King Fits: Bounds on Light New Physics from Isotope Shifts

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Physikalisch-Technische Bundesanstalt
Nationales Metrologieinstitut

axions++ 2023

28th September 2023

Outline

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Introduction to King-Plots

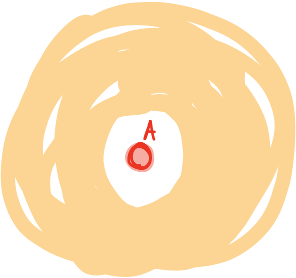
- Isotope Shifts
- The King-Plot Construction
- Bounds on New Bosons

Global Fit to King-Plot Data

Ytterbium King-Plots

Isotope Shifts

Here's an atom:



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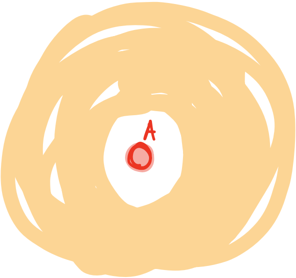

$$\frac{m_e}{m_A} \ll 1, \qquad \frac{r_{\text{nucleus}}}{r_{\text{atom}}} \ll 1$$

⇒ Can assume factorisation of electronic and nuclear contributions.

Isotope Shifts

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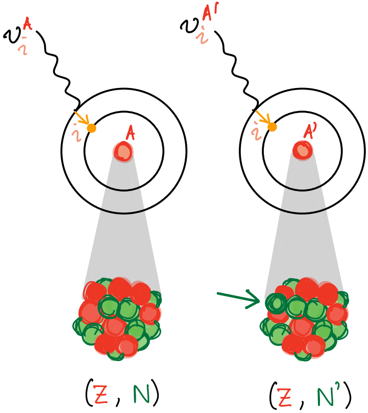

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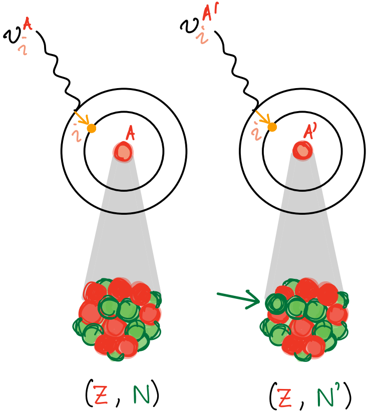


Isotope shifts:

$$\nu_i^{AA'} = K_i \mu^{AA'} + F_i \delta\langle r^2\rangle^{AA'} + \ldots$$

$\nu_i^{AA'} \equiv \nu_i^A - \nu_i^{A'}$

i: transition index

AA': isotope pair index

$K_i, F_i, \ldots$: electronic coeffs.

$\mu^{AA'}, \delta\langle r^2\rangle^{AA'}, \ldots$: nuclear coeffs.

Z: number of protons

N, N': number of neutrons in A, A'

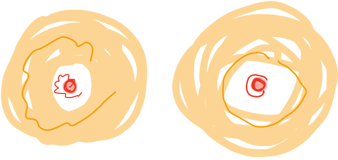
Isotope Shifts: Mass Shift & Field Shift

$$\nu_i^{AA'} = K_i \mu^{AA'} + F_i \delta\langle r^2\rangle^{AA'} + \dots$$

Mass Shift

Different motion of the nucleus
⇒ Correction to e^- kin. energy

$$\mu^{AA'} = \frac{1}{m^A} - \frac{1}{m^{A'}}$$



Field Shift

Different nucl. charge distrib.
⇒ Different contact interactions betw. e^- & nuclei

$$\delta\langle r^2\rangle^{AA'} = \langle r^2\rangle^{A} - \langle r^2\rangle^{A'}$$

The King-Plot: Invert a System of Linear Equations

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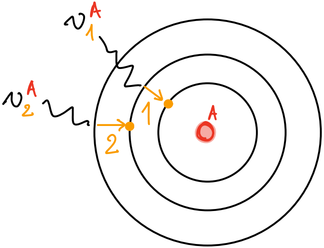

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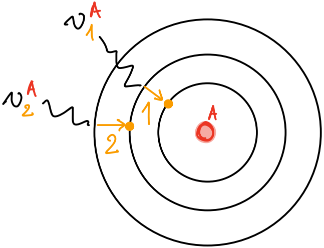


Isotope shifts:

$$\nu_1^{AA'} = K_1\mu^{AA'} + F_1\delta\langle r^2\rangle^{AA'}$$
$$\nu_2^{AA'} = K_2\mu^{AA'} + F_2\delta\langle r^2\rangle^{AA'}$$

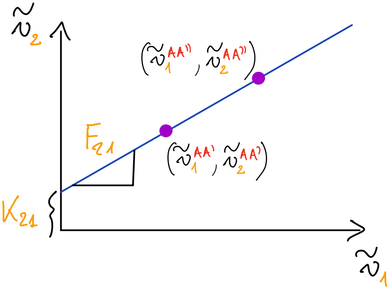
Eliminate charge radius variance $\delta\langle r^2\rangle^{AA'}$

$$\tilde{\nu}_2^{AA'} = K_{21} + F_{21}\tilde{\nu}_1^{AA'}$$

$\tilde{\nu}_i^{AA'} \equiv \nu_i^{AA'}/\mu^{AA'}$ ⇒ data

$F_{21} \equiv F_2/F_1 \quad K_{21} \equiv K_2 - F_{21}K_1$ ⇒ fit

The King-Plot: Fit to Isotope Shift Data



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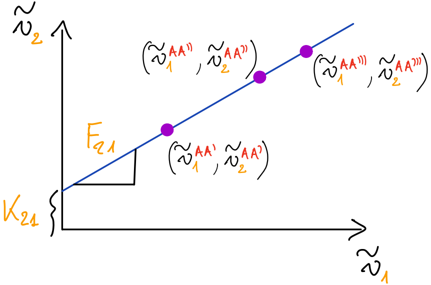

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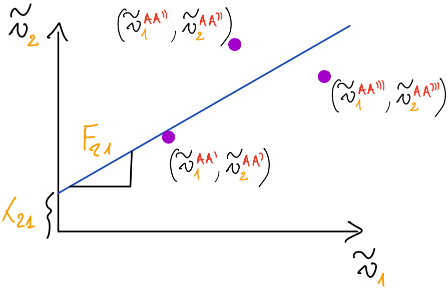

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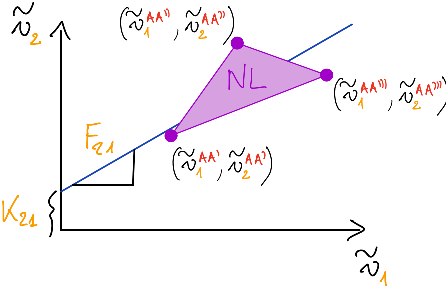

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Bounds on New Bosons

...How about using King-Plots for the search for new physics?
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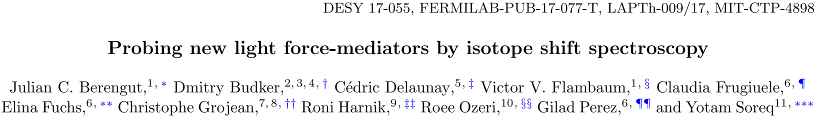

DESY 17-055, FERMILAB-PUB-17-077-T, LAPTh-009/17, MIT-CTP-4898

Probing new light force-mediators by isotope shift spectroscopy

Julian C. Berengut,[1, *] Dmitry Budker,[2, 3, 4, †] Cédric Delaunay,[5, ‡] Victor V. Flambaum,[1, §] Claudia Frugiuele,[6, ¶]
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Search for new mediator ϕ between the nucleus and bound e^-

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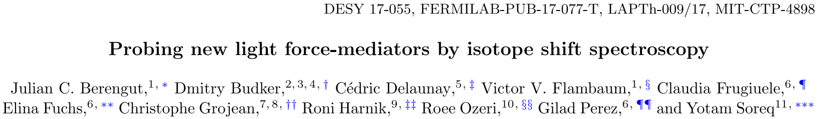

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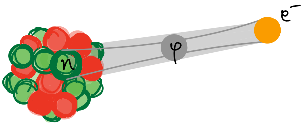
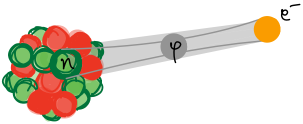


New effective Yukawa-potential

$$V_\phi(r) = -\alpha_{\text{NP}}(A - Z)\frac{e^{-m_\phi r}}{r}$$

with $\alpha_{\text{NP}} = (-1)^s \frac{y_e y_n}{4\pi}$, $s = 0, 1, 2$ (spin)

Bounds on New Bosons



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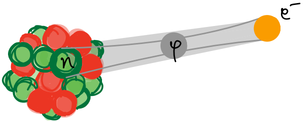
Induces new term in the King-relation:

$$\tilde{\nu}_2^{AA'} = K_{21}\tilde{\mu}^{AA'} + F_{21}\tilde{\nu}_1^{AA'} + \alpha_{\mathrm{NP}} X_{21}\tilde{\gamma}^{AA'}$$

$X_{21} = X_2 - F_{21}X_1$: NP electronic coefficient

$\tilde{\gamma}^{AA'} \equiv (A - A')/\mu^{AA'}$: NP nucl. coeff.

Bounds on New Bosons



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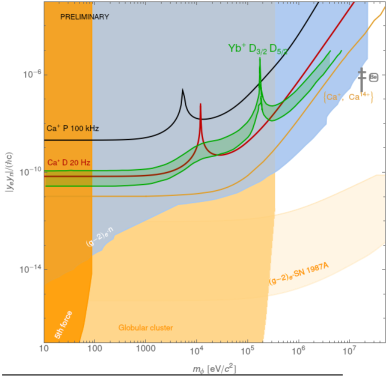
⇒ Extract α_{NP} from fraction of volumes spanned by frequency vectors:

$$\alpha_{\mathrm{NP}} = \frac{Vol.}{Vol.|_{th,\alpha_{\mathrm{NP}}=1}} = \frac{\det\left(\vec{\tilde{\nu}}_1, \vec{\tilde{\nu}}_2, \vec{\tilde{\mu}}\right)}{\varepsilon_{ijk}\det\left(X_i\vec{\tilde{\gamma}}, \vec{\nu}_j, \vec{\nu}_k\right)}$$

$\{\vec{\nu}_i\}$: data vectors in isotope-pair space, $\vec{\tilde{\mu}} \equiv (1,1,1)$, X_i, $\vec{\tilde{\gamma}}$: theory input

The “Money Plot”: α_{NP} vs. m_ϕ

Upper bounds on α_{NP} vs. the mass of the new mediator between neutrons and bound electrons: *



- $m_\phi \to 0$: characteristic length scale > size of atom
- $m_\phi \to \infty$: King-Plots are not sensitive to contact interactions
- “Peaks” due to cancellations among electronic coefficients

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*black: [PRL 115 053003], red: [PRL 125, 123003 (2020)], green: [PRL 125, 123002 (2020)], orange: [preliminary results from PTB], Be: [PRL 117 (2016) 7, 071803, PRD 95 (2017) 3, 035017]

Choose your King-Plot

Extraction of α_{NP} using the "determinant method" requires

Type of King-Plot	Isotope-Pairs	Transitions	
Generalised King-Plot:	n	$n-1$	[PRR 2, 043444 (2020)]

$n \geq 3$ (else cannot search for nonlinearities)

$$\alpha_{\mathrm{NP}} = \frac{Vol.'}{Vol.'|_{\mathrm{th},\alpha_{\mathrm{NP}}=1}} = \frac{(n-2)!\,\det\left(\vec{\tilde{\nu}}_1, \ldots, \vec{\tilde{\nu}}_{n-1}, \vec{\tilde{\mu}}\right)}{\varepsilon_{i_1,\ldots,i_{n-1}}\det\left(X_{i_1}\vec{\tilde{\gamma}}, \vec{\tilde{\nu}}_{i_2}, \ldots, \vec{\tilde{\nu}}_{i_{n-1}}, \vec{\tilde{\mu}}_{i_n}\right)}$$

Choose your King-Plot

Extraction of α_{NP} using the “determinant method” requires

Type of King-Plot	Isotope-Pairs	Transitions	
Generalised King-Plot:	n	$n-1$	[PRR 2, 043444 (2020)]
No-Mass King-Plot:	n	n	[PRR 2, 043444 (2020)]

$n \geq 3$ (else cannot search for nonlinearities)

$$\alpha_{\mathrm{NP}} = \frac{Vol.'}{Vol.'|_{\mathrm{th},\alpha_{\mathrm{NP}}=1}} = \frac{(n-2)!\det\left(\vec{\tilde{\nu}}_1, \ldots, \vec{\tilde{\nu}}_{n-1}, \vec{\tilde{\mu}}\right)}{\varepsilon_{i_1,\ldots,i_{n-1}}\det\left(X_{i_1}\vec{\tilde{\gamma}}, \vec{\tilde{\nu}}_{i_2}, \ldots, \vec{\tilde{\nu}}_{i_{n-1}}, \vec{\tilde{\mu}}_{i_n}\right)}$$

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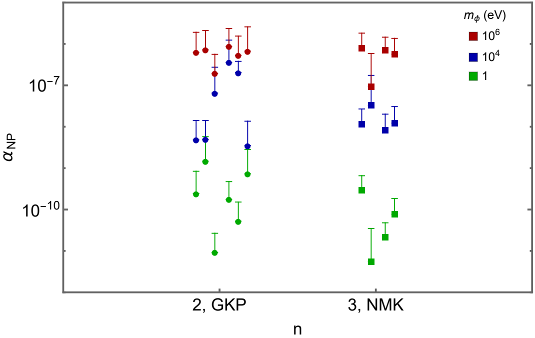
Global Fit to King-Plot Data

Ytterbium King-Plots

The Need for a Global Fit to King-Plot Data

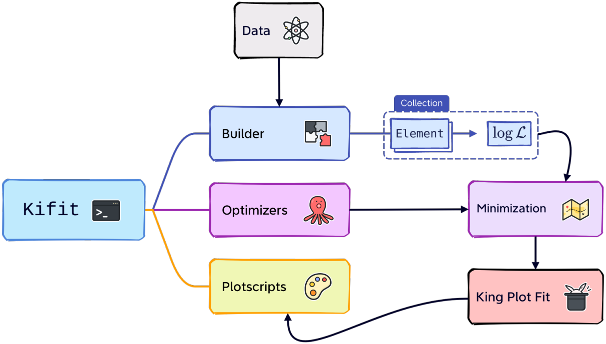
Which is the “right” upper bound?

Limits on α_{NP} *from* **Ca-data** [†]*, with* 2σ *uncertainty bars:*



[†] [PRA 100, 022514 (2019), PRL 115, 053003 (2015), Chin Phys C 45 030003 (2021)]

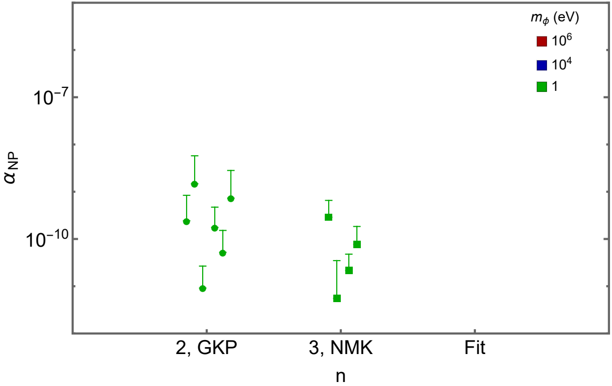
Kifit: Global Fit to King-Plot Data



in collaboration with Elina Fuchs, Agnese Mariotti and Matteo Robbiati

The Need for a Global Fit to King-Plot Data

Towards a systematic treatment of King-Plot data
Here: **Ca-data** ‡



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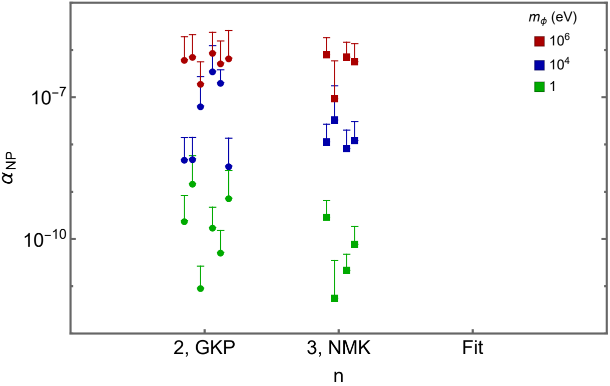

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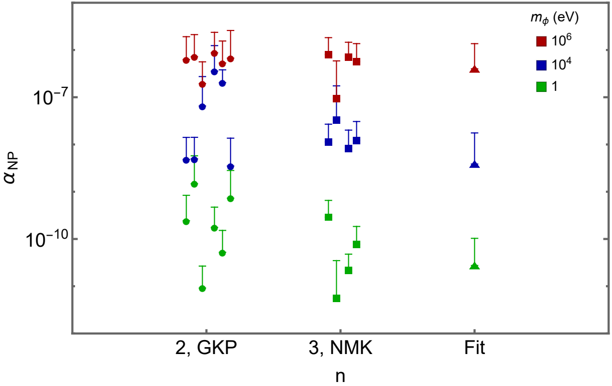

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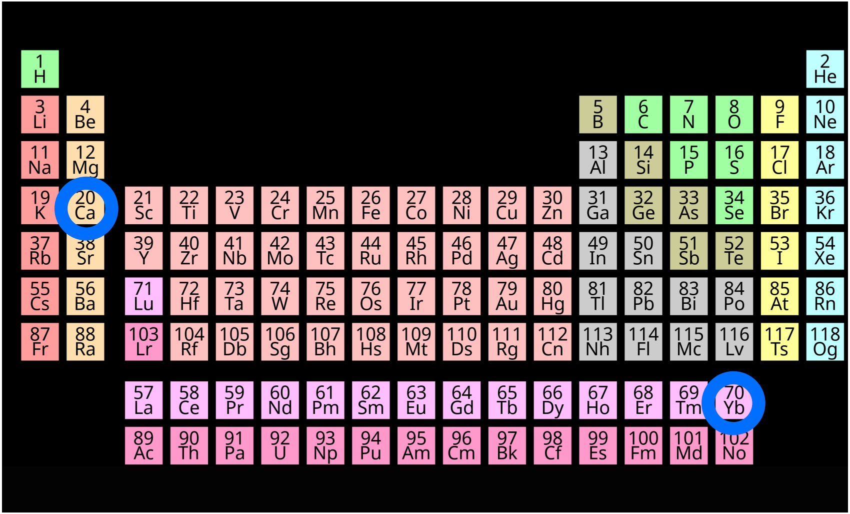
Current Precision King-Plots



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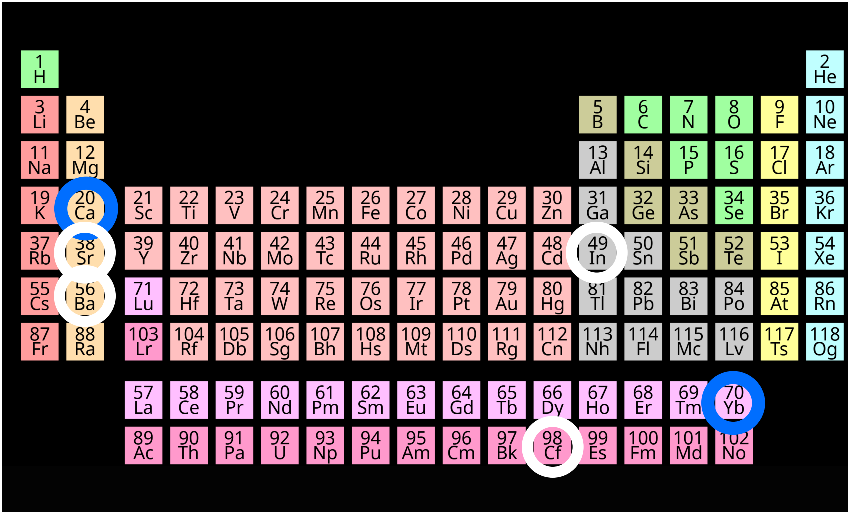
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba	71 Lu	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra	103 Lr	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
		57 La	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb		
		89 Ac	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No		

Precision King-Plots in Planning



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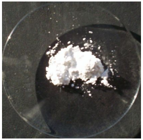

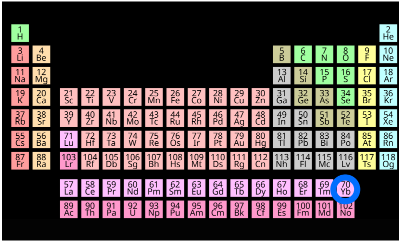

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Jean Charles Galissard de Marignac (from Geneva)

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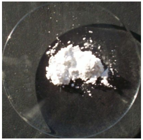

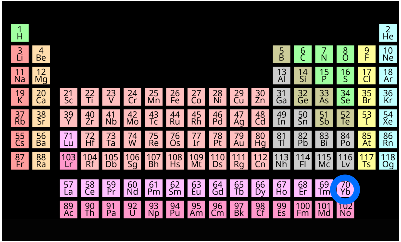

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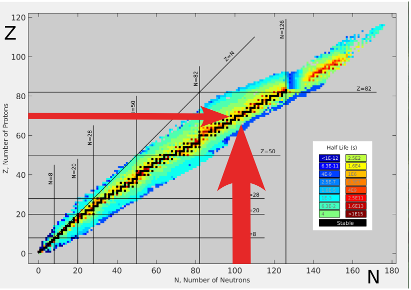
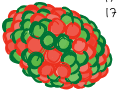


Yb

$(Z,A) = (70, \begin{matrix}168\\170\\172\\174\\176\end{matrix})$



+ 70 × ● e^- for Yb

69 × ● e^- for Yb^+

Ytterbium King-Plot Drama

PHYSICAL REVIEW LETTERS **128,** 163201 (2022)

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Evidence of Two-Source King Plot Nonlinearity in Spectroscopic Search for New Boson

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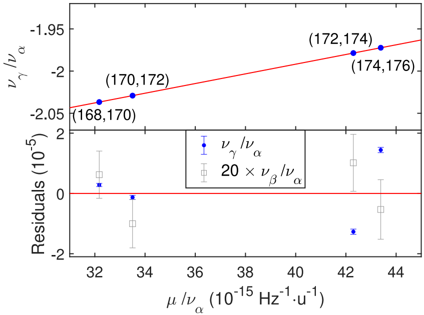
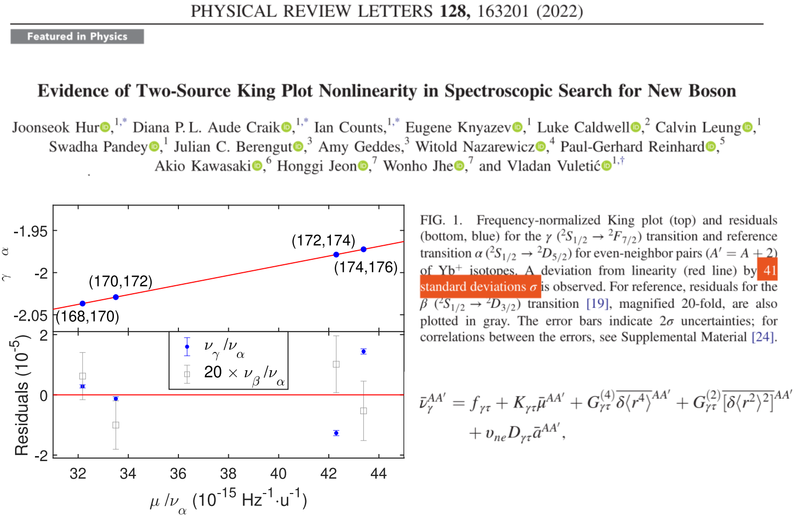


FIG. 1. Frequency-normalized King plot (top) and residuals (bottom, blue) for the γ ($^2S_{1/2}\rightarrow{}^2F_{7/2}$) transition and reference transition α ($^2S_{1/2}\rightarrow{}^2D_{5/2}$) for even-neighbor pairs ($A' = A + 2$) of Yb$^+$ isotopes. A deviation from linearity (red line) by 41 standard deviations σ is observed. For reference, residuals for the β ($^2S_{1/2}\rightarrow{}^2D_{3/2}$) transition [19], magnified 20-fold, are also plotted in gray. The error bars indicate 2σ uncertainties; for correlations between the errors, see Supplemental Material [24].

Ytterbium King-Plot Drama



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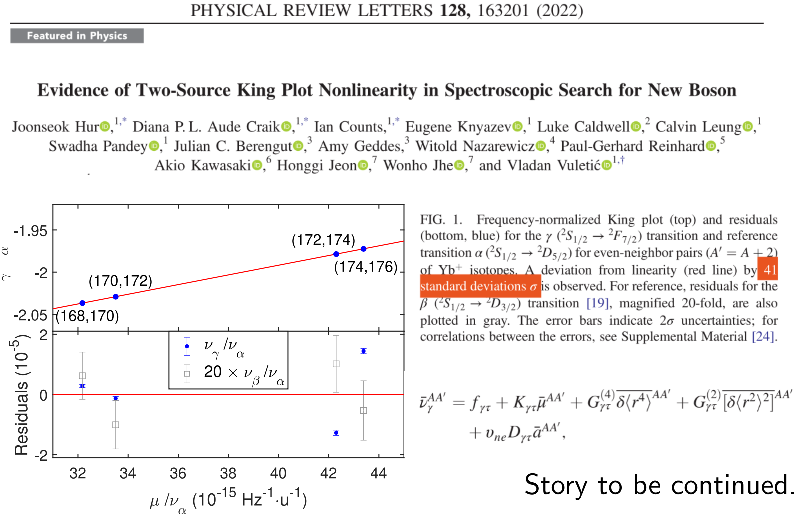
$$\bar{\nu}_\gamma^{AA'} = f_{\gamma\tau} + K_{\gamma\tau}\bar{\mu}^{AA'} + G^{(4)}_{\gamma\tau}\overline{\delta\langle r^4\rangle}^{AA'} + G^{(2)}_{\gamma\tau}\overline{[\delta\langle r^2\rangle^2]}^{AA'} + \upsilon_{ne}D_{\gamma\tau}\bar{a}^{AA'},$$

Ytterbium King-Plot Drama



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Story to be continued...

Stay tuned for isotope shift news:

- Kifit: Global King Fits
- Ytterbium King-Plot Analysis

...with fresh record-breaking precision data from PTB (frequencies) and Heidelberg (isotope masses) + theory & interpretation from Darmstadt, Sydney, Hannover and PTB

Thank you for your attention.

α_{NP} from Determinants

(No-Mass King-Plot:)

$$\begin{aligned}
\vec{\nu}_1 =& K_1\vec{\mu} + F_1\overrightarrow{\delta\langle r^2\rangle} + \alpha_{\text{NP}} X_1\vec{\gamma} \\
\vec{\nu}_2 =& K_2\vec{\mu} + F_2\overrightarrow{\delta\langle r^2\rangle} + \alpha_{\text{NP}} X_2\vec{\gamma} \\
\vec{\nu}_3 =& K_3\vec{\mu} + F_3\overrightarrow{\delta\langle r^2\rangle} + \alpha_{\text{NP}} X_3\vec{\gamma} \\
\Rightarrow \det\left(\vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3\right) =& \alpha_{\text{NP}} \det\left(\vec{K}, \vec{F}, \vec{X}\right) \det\left(\vec{\mu}, \overrightarrow{\delta\langle r^2\rangle}, \vec{\gamma}\right) \\
\Rightarrow \alpha_{\text{NP}} =& \frac{Vol}{Vol|_{th,\alpha_{\text{NP}}=1}} = \frac{\det\left(\vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3\right)}{\det\left(\vec{K}, \vec{F}, \vec{X}\right) \det\left(\vec{\mu}, \overrightarrow{\delta\langle r^2\rangle}, \vec{\gamma}\right)} \\
=& \frac{\det\left(\vec{\nu}_1, \vec{\nu}_2, \vec{\nu}_3\right)}{\frac{1}{2}\varepsilon_{ijk}\det\left(X_i\vec{\gamma}, \vec{\nu}_j, \vec{\nu}_k\right)}
\end{aligned}$$

X Coefficients

Overlap of new physics potential and electronic wavefunction

$$X_i = \int \mathrm{d}^3 r \frac{e^{-m_\phi r}}{r} \left[|\psi_b(r)|^2 - |\psi_a(r)|^2 \right]$$

$|\psi(r)|^2$: electron density in absence of new physics,

a, b initial, final states

Requirement for searches for new light bosons:

- At least one of ψ_a or ψ_b should have good overlap with new potential.
- For tight bounds on α_{NP}, one X_i needs to be large.