Large Charge 't Hooft Limit of $\mathcal{N} = 4$ SYM

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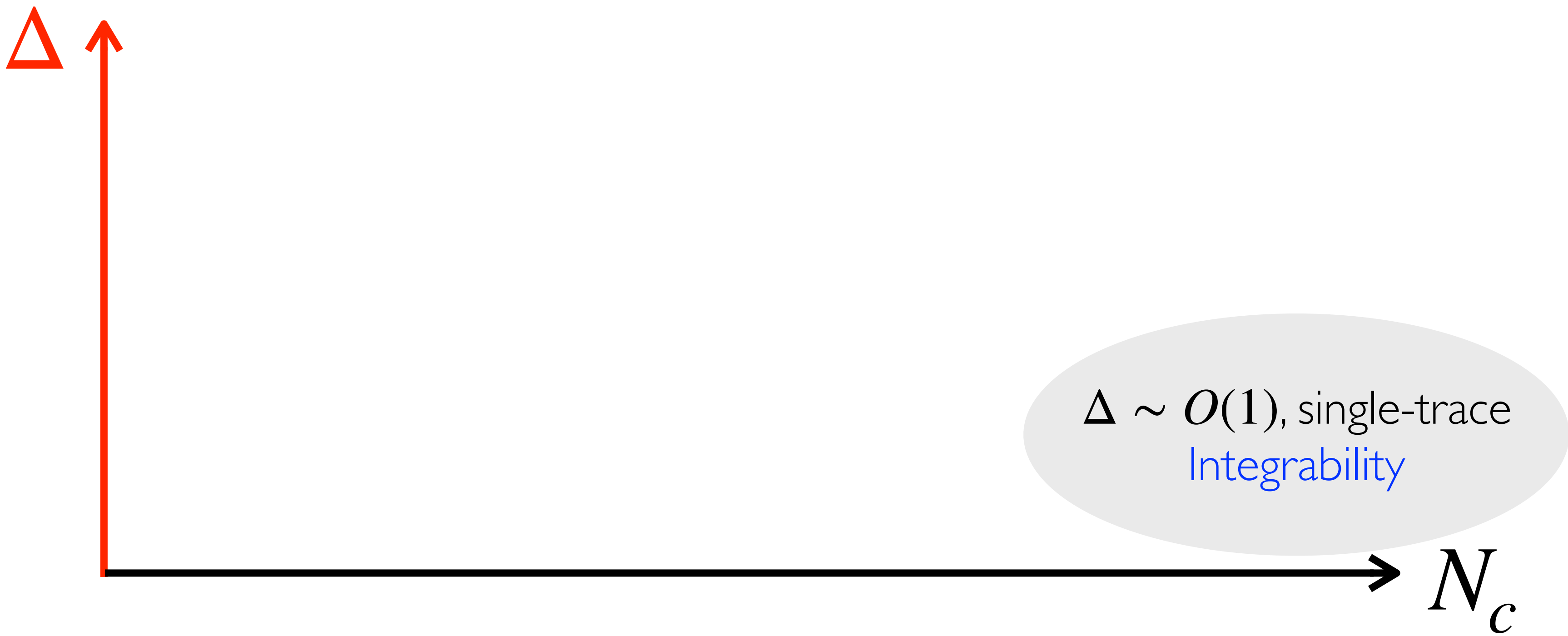
Based on **arXiv: 2306.00929** with **Joao Caetano** (CERN), **Yifan Wang** (NYU)
& works / discussion in progress + Jingxiang Wu (Oxford) (1/16-BPS)
Nicola Dondi (Bern), Francesco Galvagno (ETH) ("dual" description)

A big goal, dream, or hallucination



- **Integrability** has been extremely powerful in determining the planar CFT data of single-trace ops.

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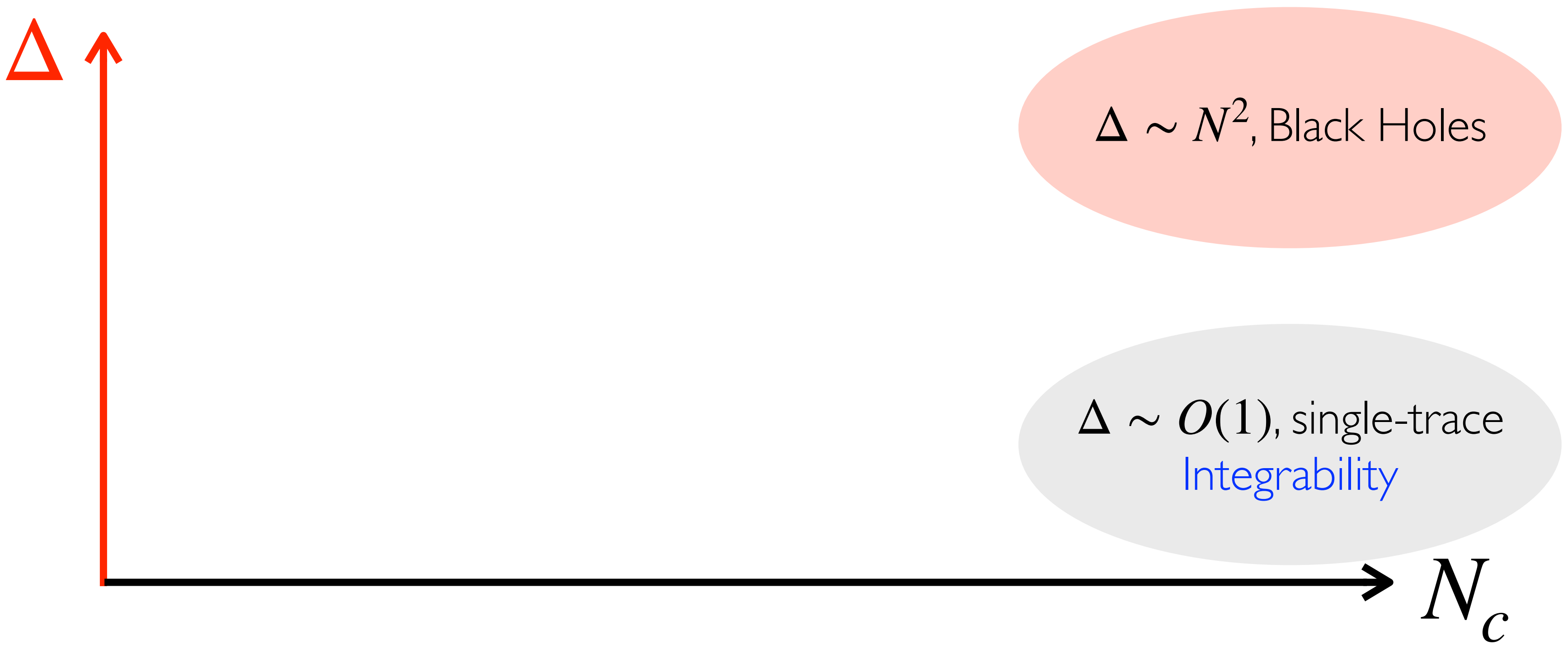
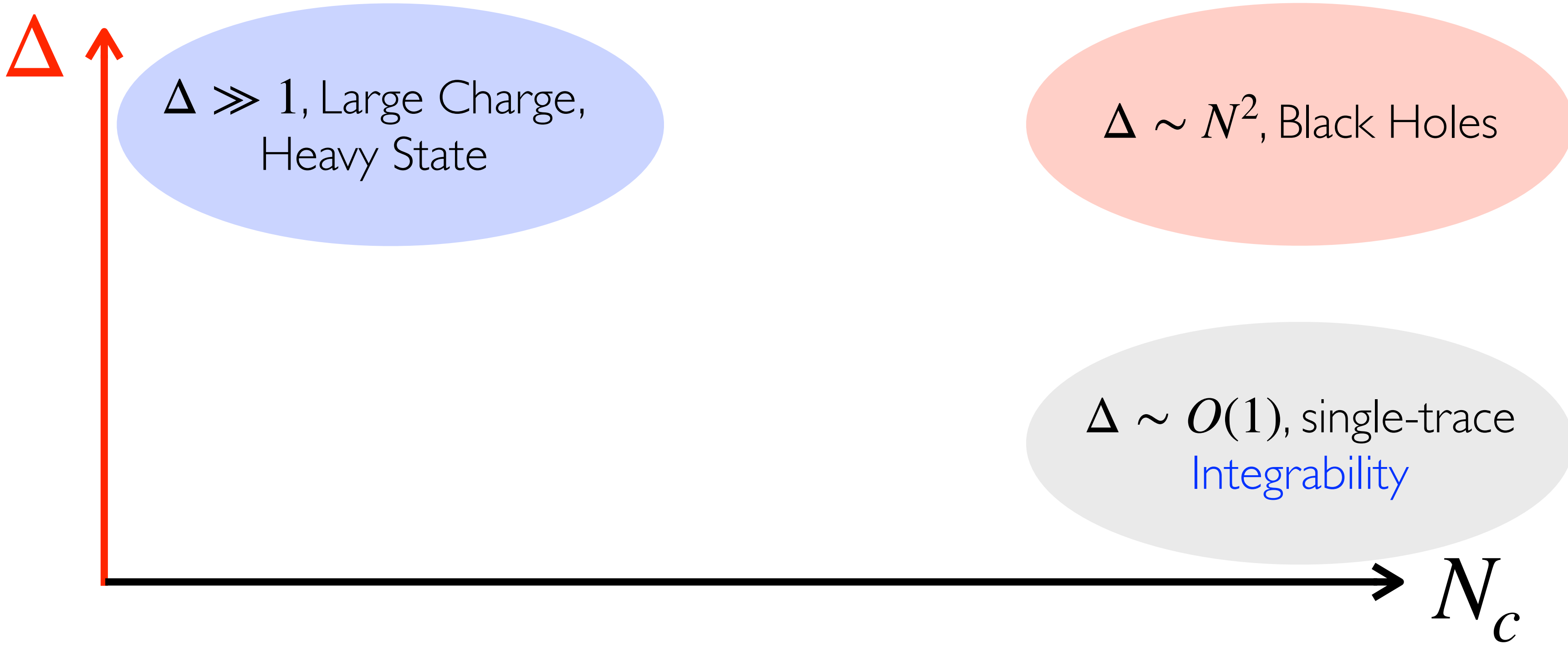


- **Integrability** has been extremely powerful in determining the planar CFT data of single-trace ops.

- **An important but difficult question:** Understand black-hole states from field theory (beyond BPS micro-state counting). Integrability seems useless.

A big goal, dream, or hallucination



- **One possible strategy:** First study $\Delta \gg 1$ operators at finite N_c . (Large quantum number expansion). And combine it with insights we learned from integrability.
- **For this strategy to work,** at least we need to see similar features in two regimes (planar limit & large charge).

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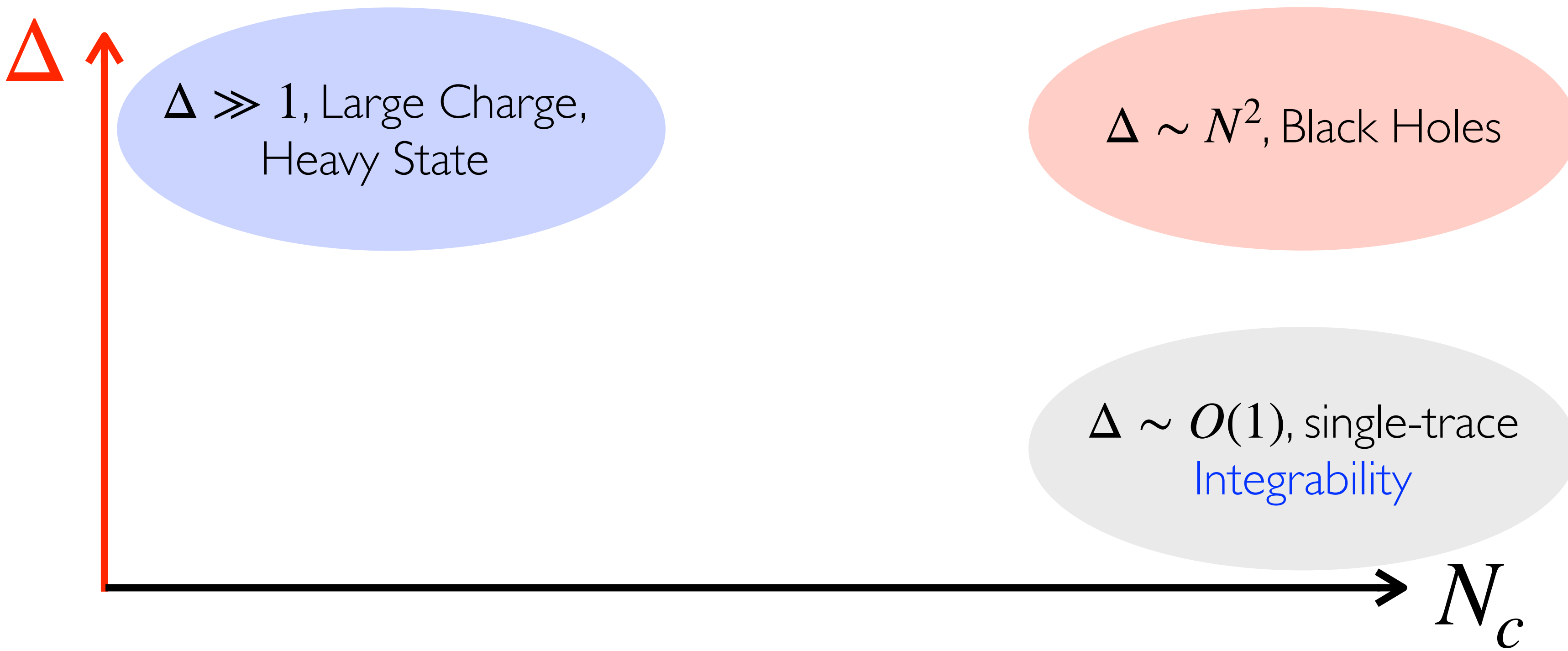

- **Punchline:** There is one common feature (for 1/2 BPS states), which constrains the dynamics in both regimes.

Centrally-extended $\mathfrak{psu}(2|2)^2$ [Beisert '06]

(A similar story likely holds also for 1/16 BPS operators)

Outline

1. Intro / motivation
2. Large charge limit vs large charge ’t Hooft limit
3. Spectral problem at large charge
4. Higher-point functions
5. Conclusion and future directions

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Systems with large # of d.o.f

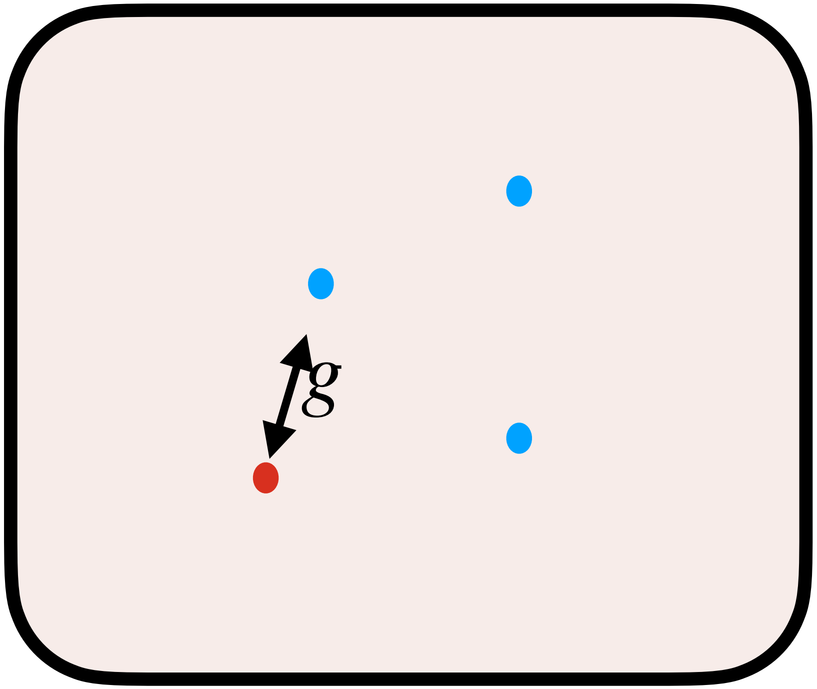
- Systems with large # of d.o.f often exhibit emergent phenomena.

- Two ways to introduce large # of d.o.f.

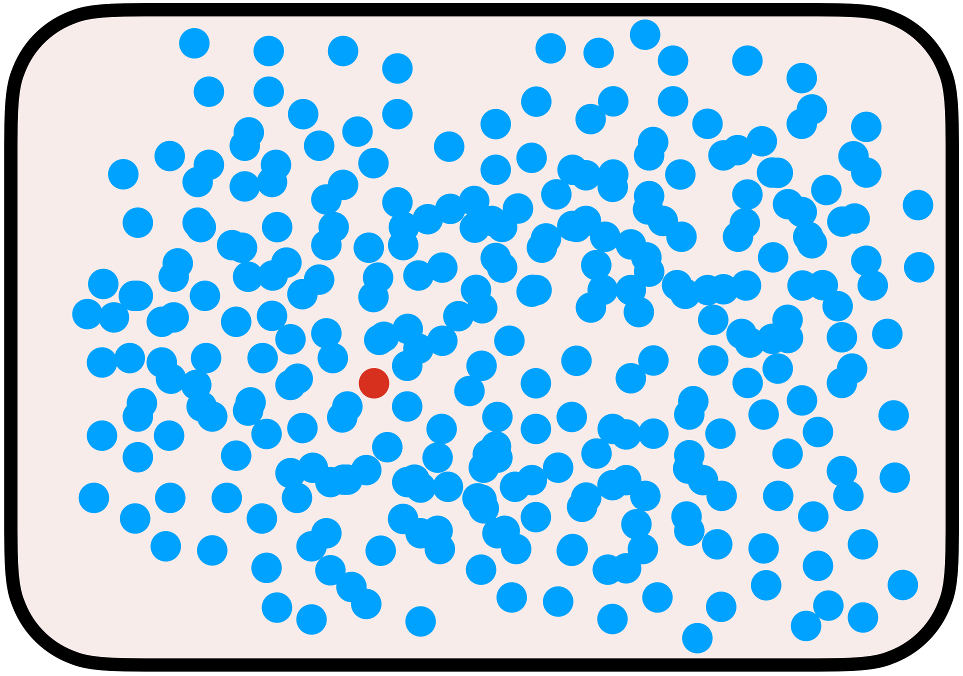
  1. Consider a family of theories (parametrized by "N_c") and take $N_c \to \infty$.
  2. Consider a state in a given theory in which a large number of particles are excited.

- The former is the large N_c limit. The latter is common in cond-mat.

Large # of particles



For g and N small:
perturbation around free system



Effective interaction strength
$\lambda_{\text{eff}} \sim g\,N$

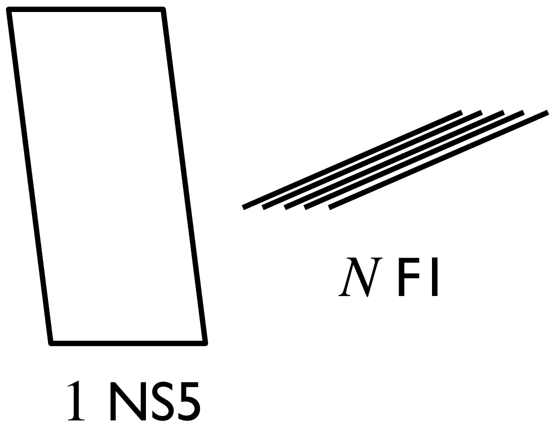
Large # of particles

$$\lambda_{\text{eff}} \sim g\,N$$

- Even if the fundamental interaction is weak ($g \ll 1$), the sector with a large # of particles can be strongly coupled.
- Suggests a double scaling limit, λ_{eff} : fixed, and $N \to \infty$.
- Formally this looks like a 't Hooft limit…..
- Is there any similarity with the standard 't Hooft limit….?

"Duality"

[Polchinski, Silverstein '12]



$$\longleftrightarrow \quad AdS_3 \times S^3 \times T^4$$

- When viewed from fundamental strings, this is a standard large N limit of 2d theory.
- When viewed from NS5, this is a large charge state in 6d.
- It suggests that the large charge and the large N_c can be sometimes dual descriptions of the same system.
- There is also a close connection to open-closed-open triality.

[Gopakumar, unpublished], [Gopakumar, Mazenc '22]
[Goel, Verlinde '21], [WIP]

Set up: $\mathcal{N} = 4$ SYM at large charge

- 4d $\mathcal{N} = 4$ SYM with $SU(2)$ gauge group.

- 1/2 BPS operator $\mathrm{tr}\left(\phi^J\right)$ with large R-charge J and take

$$J \to \infty \text{ with fixed } \lambda_J = g_{\mathrm{YM}}^2 J.$$

[Bourget,Rodriguez-Gomez, Russo'18]

- Study near BPS spectrum and correlation functions (heavy-heavy-light-light etc) in this limit.

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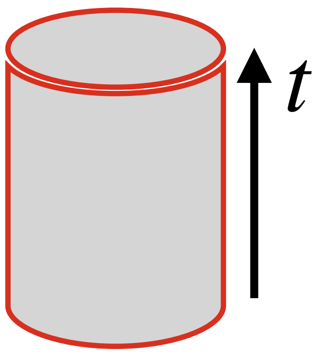
Physics of large charge limit in CFT_d

- $\langle \mathcal{O}_J \bar{\mathcal{O}}_J \rangle$ at large charge $\leftrightarrow$ large charge state on $R_t \times S_L^{d-1}$

  $\mathcal{O}_J$: minimal dim op. for a given charge J



- $$E_{\text{state}} = \frac{\Delta_{\min}}{L} \quad \rightarrow \quad \epsilon_{\text{state}} = \frac{\Delta_{\min}}{L^d}, \quad j_{\min} = \frac{J}{L^{d-1}}$$

- Large charge limit: $J \rightarrow \infty, L \rightarrow \infty$ with $j_{\min}$ finite.

- Lowest energy state in flat space with finite charge density.

Physics of large charge limit in CFT_d

$$\epsilon_{\text{state}} = \frac{\Delta_{\min}}{L^d}, \quad j_{\min} = \frac{J}{L^{d-1}}$$

- Generic (nonsupersymmetric) CFT_d : $j_{\text{state}} \sim \epsilon_{\text{state}} \sim O(1)$

$$\Delta \overset{J\to\infty}{\sim} J^{\frac{d}{d-1}}$$

[Alvarez-Gaume, Cuomo, Dondi, Giombi, Hellerman, Kalogerakis, Loukas, Monin, Orlando, Pirtskhalava, Rattazzi, Reffert, Sannino, Watanabe. . . .]

- CFT with a moduli space of vacua: $j_{\text{state}} \sim O(1), \ \epsilon_{\text{state}} = 0.$

e.g. BPS operator in SUSY CFT: $\Delta \propto J$

[Arias-Tamargo, Beccaria, Bourget, Hellerman, Maeda, Orlando, Rodriguez-Gomez, Reffert, Russo, Watanabe. . . .]

Large charge limit in SUSY CFT

- Coulomb branch chiral ring in $\mathcal{N}=2$ SCFT $O_J = \mathrm{tr}(\phi^J)$
- Large charge insertion → nontrivial profile of scalar field

$$\int \mathscr{D}\phi \exp\left(-S + \underline{J\log(\phi)\delta^d(x-x_1) + J\log(\bar{\phi})\delta^d(x-x_2)}\right)$$

- In theories with marginal coupling g_{YM} : $\langle\phi\rangle \sim g_{\mathrm{YM}}\sqrt{J}$

  The theory is effectively on the Coulomb branch.

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- Mass of BPS W-bosons: $m_W \sim g_{\mathrm{YM}}\sqrt{J} \to \infty \ (J \to \infty)$

- Derivative expansion of Coulomb branch EFT = $1/J$ expansion

$$\frac{1}{p^2 + m_W^2} = \frac{1}{m_W^2} - \frac{p^2}{m_W^4} + \cdots \sim \frac{\#}{J} + \frac{\# p^2}{J^2} + \cdots$$

Large charge 't Hooft limit

- Large charge limit + EFT : powerful, universal predictions
- But it is insensitive to physics of massive (BPS) excitations
- Alternative limit: [Bourget,Rodriguez-Gomez, Russo'18]

$$J \to \infty \text{ with } \lambda_J = g_{\rm YM}^2 J \text{ fixed.}$$

- $m_W \sim \lambda_J$ is finite. They contribute to obs even at $J \to \infty$.
- Results in the literature so far: BPS correlation functions.

[Arias-Tamargo, Beccaria, Bourget, Grassi, Komargodski, Hellerman, Maeda, Orlando, Rodriguez-Gomez, Reffert, Russo, Tizzano, Watanabe….]

- Goal: Study non-BPS (near BPS) spectrum.

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Simple weak coupling analysis

- Focus on operators made out of complex scalars X and Z.
- Thanks to the $SU(2)$ trace relation, all operators are generated by

$$\mathrm{tr}(Z^2), \mathrm{tr}(ZX), \mathrm{tr}(X^2)$$

- Near BPS operator at large charge: $\left(\mathrm{tr}(Z^2)\right)^{J/2}(1 + \text{corrections})$
- Full non-planar dilatation operator up to two loops was computed.

[Beisert '05]

$$D_{\text{1-loop}} \sim g_{\mathrm{YM}}^2 \mathrm{tr}(ZX \frac{\delta}{\delta Z}\frac{\delta}{\delta X})$$

- If $\delta/\delta Z$ acts on the "vacuum" $(\mathrm{tr}(Z^2))^{J/2}$, $D_{\text{1-loop}} \sim \lambda_J$. Otherwise $1/J$ suppressed.

Result

- Lightest non-BPS operator around large charge vacuum:

$$\Delta - J = 2 + 16\lambda_J - 64\lambda_J^2 + \cdots$$

- Obviously, this is an expansion of

$$\Delta - J = 2\sqrt{1 + 16\lambda_J}$$

- Coincides "magnon dispersion relation" in the planar limit:

$$E_{\mathsf{magnon}} = \sqrt{1 + 16\lambda \sin^2 \frac{p_{\mathsf{magnon}}}{2}}$$

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- Of course, we have to be scientific.

Science: Centrally extended $\mathfrak{psu}(2|2)^2$

Centrally-extended $\mathfrak{psu}(2|2)^2$

- Consider a large charge BPS op and perturb it:

$$|Z\ldots Z\chi Z\ldots Z\chi Z\ldots Z\rangle$$

- Subgroup of superconformal symmetry preserved by large charge 1/2 BPS state → $\mathfrak{psu}(2|2)^2$

- Excitations are classified by irreps of $\mathfrak{psu}(2|2)^2$.

- But (as expected) it is not powerful enough to constrain the dynamics. No g_{YM} dependence.

- The actual symmetry is larger: centrally-extended $\mathfrak{psu}(2|2)^2$

Centrally-extended $\mathfrak{psu}(2|2)^2$

- In general superconformal algebra, $\{Q, Q\} = 0$.

- This is true when acting on gauge inv states. But not true for individual fields.

$$\{Q, Q\}\chi \sim [Z, \chi]$$

- Maximally centrally extended $\mathfrak{psu}(2|2)^2$

$$\{Q, S\} \sim D - \hat{J}$$

$$\{Q^a{}_\alpha, Q^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\, P\,, \quad \{S^\alpha{}_a, S^\beta{}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\, K$$

$$P \cdot \chi \equiv [Z, \chi]\,, \quad K \cdot \chi \equiv [Z^{-1}, \chi]$$

Centrally-extended $\mathfrak{psu}(2|2)^2$

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$$P \cdot \chi \equiv [Z, \chi]\,, \quad K \cdot \chi \equiv [Z^{-1}, \chi]$$

- In the planar limit, P and K can be identified with translations on spin chain.

- In the large charge 't Hooft limit, the action of P and K are determined by VEV of Z induced by the charge.

$$\langle Z \rangle = \begin{pmatrix} \frac{g_{\mathrm{YM}}\sqrt{J}}{2\pi} e^{i\varphi} & 0 \\ 0 & -\frac{g_{\mathrm{YM}}\sqrt{J}}{2\pi} e^{i\varphi} \end{pmatrix}$$

- $\{Q, Q\}\, m^{\pm} \sim \pm\, 2\lambda m^{\pm}$, $\{Q, Q\}\, m^0 = 0\,.$

$$\chi = \begin{pmatrix} m^0 & m^+ \\ m^- & -m^0 \end{pmatrix}$$

Centrally-extended $\mathfrak{psu}(2|2)^2$

$$\{Q^a{}_\alpha, Q^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\, P\,, \quad \{S^\alpha{}_a, S^\beta{}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\, K$$

$$\{Q, S\} \sim D - \hat{J}$$

- By requiring that the centrally-extended algebra closes on $|Z\ldots m^{\pm}\ldots Z\rangle$ (BPS rep)

$$(D - \hat{J})|Z\ldots m^{\pm}\ldots Z\rangle = \sqrt{1 + 16\,\lambda}\,|Z\ldots m^{\pm}\ldots Z\rangle$$

$$(D - \hat{J})|Z\ldots m^{0}\ldots Z\rangle = |Z\ldots m^{0}\ldots Z\rangle$$

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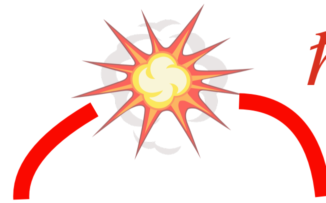
- To construct a gauge inv state, we require $\# m^+ = \# m^-$

e.g. $\quad |Z \ldots Z\, m^- \, m^- \, m^0 \, Z \ldots Z\, m^+ \, m^+ \, Z \ldots Z\rangle$

Energy = $\sum$ individual energies $\qquad \Delta - J = 1 + 4\sqrt{1 + 16\,\lambda}$

(interactions $\sim 1/J$)

Spectrum at $1/J$

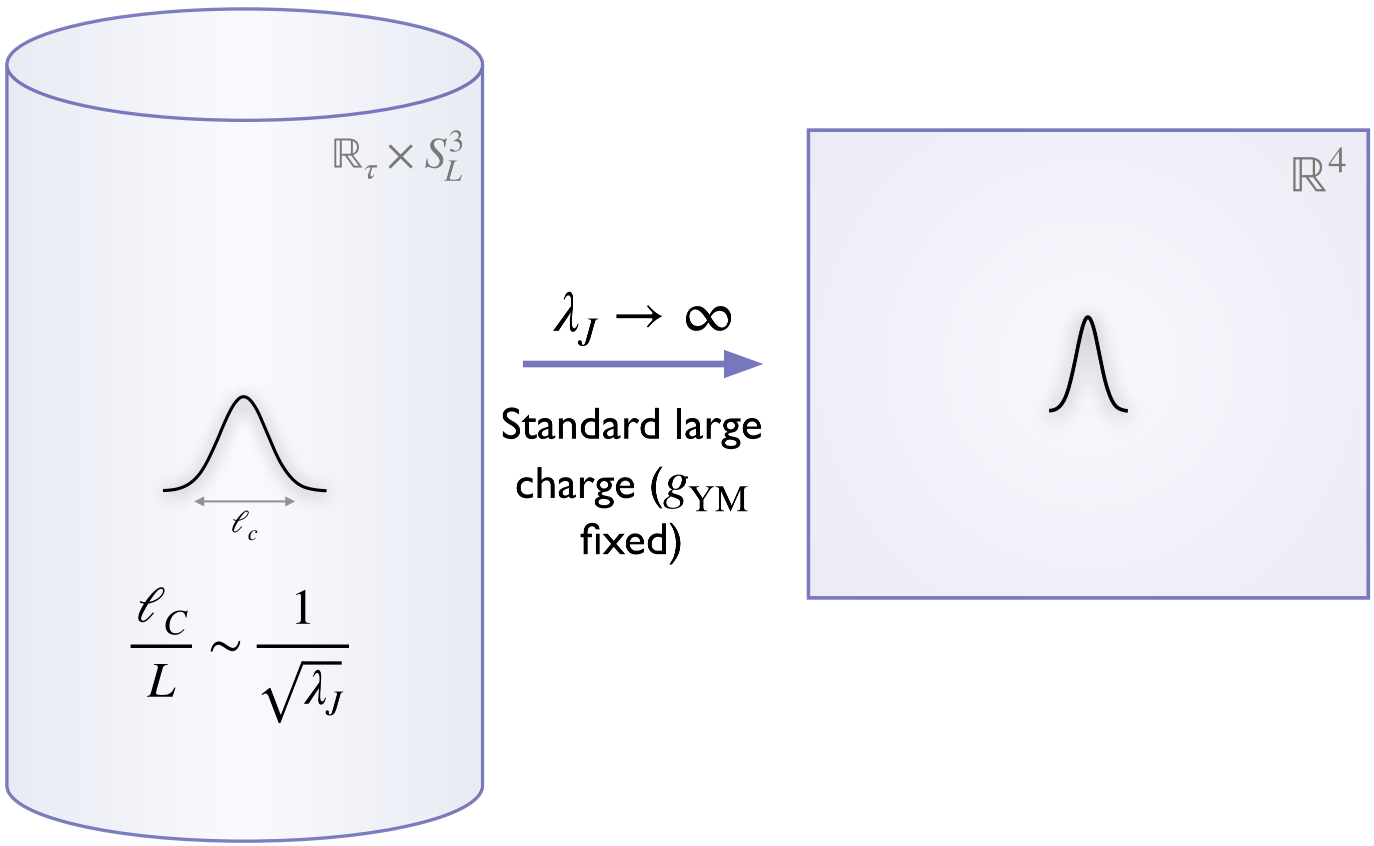


$$\hbar \sim \frac{1}{J}$$

$$|\, Z \ldots Z\, m^{-}\, m^{-}\, m^{0}\, Z \ldots Z\, m^{+}\, Z \ldots Z\, m^{+}\, Z \ldots Z \rangle$$

- At $1/J$, there is 2-body interaction among excitations (“magnons”).
- Since we are not in the planar limit, the interaction is “all-to-all”.
- Centrally-extended $\mathfrak{psu}(2|2)$ is still powerful: It determines the interaction Hamiltonian up to a few overall coefficient.

Relation to Standard Large Charge



- Centrally-extended $\mathfrak{psu}(2|2) \to$ Centrally-extended Poincare SUSY
- BPS rep $\mathfrak{psu}(2|2) \to$ BPS particles of Poincare SUSY

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Higher-point functions

- Consider higher-point functions (HHLLLL….)

$$\langle O_J(0) O_{i_1}(x_1)\dots O_{i_n}(x_n) O_J(\infty)\rangle = \langle J | O_{i_1}(x_1)\dots O_{i_n}(x_n) | J\rangle$$

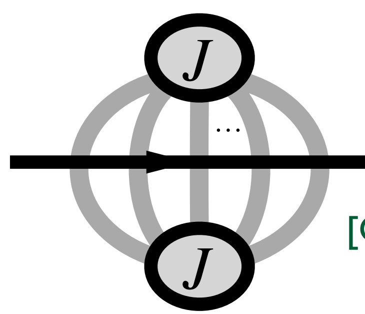
- **Examples**: light BPS $\mathbf{tr}\left((Y^I\phi_I)^\ell\right)$, Konishi $K \sim \mathbf{tr}\left(\phi^I\phi_I\right)$

- Large charge 't Hooft limit:

$$\langle J | O_{i_1}(x_1)\dots O_{i_n}(x_n) | J\rangle \to \langle O_{i_1}(x_1)\dots O_{i_n}(x_n)\rangle_{\text{large charge bkd}}$$

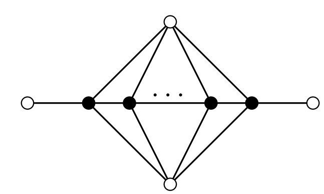
- Basic building block: propagator in the background

$$\langle \phi\phi\rangle_{\text{bkd}} \sim \sum$$



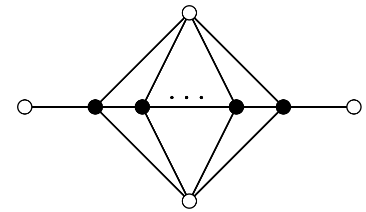
$$= \sum_k \lambda^k F^{(k)}(z,\bar{z})$$

[Giombi, Hyman '20]



Propagator in the large charge bkd.

$$\langle \phi\phi \rangle_{\text{bkd}} = \sum_k \lambda^k F^{(k)}(z, \bar{z})$$



- This resummation was studied by [Broadhurst, Davydychev '10] without concrete physical application.

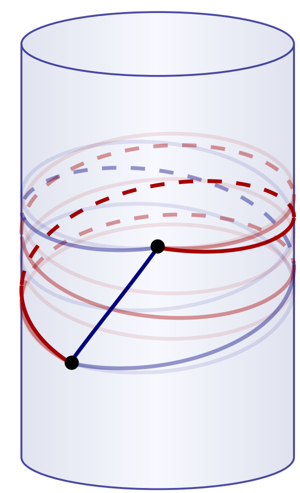
$$\langle \phi\phi \rangle_{\text{bkd}} \sim e^{-\sqrt{\lambda}}$$

cf. [Arkani-Hamed, Henn, Trnka '21]

- We found a more refined formula:

$$\frac{(1-z)(1-\bar{z})}{\sqrt{z\bar{z}}} \sum_{n=0}^{\infty} W(\varphi + 2\pi n) + W(2\pi - \varphi + 2\pi n)$$

$$W(x) = \frac{x K_1(\sqrt{\lambda}\sqrt{x^2+\sigma^2})}{\sin x \sqrt{x^2+\sigma^2}}$$



- Infinite sum of massive propagator in flat space; worldline instantons

[Hellerman, Orlando]
[Dondi, Kalogerakis, Orlando, Reffert]

Result for Konishi 3pt

- Using the propagator, we computed the 3pt function $\langle \mathcal{O}_J \, K \, \mathcal{O}_J \rangle$

$$\left(K \sim \mathrm{tr}\left(\phi^I \phi_I\right) \right)$$

$$C_{KJJ} = -8\lambda + 4\sqrt{\lambda}\int_0^\infty dw \frac{4\sqrt{\lambda}w - J_1(8\sqrt{\lambda}w)}{\sinh^2(w)}$$

- Weak coupling expansion: finite radius of convergence $|\lambda| < 1/16$.

  "magnon" becoming massless, tachyonic instability $\sqrt{1+16\lambda}$

- Strong coupling expansion:

$$C_{KJJ} = \frac{\lambda}{2\pi^2}\left(\gamma_E + \log\frac{\lambda}{4\pi^2}\right) - \frac{2\lambda}{\pi^2}\sum_{n=1}^{\infty}\left(K_1(2n\sqrt{\lambda}) + K_0(2n\sqrt{\lambda})\right)$$

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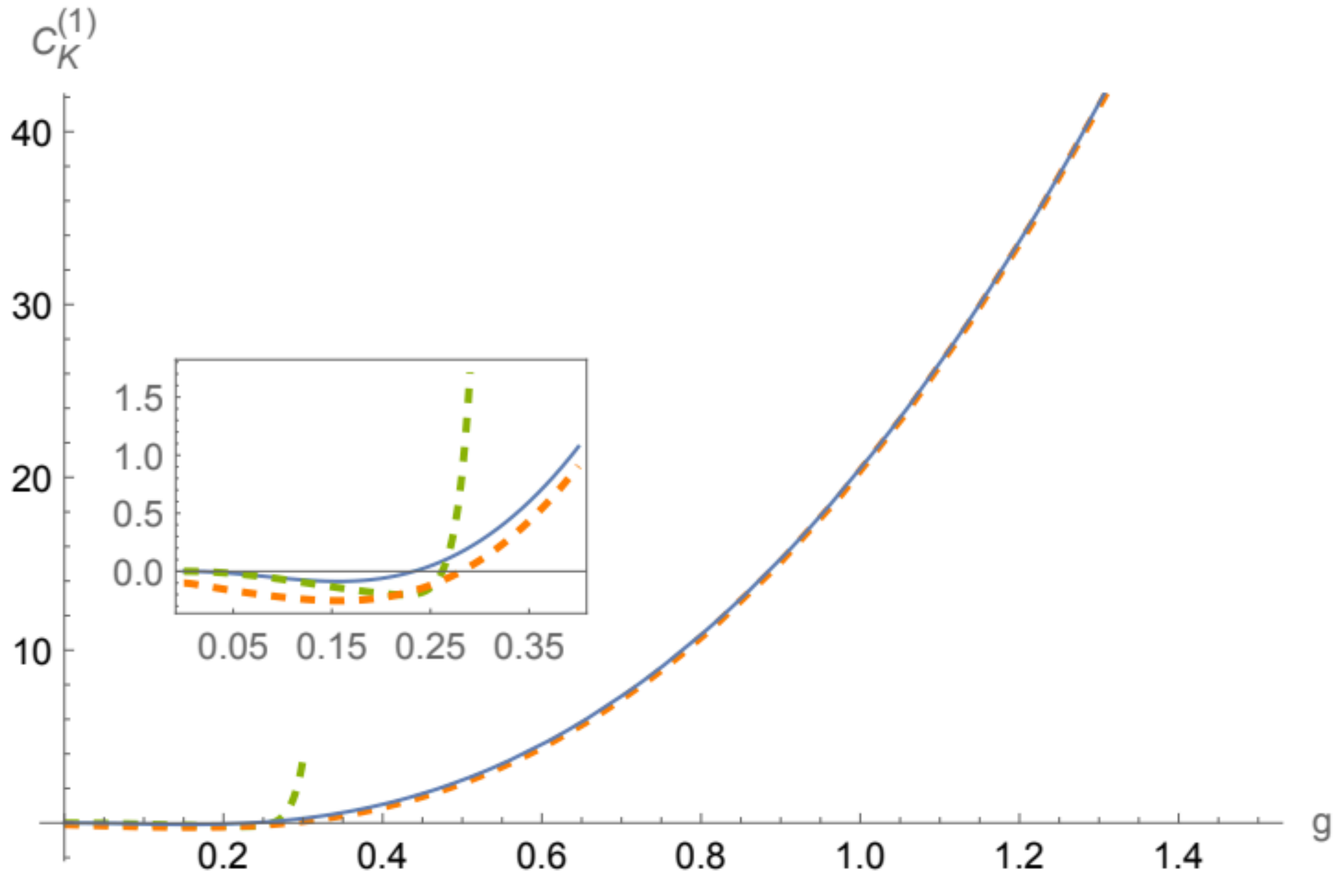

Comparison with BPS 2pt in $\mathcal{N} = 2$ SCFT

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- Similar integral (of Bessl J) appear in the planar limit.

Cusp anomalous dimension, Octagon (large charge 4pt) [Basso, Korchemsky] [Bargheer, Coronado, Vieira]

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  Cusp anomalous dimension, Octagon (large charge 4pt) [Basso, Korchemsky] [Bargheer, Coronado, Vieira]

- Very similar expressions for large charge 2pt functions of Coulomb branch BPS ops. [Hellerman, Maeda] [Grassi, Komargodski, Tizzano]

- There, the coefficient of $\log\lambda$ is given by a-anomaly.

- Here, one can show that it is given by the anomalous dim of Konishi operator.

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Conclusion

- Large charge 't Hooft limit provides an interesting solvable corner of $\mathcal{N}=4$ SYM.
- Underlying centrally extended symmetry.
- Various observables can be computed exactly as a function of λ_J.

  $J \times J$ matrix model reformulation of integrated correlators.

  cf. Gromov, Sever, Giombi, SK, Grassi, Komargodski, Tizzano…
- Many more observables to study.

  Heavy-heavy-heavy 3pt
  Higher rank
  Large spin 't Hooft limit $g_{\rm YM}^2 \log S$

1/16 BPS states

- Symmetry analysis can be easily extended.

We expect to have centrally-extended su(1|1)

$$\{Q,Q\} = P, \quad \{S,S\} = K$$

- Taking the standard large charge limit, this would become centrally-extended SUSY in flat space.

- **Puzzle**: the same is true for $\mathcal{N}=1$ SCFT. But the central-extension & BPS particles normally require $\mathcal{N} \geq 2$ SUSY.

- **Resolution**: a typical 1/16 BPS op.

$$\mathcal{O} \sim \mathrm{tr}(F_{++}^2)^J$$ [Choi, Kim, Lee, Lee, Park '22]

- Large charge limit gives a theory in flat space with magnetic field.

- Counting of BPS particles in the presence of magnetic field? Magnetic (or "Landau") branch instead of Coulomb branch?