The shift-invariant orders of an ALP

based on PRL **130**, 111803 | arXiv:2206.04182 in collaboration with Quentin Bonnefoy and Christophe Grojean

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HELMHOLTZ

Reminder: PQ solution to the strong CP problem

- axion arises as the Goldstone boson of the spontaneously broken $U(1)_{PQ}$ Peccei-Quinn symmetry
- PQ symmetry implies a **shift symmetry for the axion** $a \to a + \epsilon f$

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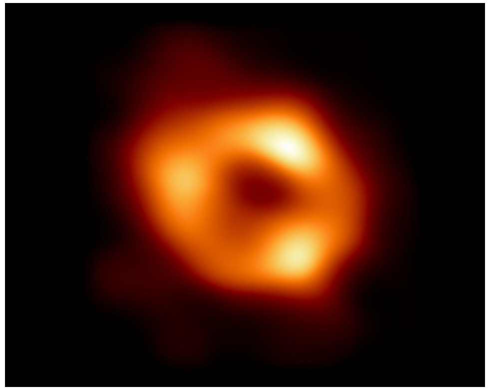
- axion arises as the Goldstone boson of the spontaneously broken $U(1)_{PQ}$ Peccei-Quinn symmetry
- PQ symmetry implies a **shift symmetry for the axion** $a \to a + \epsilon f$
- Can either look at specific models (like DFSZ, KSVZ) or use an EFT approach
- Only **derivative couplings** for ALP due to shift symmetry (dictated by CCWZ):

$$\mathcal{L}_a = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{m_{a,0}^2}{2}a^2 + \frac{\partial_\mu a}{f}\sum_{\psi\in\mathrm{SM}}\bar{\psi}c_\psi\gamma^\mu\psi + C_{G\tilde{G}}\frac{a}{f}G^a_{\mu\nu}\tilde{G}^{a\mu\nu}$$
$$+ C_{B\tilde{B}}\frac{a}{f}B_{\mu\nu}\tilde{B}^{\mu\nu} + C_{W\tilde{W}}\frac{a}{f}W^A_{\mu\nu}\tilde{W}^{A\mu\nu} + \mathcal{O}\left(\frac{1}{f^2}\right)$$

[Georgi, Kaplan, Randall, 1986]

Explicit PQ breaking

There are several reasons to believe that explicit PQ breaking is interesting. For example,



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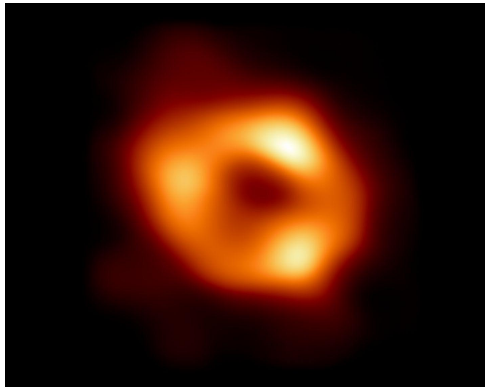

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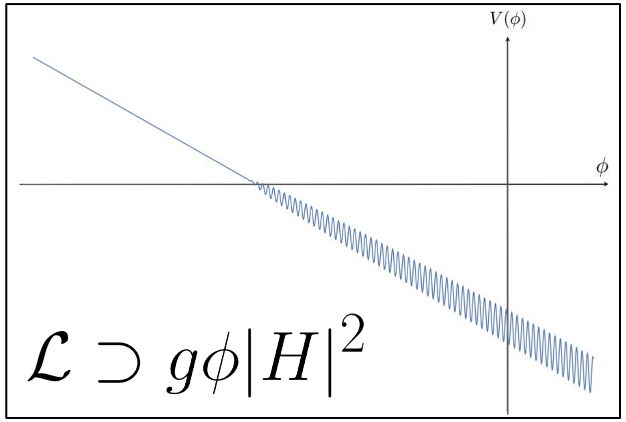

[Graham et al., 1504.07551]

- It can be interesting to allow for some explicit breaking from a **model building** perspective.

Hence, it is interesting to allow for some explicit breaking and to understand how to take the **limit from** the **non-shift symmetric to shift-symmetric EFT**.

Two bases for an ALP EFT

- If we also want to capture **shift breaking couplings** we have to consider the following Lagrangian

$$\mathcal{L}_a = \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{m_{a,0}^2}{2}a^2 - V\left(C_{V,i}, a, H\right) - \frac{a}{f}\left(\bar{Q}\tilde{Y}_u\tilde{H}u + \bar{Q}\tilde{Y}_d H d + \bar{L}\tilde{Y}_e H e + \text{h.c.}\right)$$

$$+ C_{G\tilde{G}}\frac{a}{f}G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + C_{B\tilde{B}}\frac{a}{f}B_{\mu\nu}\tilde{B}^{\mu\nu} + C_{W\tilde{W}}\frac{a}{f}W^A_{\mu\nu}\tilde{W}^{A\mu\nu}$$

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allowing for **shift-symmetric** and **shift-breaking** couplings.

- Shift-symmetric limit? Bosonic sector: simply take some $C_i \to 0$

Fermionic sector: ???

Two bases for an ALP EFT

- Can map the explicitly symmetric couplings on other basis

[Chala et al., 2012.09017]
[Bauer et al., 2012.12272]

$$\mathcal{L} \supset \frac{\partial_\mu a}{f} \sum_{\psi \in \mathrm{SM}} \bar{\psi} c_\psi \gamma^\mu \psi + \mathcal{O}\left(\frac{1}{f^2}\right) \longrightarrow \mathcal{L} \supset -\frac{a}{f}\left(\bar{Q}\tilde{Y}_u \tilde{H} u + \bar{Q}\tilde{Y}_d H d + \bar{L}\tilde{Y}_e H e + \mathrm{h.c.}\right) + \mathcal{O}\left(\frac{1}{f^2}\right)$$

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with the following relations

$$\tilde{Y}_u = i(Y_u c_u - c_Q Y_u) \qquad \tilde{Y}_d = i(Y_d c_d - c_Q Y_d) \qquad \tilde{Y}_e = i(Y_e c_e - c_L Y_e)$$

where c_Q, c_u, c_d, c_L, c_e hermitian.

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- But those relations are **implicit, flavour variant** and it is **unclear how to implement different power countings** in both sectors. Typically,

$$f \ll \Lambda_{\not{PQ}} \sim M_{\mathrm{Pl}}$$

Spontaneous PQ breaking (→ f)

Explicit PQ breaking (→ $\Lambda_{\not{PQ}}$)

Flavour invariants as order parameters

- We want to find equivalent relations that **vanish** for an **unbroken shift symmetry** and are **non-zero** for a **broken shift symmetry** $\Longrightarrow$ order parameter

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- Use language of flavour invariants, i.e. promote flavourful couplings to spurions under flavour transformations

	$SU(3)_Q$	$SU(3)_u$	$SU(3)_d$	$SU(3)_L$	$SU(3)_e$
$Y_u, \tilde{Y}_u$	$\mathbf{3}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$Y_d, \tilde{Y}_d$	$\mathbf{3}$	$\mathbf{1}$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$
$Y_e, \tilde{Y}_e$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\bar{\mathbf{3}}$

and build flavour-invariant combinations, e.g. $\mathrm{Tr}(X_u) \to \mathrm{Tr}(U_Q X_u U_Q^\dagger) = \mathrm{Tr}(U_Q^\dagger U_Q X_u) = \mathrm{Tr}(X_u)$

$$X_{u,d,e} = Y_{u,d,e} Y_{u,d,e}^\dagger$$

Example: order parameter in the SM

[Jarlskog, 1985]

- Famous example in the SM:

  all **CP breaking** in the SM is given by one flavour-invariant quantity:

$$J_4 \equiv \mathrm{Im}\mathrm{Tr}\left([X_u, X_d]^3\right) = 6(y_t^2 - y_c^2)(y_t^2 - y_u^2)(y_c^2 - y_u^2)(y_b^2 - y_s^2)(y_b^2 - y_d^2)(y_s^2 - y_d^2)\mathcal{J}$$

where $$X_{u,d} = Y_{u,d}Y_{u,d}^\dagger \qquad \mathcal{J} = \mathrm{Im}\left(V_{us}V_{cb}V_{ub}^*V_{cs}^*\right)$$

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where $X_{u,d} = Y_{u,d} Y_{u,d}^\dagger$ $\qquad \mathcal{J} = \mathrm{Im}\left(V_{us} V_{cb} V_{ub}^* V_{cs}^*\right)$

We want to find a **set of order parameters** which **capture all physical shift-breaking degrees** of freedom in all degenerate cases like the Jarlskog invariant.

Order parameters for an axion shift symmetry

- One can construct the following set of order parameters that are equivalent to the implicit relations:

$$I_e^{(1)} = \mathrm{Re}\,\mathrm{Tr}\left(\tilde{Y}_e Y_e^\dagger\right), \qquad I_e^{(2)} = \mathrm{Re}\,\mathrm{Tr}\left(X_e \tilde{Y}_e Y_e^\dagger\right), \qquad I_e^{(3)} = \mathrm{Re}\,\mathrm{Tr}\left(X_e^2 \tilde{Y}_e Y_e^\dagger\right)$$

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Minimal set capturing all conditions: $I_e^{(1)}, I_e^{(2)}, I_e^{(3)}, I_u^{(1)}, I_u^{(2)}, I_d^{(1)}, I_d^{(2)}, I_u^{(3)} + I_d^{(3)}, I_{ud}^{(1)}, I_{ud,u}^{(2)}, I_{ud,d}^{(2)}, I_{ud}^{(3)}, I_{ud}^{(4)}$

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- For schematic breaking of PQ: $\mathcal{L} = \mathcal{L}_{\mathrm{PQ}}(\vec{C}_{PQ}) + \epsilon \mathcal{L}_{\not{\mathrm{PQ}}}(\vec{C}_{\not{PQ}})$

  All invariants proportional to spurion explicitly breaking PQ: $I_i \propto \epsilon f(\vec{C}_{PQ}, \vec{C}_{\not{PQ}})$

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Collectivity: up- & down-couplings have to conspire to give PQ breaking

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For non-linear EWSB: correlations from linear EWSB in EFT disappear and all shift-breaking effects are captured by 9 invariants

$$X_{u,d,e} = Y_{u,d,e} Y_{u,d,e}^\dagger$$

CP properties of the invariants

$\frac{a}{f}\bar{\psi}_L \tilde{Y}_\psi H \psi_R$

- In quark sector, 9 CP-odd and 1 CP-even relation (c.f. Re vs Im)

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CP conservation **almost** implies shift invariance

CP properties of the invariants

$$\frac{a}{f}\bar{\psi}_L \tilde{Y}_\psi H \psi_R$$

- In quark sector, 9 CP-odd and 1 CP-even relation (c.f. Re vs Im)

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CP conservation **almost** implies shift invariance

- Studied the RGEs of the invariants. E.g., running of CP-even invariant:

$$\dot{I}_{ud}^{(4)} = 6\left(\gamma_u + \gamma_d + \frac{1}{2}\mathrm{Tr}(X_u + X_d)\right) I_{ud}^{(4)} - \mathrm{Im\,Tr}\left([X_u, X_d]^3\right)\left(I_u^{(1)} + I_d^{(1)}\right)$$

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Applications – EDMs

[Di Luzio et al., 2010.13760]

- We can identify the breaking parameters in low-energy observables like EDMs.
- Example: **EDM of mercury** can be expressed as follows:

$$\mathcal{L} \supset -\frac{G_F}{\sqrt{2}} C_S \bar{N} N \bar{e} i \gamma_5 e$$

$$d_{\rm Hg} \simeq 4.0 \cdot 10^{-4} d_n - [2.8\, C_S - 2.1\, C_P]\, 10^{-22}\, e\,{\rm cm}$$

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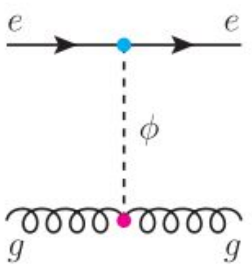
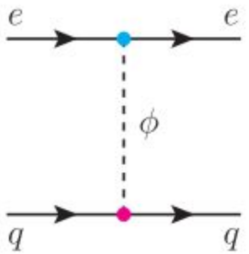
$$\frac{C_S}{v^2} = -17(C_{ue} + C_{de}) + 4.7 \text{ GeV } C_{Ge} \qquad \frac{C_P}{v^2} = 350(C_{eu} + C_{ed}) + 1.1 \text{ GeV } C_{\tilde{G}e}$$





- Matching the ALP EFT to those operators gives

$$C_{ij} \simeq \frac{v^2}{\Lambda^2} \frac{y_S^{ii} y_P^{jj}}{m_\phi^2}, \qquad C_{Ge} = \frac{4\pi}{m_\phi^2} \frac{v}{\Lambda^2} C_g y_P^{ee}$$

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More complete analysis in
[Das Bakshi et al., 2306.08036]
→talk by Maria

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Outlook: beyond the leading order

[Grojean, JK, Yao, 2307.08563]

- Can we extend this discussion beyond the leading order in $1/f$?

- Used Hilbert series to derive operator basis up to dim-8 for the ALP EFT

- Find that there are no further EOM redundancies up to dim-15 like the one at dim-5

$$\mathcal{L} \supset \frac{\partial_\mu a}{f} \sum_{\psi \in \mathrm{SM}} \bar{\psi} c_\psi \gamma^\mu \psi + \mathcal{O}\left(\frac{1}{f^2}\right) \longrightarrow \mathcal{L} \supset -\frac{a}{f}\left(\bar{Q}\tilde{Y}_u \tilde{H} u + \bar{Q}\tilde{Y}_d H d + \bar{L}\tilde{Y}_e H e + \mathrm{h.c.}\right) + \mathcal{O}\left(\frac{1}{f^2}\right)$$

- Have to keep track of field redefinition that removes redundancy at higher mass dimension, e.g. for $\mathcal{L} \supset \frac{a^2}{f^2} \bar{Q} C_{a^2 u} \tilde{H} u + \mathrm{h.c.}$:

$$C_{a^2 u} = \frac{1}{2}\left(C_q^2 Y_u + Y_u c_u^2\right) - c_Q Y_u c_u$$

Summary

- Have considered **shift-breaking effects** in **ALP EFTs**
- Derived **order parameters** for shift-breaking that vanish for shift symmetry and are non-zero for shift-breaking
- Have studied **interesting properties** of these order parameters under **CP transformations**, in the **EFT below the EW scale** and for **different scenarios of EWSB**
- Can use the invariants to **differentiate between a generic pseudoscalar and an ALP** by identifying the invariants **in calculation of observables**
- Work in progress: extend construction to higher order in EFT

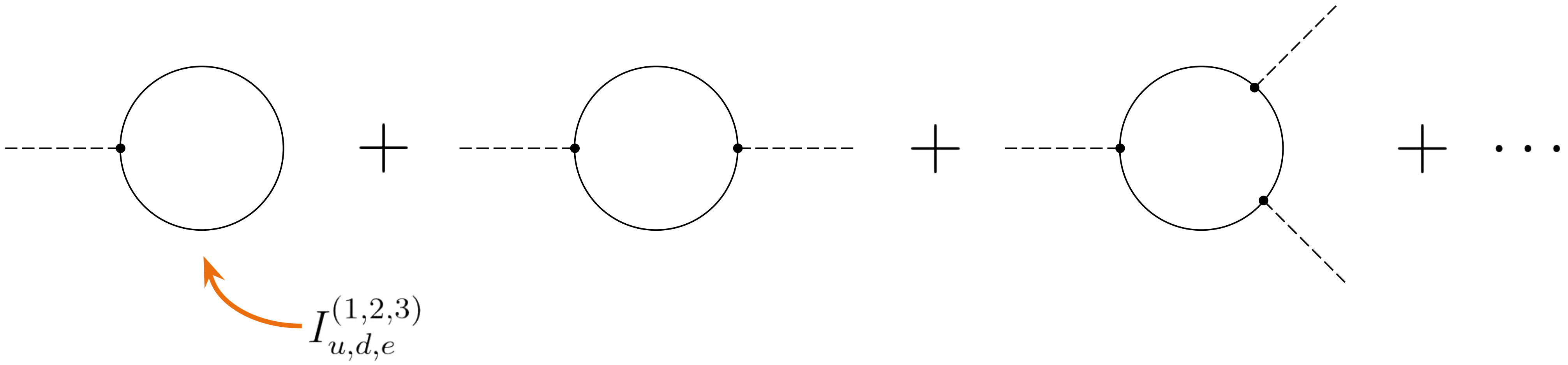
Thank you!

Backup

Applications – Coleman-Weinberg potential

- We expect that all PQ-breaking quantities in the theory are proportional to the shift-breaking invariants.
- Example: Coleman-Weinberg potential of an ALP [Bonnefoy, 2212.00102]

Calculate the 1-loop corrections to the potential to all orders in external fields



One can show that leading correction to potential is proportional to a subset of the shift-breaking invariants. This correction can change the minimum of the potential, the physical theta term.

Shift-symmetry in non-shift symmetric basis

We can check that the ALP EFT in the Yukawa basis can be made shift symmetric by itself.
First, shift the ALP in the ALP-fermion sector of the Lagrangian:

$$\mathcal{L}_a = -\left(\bar{Q}Y_u\tilde{H}u + \bar{Q}Y_d H d + \bar{L}Y_e He + \text{h.c.}\right) - \frac{a}{f}\left(\bar{Q}\tilde{Y}_u\tilde{H}u + \bar{Q}\tilde{Y}_d H d + \bar{L}\tilde{Y}_e He + \text{h.c.}\right)$$

$$\to -\bar{Q}Y_u\tilde{H}u - \bar{Q}Y_d H d - \bar{L}Y_e He - \frac{a+c}{f}\left(-\bar{Q}\tilde{Y}_u\tilde{H}u + \bar{Q}\tilde{Y}_d H d + \bar{L}\tilde{Y}_e He\right) + \text{h.c.}$$

Now, we can make a field redefinition $\psi \to \psi + i\frac{c}{f}c_\psi\psi$ on the fermions

$$\to \mathcal{L}_{\text{SM}} - \bar{Q}i\left(Y_u c_u - c_Q Y_u\right)\tilde{H}u - \bar{Q}i\left(Y_d c_d - c_Q Y_d\right)Hd - \bar{L}i\left(Y_e c_e - c_L Y_e\right)He$$
$$-\frac{a+c}{f}\left(-\bar{Q}\tilde{Y}_u\tilde{H}u + \bar{Q}\tilde{Y}_d H d + \bar{L}\tilde{Y}_e He\right) + \text{h.c.}$$

where the c_ψ are hermitian s.t. the kinetic term is invariant. We can absorb the shift iff

$$\tilde{Y}_u = i(Y_u c_u - c_Q Y_u) \qquad \tilde{Y}_d = i(Y_d c_d - c_Q Y_d) \qquad \tilde{Y}_e = i(Y_e c_e - c_L Y_e)$$

Counting of physical parameters

- Before we can construct the explicit relations we have to count how many physical parameters exist in both bases.

	Explicitly shift-symmetric basis		Non-shift-symmetric basis	
Leptons	c_L, c_e	$2 \times 6 - \underline{3} = 9$ CP-even $2 \times 3 - \underline{2} = 4$ CP-odd	$\tilde{Y}_e$	$1 \times 9 = 9$ CP-even $1 \times 9 - \underline{2} = 7$ CP-odd
	$\frac{\partial_\mu a}{f} \overline{\psi} c_\psi \gamma^\mu \psi$		$\frac{a}{f} \overline{\psi}_L \tilde{Y}_\psi H \psi_R$	$\underline{U(1)^3_{L_i} \text{ reph.}}$ $\underline{\partial_\mu j^\mu = 0}$
Quarks	c_Q, c_u, c_d	$3 \times 6 - \underline{1} = 17$ CP-even $3 \times 3 = 9$ CP-odd	$\tilde{Y}_u, \tilde{Y}_d$	$2 \times 9 = 18$ CP-even $2 \times 9 = 18$ CP-odd

We expect **3 CP-odd** relations in **lepton** sector and **9 CP-odd** and **1 CP-even** relation in **quark** sector.

Construction of invariants

- Start from relation obtained from enforcing shift invariance via field redefinitions:

$$\tilde{Y}_{u,d} = i(Y_{u,d}c_{u,d} - c_Q Y_{u,d}) \quad \Longleftrightarrow \quad c_{u,d} = -iY_{u,d}^{-1}\left(\tilde{Y}_{u,d} + ic_Q Y_{u,d}\right)$$

- Then, enforce hermiticity of $c_{u,d}$

$$X_{u,d} = Y_{u,d}Y_{u,d}^{\dagger}$$

$$c_{u,d}^{(ah)} \sim c_{u,d} - c_{u,d}^{\dagger} = 0 \quad \Longrightarrow \quad [c_Q, X_{u,d}] = i\left(\tilde{Y}_{u,d}Y_{u,d}^{\dagger} + Y_{u,d}\tilde{Y}_{u,d}^{\dagger}\right)$$

- Use well-known commutator relations to construct invariants.

E.g. using $\mathrm{Tr}\left(A^n\left[A,B\right]\right) = 0 \quad \forall n \in \mathbb{Z}$ can get

$$-i\,\mathrm{Tr}\left(X_x^n\left[c_Q, X_x\right]\right) = \mathrm{Tr}\left(X_x^n\left(\tilde{Y}_x Y_x^{\dagger} + Y_x\tilde{Y}_x^{\dagger}\right)\right) = 0$$

Since we started with the equations which characterize shift-symmetric ALP-fermion interactions the **last equation is only zero iff the couplings are shift-symmetric.**

Redundancies in invariants

During the construction of the invariants and the calculation of their RGEs other flavour invariants arise which **look different** from the invariants **in the minimal set** but **can be expressed in terms of them.** Example: Can write down another beyond invariants $I_e^{(1,2,3)} = \mathrm{ReTr}\left(X_e^{0,1,2}\tilde{Y}_e Y_e^\dagger\right)$ in minimal set.

$$I_e^{(4)} = \mathrm{ReTr}\left(X_e^3 \tilde{Y}_e Y_e^\dagger\right)$$

- Use Cayley-Hamilton theorem for 3 x 3 matrices:

$$A^3 = A^2\,\mathrm{Tr}\,A - \frac{1}{2}A\left((\mathrm{Tr}\,A)^2 - \mathrm{Tr}\,A^2\right) + \frac{1}{6}\mathbb{1}\left((\mathrm{Tr}\,A)^3 - 3\,\mathrm{Tr}\,A^2\,\mathrm{Tr}\,A + 2\,\mathrm{Tr}\,A^3\right)$$

- Then, can write additional invariant in terms of invariants in minimal set

$$I_e^{(4)} = \mathrm{Tr}(X_e) I_e^{(3)} - \frac{1}{2}\left((\mathrm{Tr}\,X_e)^2 - \mathrm{Tr}\,X_e^2\right) I_e^{(2)} + \frac{1}{6}\left((\mathrm{Tr}\,X_e)^3 - 3\ \mathrm{Tr}\,X_e^2\,\mathrm{Tr}\,X_e + 2\,\mathrm{Tr}\,X_e^3\right) I_e^{(1)}$$

- Further example:

$$\begin{aligned} I_u' =& \frac{1}{2} I_u^{(1)} \left(\mathrm{Tr}(X_u)^2\,\mathrm{Tr}(X_d) - \mathrm{Tr}\left(X_u^2\right)\mathrm{Tr}(X_d) + 2\,\mathrm{Tr}\left(X_u^2 X_d\right) - 2\,\mathrm{Tr}(X_u)\,\mathrm{Tr}(X_u X_d)\right) \\ &+ 2 I_u^{(2)} \left(\mathrm{Tr}(X_u X_d) - \mathrm{Tr}(X_u)\,\mathrm{Tr}(X_d)\right) + 2\,\mathrm{Tr}(X_d) I_u^{(3)} + \frac{1}{2}\left(\mathrm{Tr}\left(X_u^2\right) - \mathrm{Tr}(X_u)^2\right) I_{ud}^{(1)} \\ &+ \mathrm{Tr}(X_u) I_{ud,u}^{(2)} + \frac{1}{6}\left(\mathrm{Tr}(X_u)^3 - 3\,\mathrm{Tr}\left(X_u^2\right)\mathrm{Tr}(X_u) + 2\,\mathrm{Tr}\left(X_u^3\right)\right) I_d^{(1)}\ , \end{aligned}$$

Invariants for degenerate mass spectra

Degeneracies in the SM **spectrum** can lead to **enhanced flavour symmetries**. Thus, **parameters become unphysical** and can be removed with symmetry transformations. This **reduces** the **number of expected relations**.

	Shift-symmetric Wilson coefficients $c_{Q,u,d}$				Generic Wilson coefficients $\tilde{Y}_{u,d}$				Number of constraints			
	All		Primary		All		Primary		All		Primary (# of indep. invariants)	
Flavor symmetries of the quark sector of the SM	CP-even	CP-odd	CP-even	CP-odd	CP-even	CP-odd	CP-even	CP-odd	CP-even	CP-odd	CP-even	CP-odd
$U(1)_B$	17	9	17	9	18	18	18	18	1	9	1	9
$U(1)^2$	16	8	10	3	18	17	10	10	2	9	0	7
$U(1)^3$	15	7	6	0	18	16	6	6	3	9	0	6
$U(2) \times U(1)$	13	5	4	0	17	15	4	4	4	10	0	4
$U(3)$	9	1	2	0	15	13	2	2	6	12	0	2

C.f.: CP violation in the SM. CKM phase can be removed for degenerate SM mass spectrum implying vanishing Jarlskog invariant:

$$J_4 \equiv \mathrm{ImTr}\left([X_u, X_d]^3\right) = 6(y_t^2 - y_c^2)(y_t^2 - y_u^2)(y_c^2 - y_u^2)(y_b^2 - y_s^2)(y_b^2 - y_d^2)(y_s^2 - y_d^2)\mathcal{J}$$

$$X_{u,d} = Y_{u,d} Y_{u,d}^\dagger \qquad \mathcal{J} = \mathrm{Im}\left(V_{us} V_{cb} V_{ub}^* V_{cs}^*\right)$$

Invariants in LEFT+a

- We can repeat our **discussion in the EFT below the EW** scale where all **heavy particles** (H,W,Z,t) are **integrated out**. We have following Lagrangian for the fermion sector

$$\mathcal{L} \supset -\bar{u}_L m_u u_R - \bar{d}_L m_d d_R - \bar{e}_L m_e e_R + h.c. \qquad \mathcal{L} \supset -\frac{a}{f}\left(\bar{u}_L \tilde{m}_u u_R + \bar{d}_L \tilde{m}_d d_R + \bar{e}_L \tilde{m}_e e_R + h.c.\right)$$

- Can write down the following invariants which are the same as 'lepton-like' invariants in SMEFT

$$I_x^{(i+1,IR)} \equiv \mathrm{Tr}\left(X_x^{i=0,1,\dots,N_x-1} \tilde{m}_x m_x^\dagger\right) = 0 \qquad x = u, d, e,\ N_u = 2, N_{d,e} = 3$$

- Can use LEFT operators with EFT power counting E/v to build more invariants. Then, we find at $\mathcal{O}\left(\frac{1}{fv^2}\right)$:

$$I_{ud}^{(1,IR)} \propto \mathrm{Re}\left(L^{V1,LL}_{uddu,prst}\left[\left(m_d m_d^\dagger\right)_{rs}\left(\tilde{m}_u m_u^\dagger\right)_{tp} + \left(\tilde{m}_d m_d^\dagger\right)_{rs}\left(m_u m_u^\dagger\right)_{tp}\right]\right)$$

- Matching the SMEFT to the LEFT allows us to identify this with the following invariant:

$$I_{ud}^{(1,IR)} \equiv \mathrm{Re}\,\mathrm{Tr}\left(V_{\mathrm{CKM}} m_d m_d^\dagger V_{\mathrm{CKM}}^\dagger \tilde{m}_u m_u^\dagger + V_{\mathrm{CKM}} \tilde{m}_d m_d^\dagger V_{\mathrm{CKM}}^\dagger m_u m_u^\dagger\right) = 0$$

EFT operators encode intermediate UV behaviour like the collectivity of the SMEFT due to linear EWSB. **Collective effects** are therefore **suppressed in the LEFT power counting.**

Non-linear EWSB

Another viable option for EWSB beyond the linear EWSB with a Higgs and fermion doublet are a class of **non-linear** scenarios described by **HEFT-like EFTs** where the **dofs in the doublets** of linear EWSB are **independent** of one another.

- For non-linear EWSB can package Goldstones into matrix as $U = e^{i\pi^a\sigma^a/v}$
- Treating the Higgs and U as independent we can write down more interactions

$$\frac{\partial_\mu a}{f}\sum_{\psi=Q,L}\bar{\psi}\tilde{c}_\psi T\gamma^\mu\psi\ , \quad T \equiv U\sigma_3 U^\dagger$$

and

$$\frac{a}{f}\left(\bar{Q}_L U\left[K_Q + \sigma_3\tilde{K}_Q\right]\begin{pmatrix}u_R\\ d_R\end{pmatrix} + \bar{L}_L U\left[K_L + \sigma_3\tilde{K}_L\right]\begin{pmatrix}0\\ e_R\end{pmatrix}\right)$$

Repeating the counting now gives **3 CP-odd** relations in the **lepton sector** and **6 CP-odd** relations in the **quark sector** which look like

$$I^{(1,2,3)}_{u,d,e} = \mathrm{ReTr}\left(X^{0,1,2}_{u,d,e}\,\tilde{Y}_{u,d,e}Y^\dagger_{u,d,e}\right)$$

Non-linear EWSB decorrelates the dofs in the doublets and **removes the collective effects**.

Matching: Two Higgs Doublet Model

- Start with Lagrangian of 2HDM which fixes PQ charges

$$-\mathcal{L} = \bar{Q} Y_u^{(1)} \tilde{H}_1 u + \bar{Q} Y_d^{(2)} H_2 d + \text{h.c.}$$

- Lagrangian gives rise to the following SM and axion Yukawa couplings

$$Y_u = \frac{v_1}{v} Y_u^{(1)} \,, \quad Y_d = \frac{v_2}{v} Y_d^{(2)} \,, \quad \tilde{Y}_u = -i q_{H_1} Y_u \,, \quad \tilde{Y}_d = i q_{H_2} Y_d$$

- Evaluating our invariants gives $I_i = 0 \; \forall i$ as expected

- Add PQ breaking interactions: $-\mathcal{L}_{\not{PQ}} = Y_{u,ij}^{(2)} \bar{Q}_i \tilde{H}_2 u_j + \text{h.c.}$ with $Y_{u,ij}^{(2)} = \delta_{i1}\delta_{j1} Y_{u,11}^{(2)}$

  giving the following couplings: $Y_u = \frac{v_1}{v} Y_u^{(1)} + \frac{v_2}{v} Y_u^{(2)} \,, \qquad \tilde{Y}_u = -i q_{H_1} \frac{v_1}{v} Y_u^{(1)} - i q_{H_2} \frac{v_2}{v} Y_u^{(2)}$

- Then, for instance: $I_u^{(1)} = -(q_{H_1} - q_{H_2}) \frac{v_1 v_2}{v^2} \text{Im}(Y_{u,11}^{(2)} Y_{u,11}^{(1)\,*})$

Applications - SMEFT RG running

[Galda et al., 2105.01078]

- Can study RG running of SMEFT operators induced by ALP couplings. Define source terms

$$\mu \frac{dC_i^{\rm SMEFT}}{d\mu} - \gamma_{ji}^{\rm SMEFT} C_j^{\rm SMEFT} \equiv \frac{S_i}{(4\pi f)^2}$$

$$\mathcal{O}_{uG} = \bar{Q}\sigma^{\mu\nu}\tilde{H}T^a u G^a_{\mu\nu}$$

$$S_{uG} = -4ig_s\tilde{Y}_u C_{GG}$$

- Can write down the following sum rules for source terms

$$\mathrm{Im\,Tr}\left(X_x^{0,1,2} S_{xG} Y_x^\dagger\right) = -4g_s C_{GG} I_x^{(1,2,3)}$$

$$\mathrm{Im\,Tr}\left(X_x^{0,1,2} S_{xW} Y_x^\dagger\right) = -g_2 C_{WW} I_x^{(1,2,3)}$$

$$\mathrm{Im\,Tr}\left(X_x^{0,1,2} S_{xB} Y_x^\dagger\right) = -g_1 (y_Q + y_x) C_{BB} I_x^{(1,2,3)}$$

$$\mathrm{Im\,Tr}\left(X_d X_u X_d S_{uG} Y_u^\dagger + X_u X_d X_u S_{dG} Y_d^\dagger\right) = -4g_s C_{GG} I_{ud}^{(3)}$$

- Expect some **non-trivial zeros** if the ALP is **exactly shift-symmetric**.
- Observations compatible with SMEFT RGEs would suggest weak PQ breaking, while uncertainty of measurements implies bound on breaking