Revisiting logarithmic corrections to supersymmetric black hole entropy

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Bekenstein-Hawking formula for black hole entropy is universal

$$S_0 = \frac{A}{4} \qquad \text{in } \hbar = c = G_N = k_B = 1 \text{ unit}$$

A: area of the event horizon

In Einstein-Maxwell theory or (extended) supergravity, the black hole can carry electric charges Q, magnetic charges P, angular momentum J_3 and mass M

Then

$$S_0 = f_0(Q, P, M, J_3)$$

In D=4,

$$f_0(\lambda Q, \lambda P, \lambda M, \lambda^2 J_3) = \lambda^2 f_0(Q, P, M, J_3)$$

For D>4, there are different scaling laws, but we shall stick to D=4 for this talk

Q,P: n-dimensional vectors if we have n U(1) gauge fields

$$f_0(\lambda Q, \lambda P, \lambda M, \lambda^2 J_3) = \lambda^2 f_0(Q, P, M, J_3)$$

To take macroscopic limit, we parametrize M, Q, P, J_3 as

$$M = m\,\lambda, \qquad Q = q\,\lambda, \qquad P = p\,\lambda, \qquad J_3 = j\,\lambda$$

and take λ large, keeping m,q,p,j fixed

Then

$$S_0 = \lambda^2 f_0(q, p, m, j)$$

In this limit the fields associated with the black hole also has simple dependence on λ, e.g.

$$g_{\mu\nu}(Q, P, M, J_3) = \lambda^2 g_{\mu\nu}(q, p, m, j)$$

The Bekenstein-Hawking formula is expected to receive corrections due to stringy effects and quantum effects

General structure:

$$S = \lambda^2 f_0 + (\ln\lambda) f_1 + f_2 + \cdots$$

$f_0, f_1, f_2, \cdots$ are all functions of m,q,p,j

In the large λ limit the dominant correction is the term proportional to $\ln\lambda$

– focus of attention in today's lecture

General procedure for computing corrections to the black hole entropy (Gibbons-Hawking)

1. Perform a path integral over all fields subject to the same boundary condition that the black hole satisfies

– gives partition function

2. Construct the entropy from the partition function using the usual rules of statistical mechanics

e.g. for asymptotically flat black holes, the gravitational path integral gives grand canonical partition function

– need to take appropriate Laplace transform to get the microcanonical entropy

Both steps can generate logarithmic corrections to the entropy

A.S.: arXiv:1205.0971

Logarithmic corrections from the change of ensemble

Consider a black hole in flat space-time carrying charge and angular momentum along 3-axis

Euclidean continuation leads to a conical singularity at the horizon, unless

1. The euclidean time τ and the azimuthal angle ϕ are periodically identified as

$$(\tau, \phi) \equiv (\tau + \beta, \phi + \mathbf{i}\omega\beta)$$

2. The time components of the gauge fields take asymptotic values

$$\mathbf{A}_\tau = \mathbf{i}\,\mu$$

β, ω, μ are fixed in terms of M,Q,P,J_3 for classical black hole

Interpretation: Gibbons, Hawking

$\beta =$ inverse temperature, $\omega =$ chemical potential dual to J_3

$\mu =$ chemical potential dual to Q

Scaling:

$$\beta \sim \frac{\partial S_0}{\partial M} \sim \lambda, \quad \mu \sim \frac{1}{\beta}\frac{\partial S_0}{\partial Q} \sim 1, \quad \omega \sim \frac{1}{\beta}\frac{\partial S_0}{\partial J_3} \sim \lambda^{-1}$$

In quantum theory we treat $\beta, \omega, \mu,$ P as independent variables

– modes that change these values are dominant at ∞ compared to the modes that change M, Q, J_3

e.g. β fixes the constant part of $g_{\tau\tau}$ while M fixes the coefficient of 1/r in $g_{\tau\tau}$

Therefore the gravitational path integral with these boundary conditions give the grand canonical partition function:

$$Z = \mathrm{Tr}\left[e^{-\beta E - \beta\mu Q - \beta\omega J_3}\right]$$

$$Z = \int dM\, dQ\, dJ_3\, dk\, e^{[S - \beta M - \beta k^2/2M - \beta\omega J_3 - \beta\mu Q]}$$

e^S counts number of states with fixed mass, charge, J_3 but all $\vec{J}^2$

$k = k_z$ since $e^{-\beta\omega J_3}$ rotates $(k_x, k_y) \to (k'_x, k'_y)$ and hence $(k_x, k_y) \neq 0$ states do not contribute to the trace

The contribution to the integral is dominated by the Euclidean black hole saddle point, where

$$k = 0, \quad \frac{\partial S}{\partial M} = \beta, \quad \frac{\partial S}{\partial Q} = \beta\mu, \quad \frac{\partial S}{\partial J_3} = \beta\omega$$

At the leading order $S = S_0$ and we get back the classical relation between β, μ, ω and M, Q, J_3.

$$Z = \int dM\, dQ\, dJ_3\, dk\, e^{[S - \beta M - \beta k^2/2M - \beta\omega J_3 - \beta\mu Q]}$$

To keep track of the logarithmic corrections, we need to evaluate the gaussian integrals around the saddle point.

1. Since $\beta \sim \lambda$, $M \sim \lambda$, the coefficient of k^2 is λ independent.

$\Rightarrow$ k integral does not generate any a dependent factor

2. $S_0 \sim \lambda^2$, $Q, M \sim \lambda \quad \Rightarrow \quad \partial^2 S_0/\partial M^2 \sim \lambda^0, \quad \partial^2 S_0/\partial Q^2 \sim \lambda^0$

$\Rightarrow$ Q, M integrals do not generate any λ dependent factor

3. $S_0 \sim \lambda^2$, $J_3 \sim \lambda^2 \quad \Rightarrow \quad \partial^2 S_0/\partial J_3^2 \sim \lambda^{-2}$

$\Rightarrow$ J_3 integral generates a factor of λ

$$\ln Z \simeq S - \beta M - \beta\omega J_3 - \beta\mu Q + \ln\lambda$$

$$S \simeq \ln Z + \beta M + \beta\omega J_3 + \beta\mu Q - \ln\lambda$$

M,Q,J_3 are determined by their saddle point values

$\Rightarrow$ follows the classical relation:

$$\frac{\partial S_0}{\partial M} = \beta, \quad \frac{\partial S_0}{\partial Q} = \beta\mu, \quad \frac{\partial S_0}{\partial J_3} = \beta\omega$$

Note: e^S counts the number of states with fixed J_3 but all $\vec{J}^2$.

To find the number of states with fixed $\vec{J}^2$ we need to calculate

$$e^{S_{micro}(J)} = e^{S(J_3=J)} - e^{S(J_3=J+1)}$$

e.g. for $J = 0$ we get another $1/\lambda^2$ in the expression for $e^{S_{micro}}$,
and hence $-2\ln\lambda$ in the expression for S_{micro}

Logarithmic corrections to the partition function

$$\mathbf{S} \simeq \ln \mathbf{Z} + \beta \mathbf{M} + \beta\omega \mathbf{J_3} + \beta\mu \mathbf{Q} - \ln \lambda$$

Leading order contribution Z_0 to Z is e^{-I_0}

I_0: classical action of the Euclidean black hole

$$\mathbf{S} = -\mathbf{I_0} + \beta \mathbf{M} + \beta\omega \mathbf{J_3} + \beta\mu \mathbf{Q} - \ln \lambda + \delta \ln \mathbf{Z} = \mathbf{S_0} - \ln \lambda + \delta \ln \mathbf{Z}$$

S_0: Bekenstein-Hawking result

$\delta \ln Z$: corrections to Z

Our goal will be to pick up the logarithmic terms among the corrections to ln Z.

Stringy effects and quantum corrections involving loops of massive fields generate local corrections to the effective action

– expressed as an expansion in number of derivatives

Such corrections cannot generate logarithmic corrections to ln Z

Source of logarithmic corrections to ln Z is non-local correction to the effective action due to loops of massless fields

Fursaev, Solodukhin, $\cdots$, Review: arXiv:1104.3712 by Solodukhin

A simple power counting can be used to show that only one loop effects can generate terms $\propto$ ln λ

K_B: Kinetic operator for massless bosonic fields

K_F: Kinetic operator for massless fermionic fields

κ_b, κ_f: eigenvalues of K_B, K_F

One loop contribution to Z from massless fields:

$$(\det K_B)^{-1/2}(\det K_F)^{1/2} = \prod_b \kappa_b^{-1/2} \prod_f \kappa_f^{1/2}$$

Correction to ln Z:

$$\delta \ln Z = \frac{1}{2}\left[\sum_b \ln \kappa_b - \frac{1}{2}\sum_f \ln \kappa_f^2\right] = \frac{1}{2}\int_\epsilon^\infty \frac{ds}{s}\left[\sum_b e^{-s\kappa_b} - \frac{1}{2}\sum_f e^{-s\kappa_f^2}\right]$$

$$\delta \ln Z = \frac{1}{2}\int_{\epsilon}^{\infty} \frac{ds}{s}\left[\sum_b e^{-s\kappa_b} - \frac{1}{2}\sum_f e^{-s\kappa_f^2}\right]$$

Scaling $\Rightarrow \kappa_b = \kappa_b^0/\lambda^2$, $\kappa_f = \kappa_f^0/\lambda$ with λ independent κ_b^0, κ_f^0

Defining $u = s/\lambda^2$ we get

$$\delta \ln Z = \frac{1}{2}\int_{\epsilon/\lambda^2}^{\infty} \frac{du}{u}\left[\sum_b e^{-u\kappa_b^0} - \frac{1}{2}\sum_f e^{-u(\kappa_f^0)^2}\right]$$

1. Expand $F(u) \equiv \left[\sum_b e^{-u\kappa_b^0} - \frac{1}{2}\sum_f e^{-u(\kappa_f^0)^2}\right]$ in power series in u

2. Pick the coefficient C of the u^0 term

$$\delta \ln Z = \frac{C}{2}\ln \lambda^2 + \cdots$$

$$\delta \ln Z = \frac{C}{2} \ln \lambda^2 + \cdots$$

Using heat kernel expansion C can be expressed as

$$C = \int d^4x\, K(x)$$

K(x) is determined in terms of K_B and K_F

Seeley; de Witt; $\cdots$; Vassilevich hep-th/0306138

– depends on the background metric and gauge field configurations

e.g. for a minimally coupled scalar,

$$K(x) = \frac{1}{360 \times 16\pi^2} \left[12 D^\mu D_\mu R + 5R^2 - 2R_{\mu\nu} R^{\mu\nu} + 2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right]$$

Zero mode contribution:

K_B and / or K_F may have zero eigenvalues

– cannot be treated as part of the determinant

1. Remove their contribution from $\delta \ln Z$

e.g. a bosonic mode contributes $(\kappa_b^0/\lambda^2)^{-1/2}$ to Z

$\Rightarrow$ $\ln \lambda$ to $\ln Z$

We need to subtract ($\ln \lambda$) from $\delta \ln Z$ for each bosonic zero mode

Similarly we add ($\ln \lambda$)/2 for each fermionic zero mode

2. We need to find the actual λ dependent contribution to Z from the zero mode integrals

Zero modes typically arise from some broken symmetries

e.g. the black hole breaks translation symmetry

$\Rightarrow$ deformation of the background associated with translation of the black hole generates a zero eigenvalue of K_B

i) We change integration variable from field to symmetry parameter

– gives a jacobian that could have factors of λ

ii) We find the range of integration of the symmetry parameter and determine the λ dependence of the range

$\Rightarrow$ λ dependent result

Product $\Rightarrow$ net λ dependent contribution to Z from zero mode integral

Example: Suppose $h_{\mu\nu} \equiv \delta g_{\mu\nu}$ is the deformation associated with translation zero mode

We take integration measure over $h_{\mu\nu}$ to be $[d(\lambda^\alpha h_{\mu\nu})]$ so that

$$\int [d(\lambda^\alpha h_{\mu\nu})] \exp\left[-\int d^4x \sqrt{\det g}\, g^{\mu\rho} g^{\nu\sigma} h_{\mu\nu} h_{\rho\sigma}\right] \sim 1$$

Since $\sqrt{\det g}\, g^{\mu\rho} g^{\nu\sigma}$ scales as λ^0, we have $\alpha = 0$

Now the translation by c^μ is generated by a diffeomorphism parameter $c^\mu f(x)$ for some λ-independent $f(x)$ that vanishes at the horizon and approaches 1 at ∞

We change variable from $h_{\mu\nu}$ to c^μ using

$$h_{\mu\nu} = D_\mu(c_\nu f(x)) + D_\nu(c_\mu f(x))$$

Lowering of the index of c^μ gives a factor of λ^2 since $g_{\mu\nu} \sim \lambda^2$

Jacobian $\sim \lambda^2$

Next we have to find the range of c^μ

If we put the system in a box of physical size L, then the range of coordinates is of order L/λ since $g_{\mu\nu} \sim \lambda^2$

Range of c^μ is of order L/λ

$\Rightarrow$ Zero mode integration $\Rightarrow \lambda^2 \times \lambda^{-1} \sim \lambda = \exp[\ln \lambda]$

A similar analysis can be done for other zero modes

Note: We also need to make sure that the zero mode deformations are compatible with

$$(\tau, \phi) \equiv (\tau + \beta, \phi + \omega\beta)$$

– eliminates rotational zero modes and translational zero modes transverse to the rotation axis for Kerr-Newmann black hole

Using this method one can compute the logarithmic correction to the entropy of any black hole

e.g. for Scwarzschild black hole in D=4,

$$S_{micro} = S_0 + (C - 3)\ln\lambda = S_0 + \left(\frac{212}{45} - 3\right)\ln\lambda$$

There is no microscopic counting in string theory against which we can check this.

Supersymmetric black holes

Supersymmetric (extremal) black holes have zero temperature

$\Rightarrow$ instead of having a single large length scale, we have two different large scales

M, Q $\sim \lambda$ and $\beta = 1/T \to \infty$

Remedy: Work in the near horizon geometry $AdS_2 \times S^2$

Mann, Solodukhin hep-th/9604118; $\cdots$

$$ds^2 = v_1 \left(\frac{dr^2}{r^2 - 1} + (r^2 - 1) d\tau^2 \right) + v_2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right)$$

$v_1, v_2 \sim \lambda^2$

We can compute logarithmic correction to the partition function in this geometry following the same guidelines

Some differences:

1. The partition function computes the path integral at fixed mass, charge and angular momentum (=0) since these modes dominate as r $\to \infty$, e.g.

$A_\tau = Q\,r + \mu \Rightarrow$ the coefficient of Q grows faster than that of μ

Result: The path integral directly computes the entropy in the microcanonical ensemble and no change of ensemble is needed.

2. We integrate over modes living in the near horizon geometry

– different set of eigenvalues and eigenfunctions than those in the full geometry

Logarithmic corrections come from:

1. Non-zero modes

$\Rightarrow \ln\lambda \times \int_{AdS_2\times S^2} K(x)$

2. Zero modes

The structure of the zero modes in the near horizon geometry is quite different from that in the full geometry

But the general procedure for finding ($\ln\lambda$) terms remains the same.

Final result in theories with N $\geq$ 2 supersymmetry:

$$S = S_0 + \frac{1}{6}\,(23 + n_H - n_V)\ \ln\lambda \quad \text{for N=2}$$

n_H, n_V: number of vector and hypermultiplets

$$S = S_0 \quad \text{for N=4}$$

$$S = S_0 - 8\ln\lambda \quad \text{for N=8}$$

Banerjee, Gupta, A.S. arXiv:1005.3044, Banerjee, Gupta, Mandal, A.S. arXiv:1106.0080

A.S. arXiv:1108.3842

The results are in perfect agreement with microscopic counting formula for N=4 CHL type compatifications and N=8 compactifications

Maldacena, Moore, Strominger hep-th/9903163; Dijkgraaf, Verlinde, Verlinde hep-th/9607026;

David, A.S. hep-th/0605210; David, Jatkar, A.S. hep-th/0609109

No microscopic counting exists for black holes in N=2 theories

Recent developments

Iliesiu, Kologlu and Turiaci described a procedure for computing supersymmetric index using full black hole geometry

Iliesiu, Kologlu, Turiaci arXiv:2107.09062

Supersymmetric index:

$$I = Tr_{Q,P}\left[e^{-\beta H}(-1)^F(2J_3)^n\right]$$

The trace is taken over states at fixed Q, P

2n: number of supersymmetries broken by the black hole

$(2J_3)^n$ is needed to saturate the trace over the supermultiplet

Bachas, Kiritsis hep-th/9611205; Gregori, Kiritsis, Kounnas, Obers, Petropoulos, Pioline hep-th/9708062

The result is expected to be β independent and pick up the degeneracy of the supersymmetric states

– counted in $N = 4, 8$ supersymmetric theories

Consider the gravitational partition function with $\beta\omega = 2\pi i$ and $(2J_3)^n$ inserted

$$Z = Tr_P \left[e^{-\beta H - \mu Q - 2\pi i J_3} (2J_3)^n \right] = Tr_P \left[e^{-\beta H - \mu Q} (-1)^F (2J_3)^n \right]$$

Note: The trace is over all J_3, Q, k

We take $\beta, \mu \sim \lambda$ so that there is only one scale as before

Compare this with the index

$$I = Tr_{Q,P} \left[e^{-\beta H} (-1)^F (2J_3)^n \right]$$

In both we sum over J_3 and hence no ensemble change is needed for this

In the index I we take the trace for fixed Q, k, while in Z we integrate over Q, k keeping μ fixed

– need to change ensemble, but in D=4 this does not produce ($\ln \lambda$)

Conclusion of this analysis:

Logarithmic correction to the gravitational partition function Z with $\beta\omega = 2\pi i$ and $(2J_3)^n$ inserted

$\Rightarrow$ logarithmic correction to the supersymmetric index of the black hole.

As before, the logarithmic corrections to Z come from both non-zero modes and zero modes of massless fields.

Does this agree with the computation based on the near horizon geometry?

Non-zero mode contribution is C ln λ with

$$C = \int_{\text{full geometry}} d^4x\, K(x)$$

A surprise: All terms involving background gauge fields can be expressed in terms of the metric using equations of motion

In $N \geq 2$ supergravity, K(x) is proportional to the Gauss-Bonnet term Charles, Larsen arXiv:1505.01156; Karan, Panda arXiv:2012.12227

$\Rightarrow \int d^4x\, K(x)$ takes the same value in the

– full geometry at finite β

– the near horizon geometry $AdS_2 \times S^2$ geometry at $\beta = \infty$

since they have the same topology

The structure of the zero modes in the near horizon geometry and the full geometry are quite different

e.g. in the near horizon geometry there are infinite number of zero modes while in the full geometry there are a finite number of them

1. Integration over the translational zero modes produces ln λ for each zero mode

– cancelled by the ln λ that we need to subtract for each bosonic zero mode

2. Each of the two rotational zero modes produces 2 ln λ

After subtracting ln λ for each mode we are left with ln λ per zero mode

3. Each gravitino zero mode contributes $-\frac{1}{2}\ln\lambda + \frac{1}{2}\ln\lambda = 0$

Final result: Logarithmic correction from the zero modes are the same in the near horizon geometry and the full geometry

The $(2J_3)^n$ factor ensures that we get a finite answer after integrating over the goldstino zero modes associated with broken supersymmetry

$\Rightarrow$ the index computed from the full geometry correctly reproduces the microscopic results when they are known

e.g. in N=4,8 supersymmetric theories

Conclusion

Although this analysis has only reproduced known results, the agreement is significant due to several reasons:

1. The computation using the full geometry uses integration over the same set of modes as that for non-supersymmetric black holes

– gives us confidence in the results for non-supersymmetric black holes for which there is no independent test of the formula

2. In principle, the computation using the full geometry can be used to take into account all configurations that contribute to the index

e.g. multi-centered black holes

3. This formalism may be better suited for exact computation of supersymmetric index from gravitational path integral, e.g. via localization