NLO: Mixing, masses and couplings to $\gamma\gamma$ of pseudoscalars and axion
- Study heavier ALP in K, η, η' decays, $a \to \gamma^{(*)}\gamma$, etc.

Back-up slides

Rates ($\propto f_a^{-2}$)



For $T \gtrsim 40$ MeV the IAM Γ_a is sizeably different to the perturbative ChPT one

Cutting the $\pi^+\pi^-$ and $\pi^0\pi^0$ rates at $\sqrt{s}_{\mathrm{MAX}} = 0.8$ GeV

- Total Γ_a is reduced by a 10% for $T = 150$ MeV
- Hot dark-matter bound now is $m_a \lesssim 0.25$ eV, instead of 0.24 eV
- At $T = 150$ MeV the errrorbar for Γ_a reaches 11% instead of the previous 7%

Change of unitarization method: Spread in the ρ and σ pole positions between IAM and N/D

$\delta[\sqrt{s}_\sigma] = (1.2\%, 2.4\%)$
IAM: NLO SU(2) and SU(3), NNLO SU(2) ChPT
N/D: NLO SU(2), NNLO U(3), tree-level ChPT

$\delta[\sqrt{s}_\rho] = (1\%, 2.7\%)$
IAM: NLO SU(2) and SU(3)
N/D: NNLO U(3) ChPT

This uncertainy is doubled for the cross section

It is much smaller than the one already accounted for $\lesssim 10\%$