Hilbert Space and Symmetries of Large N Extended States

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Contribution to the Memorial Meeting
in Honor of Eugene Cremmer, Jean-Loop Gervais and Costas Kounnas

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Outline

I. Hamiltonian for Large N

II. Large N soliton States

III. Large N Thermo-field double states

IV. Implementation of Symmetries

V.*2304.11767 with X. Liu and J. Zheng

Extended States at Large N

- Large N : (Re)constructing Gravity from CFT (AdS/CFT)
- Comparison of Correlation functions; But reconstruction
  - Solitons or Coherent States
  - Thermo-field double State ⇔ Two-sided Black Holes
- Classical picture (in the sense of h~ 1/N)
  - Fluctuations (subleasing 1/N) :Hilbert Space

- **Problems**
  - **Zero modes and Broken Symmetries**
  - **Large operators ($\sim \mathcal{O}(N)$ for matrix, $\sim \mathcal{O}(\sqrt{N})$ for vector models)**
  - $1/N$ **expansion(is questionable)**
  - **Factorization of Hilbert Space**
  - **2304.11767 with X. Liu and J. Zheng**
  - **Witten:2112.12828 :Cross Product and Gravity)**

Working with J L Gervais

- **1972 : City College :**
- **Supergauges in Dual Models (1971)Gervais+Sakita**
- **1973- Strings from Field Theory (Vortex Lines (Nambu))**
- **Soliton Quantization: Semiclassical Quantization of Dashen Hasslacher and Neveu (1973)**
- **Collective Coordinates[73-74]Brookhaven**

Perturbation Expansion Around Extended Particle States in Quantum Field Theory. (1975Jean-Loup Gervais(Ecole Normale Superieure), A. Jevicki(City Coll., N.Y.), B. Sakita

- **1974 LPTENS (move from Orsay)**
- **June 16-21:ENS Conference on Non-perturbative Developments in QFT: organized by JL Gervais and A. Neveu**
- **Discussions:R Jackiw TD Lee   JL Gervais: Path Integral vs Operator method**
- **August(at Cite Universitaire):(1976)Quantum Scattering of Solitons +Canonical Transformations in Path Integral**
- **Spring: Jean Loup (visited CC and went to theater with TD Lee (operator vs path integral issue was settled***)**

- **80’s JL Gervais continued to play a big role in development of Non-perturbative QFT , CFT**

- **Schools and Workshops: Les Houches and ENS:PhysicsReports**

- **Overlap on String Field Theory and Strings from Yang-Mills**

Large N Hamiltonian

A Canonical Framework for Large N: Dynamics at Large N can be completely described though through interaction of singlet(Collective) fields Φ. [A.J. and Sakita, 1981,D Bohm+Pines 50']

- Example: Two-Matrix Quantum Mechanics at Large N

Word C built from the alphabet $\{X, Y\}$: $C = X^{n_1}Y^{m_1}X^{n_2}Y^{m_2}\ldots$

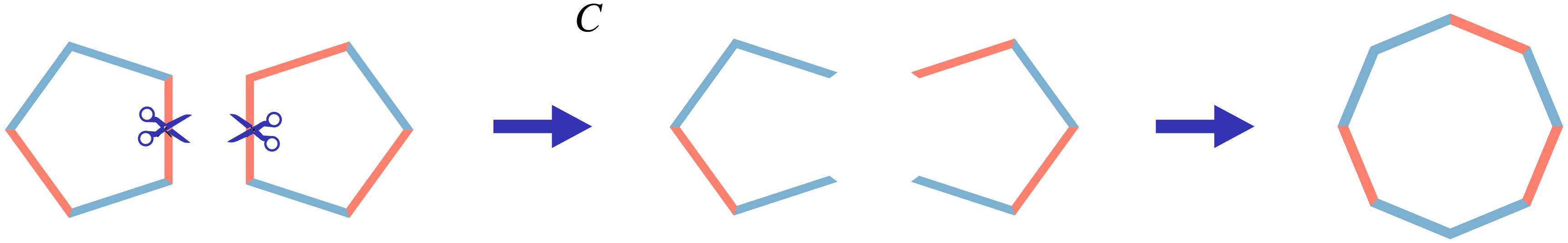
Collective fields are single traces: $\Phi(C) = \mathrm{Tr}(C)$

Obey a a closed set of equations: Migdal_Makkenko

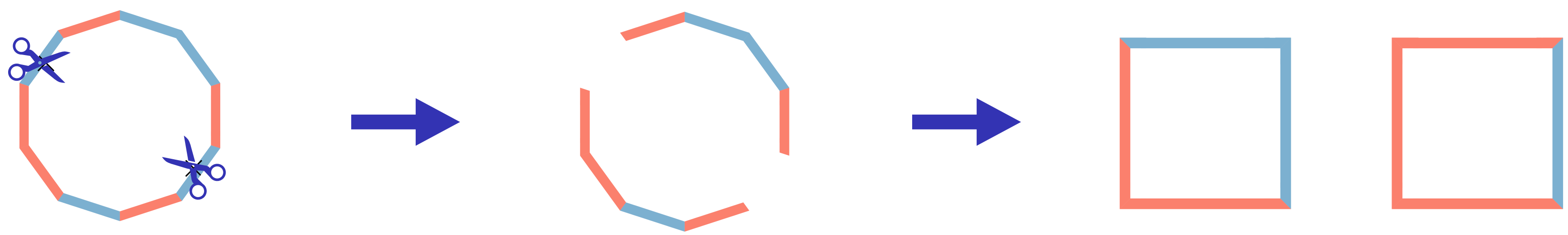
Equations of Motion <=> $H_{\text{col}} = \frac{1}{2}\Pi\,\Omega\,\Pi + V_{\text{col}}[\Phi]$

Large N Hamiltonian

Loop Joining $\Omega(C_1, C_2) = \sum_C j(C_1, C_2; C)\Phi(C)$



Loop Splitting $\omega(C) = \sum_{(C_1,C_2)} p(C; C_1, C_2)\Phi(C_1)\Phi(C_2)$



Collective potential $V_{\mathrm{col}}[\Phi] = N^2\left(\frac{1}{8}\omega\Omega^{-1}\omega + V[\Phi]\right)$

Numerical Optimization (Large N Bootstrap): V_{col}, numerically solved [d.M.Koch, A.J., Liu, Mathaba and Rodrigues, 2022].

Analytically solvable O(N) vector model will be used in the present discussion

Large N Hamiltonian: O(N) vector CFT

$$H = \int \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{m^2}{2}\varphi^2 + \frac{c}{4N}(\varphi^2)^2 \right] d^d x$$

- Two conformal points in 3d spacetime:
  - UV: $m = c = 0$
  - IR: m finite, $c = \infty$

$\Longleftrightarrow$ Higher Spin theory in AdS4

[Klebanov and Polyakov, 2002]

- Collective Hamiltonian :

$$\Psi(t, \mathbf{x}, \mathbf{y}) = \frac{1}{N}\varphi^i(\mathbf{x})\varphi^i(\mathbf{y})$$

- Bi-Local Field

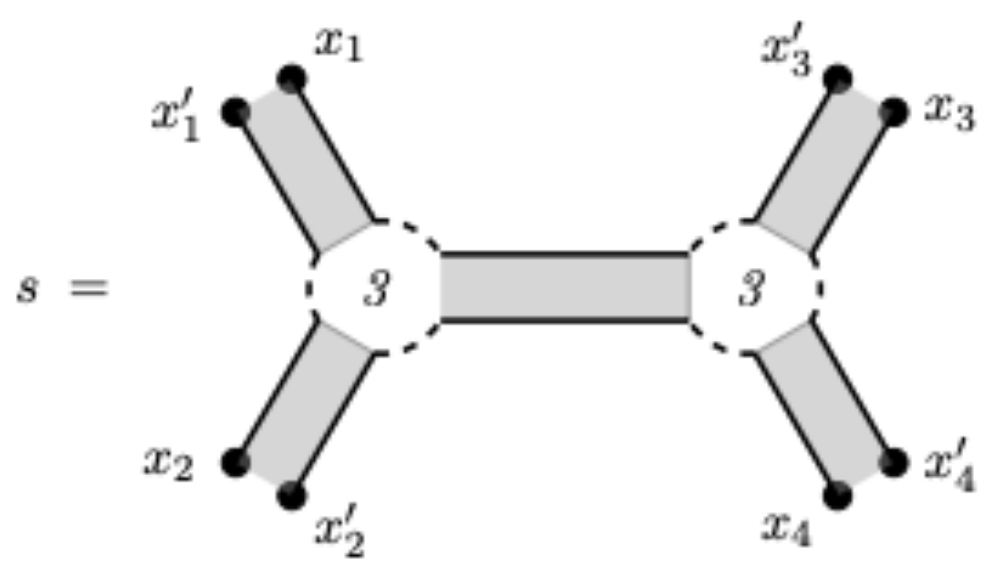
$$H_{\text{col}} = \frac{2}{N}\,\text{Tr}(\Pi\Psi\Pi) + \frac{N}{8}\,\text{Tr}(\Psi^{-1}) - \frac{N}{2}\,\text{Tr}(\nabla^2\Psi)$$

- Bulk construction(mapping) (4 ⇔ 4) [d.M. Koch, A.J., Jin and Rodrigues, 2011]

$$\mathscr{H}(\mathbf{p}, p^z, \theta) = \int d\mathbf{p}_{\text{CFT}}\, \mathrm{K}(\mathbf{p}_{\text{AdS}}; \mathbf{p}_{\text{CFT}})\, \Psi(\mathbf{p}_1, \mathbf{p}_2)$$

Constructive (AdS) Duality

- Of-shell AdS bulk Duality /Collective Fields S. R. Das and A. J, [hep-th/0304093]
  - Every Operator (in Collective Representation)= Bulk Operator
  - Coll Representation gives a 1/N Expansion
  - Bi-Local Diagrams = Ads Witten Diagrams:



- Demonstrated in :
  - R.deMello Koch, AJ, J.Yoon 1810.02332
  - also Aharony et al

Large N Solitons

- Soliton solutions exist in large N non-linear sigma models

Bifocal (coll field)

Soliton solution $$\Phi_s(x_1 + a, x_2 + a) \neq \Phi_s(x_1, x_2)$$

- Naive expansion:

$$\Phi = \Phi_s + \frac{1}{\sqrt{N}}\eta \qquad \Pi = \sqrt{N}\pi$$

$\mathcal{O}(1)$ Hamiltonian $$H = \frac{1}{2}\,\mathrm{Tr}(\pi\Omega_s\pi + \eta V\eta)$$

- Soliton State: $$|\, s, 0\rangle = |\, \Phi_s(x_1, x_2)\rangle\, e^{-\frac{1}{2}\,\mathrm{Tr}(\eta\mathcal{G}^{-1}\eta)}$$

↑

Static 2-pt function

Issues for the naive expansion

- Zero (Goldstone) mode

$$\mathcal{G} = \sum_{n=0}^{\infty} \frac{f_n^* f_n}{2\omega_n}, \qquad \Omega_s V f_n = \omega_n^2 f_n .$$

$$f_0 = \partial_{12}\Phi_s \Rightarrow V f_0 = 0$$

- Translation Op

$$P = \mathrm{Tr}(\Pi \partial_{12}\Phi) \equiv \sqrt{N} P_1 + P_2$$

- Large Operator $\quad P_1 = \mathrm{Tr}(\pi f_0) \quad$ O(N)

- Zero mode

$$[H, P] = \mathrm{Tr}[\eta V f_0] = 0$$

Issues in the naive expansion

- Issues:

1. Implementation of translations: $$P = \mathrm{Tr}(\Pi \partial_{12} \Phi) \equiv \sqrt{N} P_1 + P_2$$

$$\Phi' = \Phi'_s + \frac{1}{\sqrt{N}} \eta' \qquad \Pi' = \sqrt{N}(\Pi'_s + \pi')$$

$$e^{iaP}\,\Phi\, e^{-iaP} = \Phi_s + ia[P_1, \eta] - \frac{a^2}{2}[P_1,[P_2,\eta]] - \frac{a^2}{2}[P_2,[P_1,\eta]] + \ldots + \frac{1}{\sqrt{N}}\eta + \frac{ia}{\sqrt{N}}[P_2,\eta] - \frac{a^2}{2\sqrt{\ }}$$

- An infinite series re-summation is needed to evaluate the transformation.
- Different orders of 1/N and N mix : transformations on fluctuations will contribute to the background.

Resolution: Collective Coordinate Hilbert Space

Introduction of a collective coordinate : A position operator for the soliton $\hat{x}$, with a conjugate $\hat{p}$ ie $[\hat{x}, \hat{p}] = i$.

An Extended Hilbert space: in addition to η and π, we also have $\hat{x}$ and $\hat{p}$, which Non-linearly mix:

- Constraint: $\hat{p} - P[\Pi, \Phi] \,|\, s,0\rangle = 0$
  - Gauge condition: $\chi_{\hat{x}}[\Pi, \Phi] \,|\, s,0\rangle = 0$ $\qquad \chi_{\hat{x}} \equiv e^{-i\hat{x}P} \chi e^{i\hat{x}P}$
    - Linear gauge condition: $\int f_0 \Phi(x_1 + \hat{x}, x_2 + \hat{x}) dx_1 dx_2 \,|\, s,0\rangle = 0$
    - Canonical gauge condition: $\hat{x} - \frac{\int (x_1 + x_2)\mathscr{H}_{\text{col}} dx_1 dx_2}{H_{\text{col}}} \,|\, s,0\rangle = 0$

$$\downarrow$$

$$[\hat{x}, \hat{p}] = [KH^{-1}, P] = i$$

K is the boost operator

Resolution: Collective coordinate method

- Change of Frame $x \to x + \hat{x}$

$$\Phi'(x_1, x_2) = e^{i\hat{x}P}\Phi(x_1, x_2)e^{-i\hat{x}P} = \Phi(x_1 + \hat{x}, x_2 + \hat{x})$$

$$\Pi'(x_1, x_2) = e^{i\hat{x}P}\Pi(x_1, x_2)e^{-i\hat{x}P} = \Pi(x_1 + \hat{x}, x_2 + \hat{x})$$

- For the Soliton State

$$|s, 0\rangle' = |\Phi_s(x_1 + \hat{x}, x_2 + \hat{x})\rangle e^{-\frac{1}{2}\mathrm{Tr}(\hat{\Phi}'\mathscr{G}'^{-1}\hat{\Phi}')}$$

he Zero mode is projected out ,by the gauge condition: $\mathscr{G}' = \sum_{n=1}^{\infty} \frac{f_n^* f_n}{2\omega_n}$

- Translations: operate in the extended Hilbert space:

- Momentum eigen- state: $|s, a\rangle' = e^{ia\hat{p}}|s,0\rangle'$

  - 1-pt: $\langle s, p'|' \Phi_s'(x_1 - \hat{x}, x_2 - \hat{x})|s, p\rangle' = \int dy e^{i(p-p')y}\Phi_s'(x_1 - y, x_2 - y)$

  - 2-pt: $\langle \Phi(x_1, x_2, t)\Phi(y_1, y_2, t_0)\rangle = \langle \Phi'(x_1 - \hat{x}(t), x_2 - \hat{x}(t), t)\Phi'(y_1 - \hat{x}(t_0), y_2 - \hat{x}(t_0), t_0)\rangle$

1/ N Expansion

- The Hamiltonian becomes

$$H_{\rm col} = \frac{M_0}{2}\left(\frac{\hat{p} - {\rm Tr}[\pi'\partial_{12}\eta']}{{\rm Tr}[\partial_{12}\Phi_s'\partial_{12}\Phi']}\right)^2 + \frac{1}{2}{\rm Tr}[\pi'\Omega_s\pi' + \eta' V\eta'] - \frac{1}{8\sqrt{N}}{\rm Tr}[\Phi_s'^{-1}\eta'\Phi_s'^{-1}\eta'\Phi_s'^{-1}\eta'\Phi_s'^{-1}] + \mathcal{O}(\frac{1}{N})$$

It admits a 1/N expansion:

$$H_{\rm col} = \frac{\hat{p}^2}{2M_0} + \frac{1}{2}{\rm Tr}[\pi'\Omega_s\pi' + \eta' V\eta'] - \frac{\hat{p}}{M_0}{\rm Tr}[\pi'\partial_{12}\eta'] - \frac{\hat{p}^2}{M_0^2}{\rm Tr}[\partial_{12}\Phi_s'\partial_{12}\eta'] - \frac{1}{8\sqrt{N}}{\rm Tr}[\Phi_s'^{-1}\eta'\Phi_s'^{-1}\eta'\Phi_s'^{-1}\eta'\Phi_s'^{-1}] + \mathcal{O}(\frac{1}{N})$$

- 1/N Expansion[for States and Correlation functions in the Extended Hilbert space

Collective coordinate method: Recap

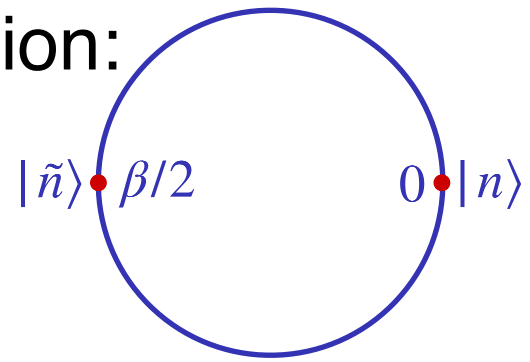
Large N backgrounds that break certain symmetries lead to zero modes and large operators. A naive 1/N expansion with these backgrounds is problematic, and the implementation of symmetry transformations is .

Collective coordinate method solves these problems through introducing new degrees of freedom, and an extended Hilbert space . Nonlinear Constraints (and gauge conditions)are applied on this extended Hilbert space.

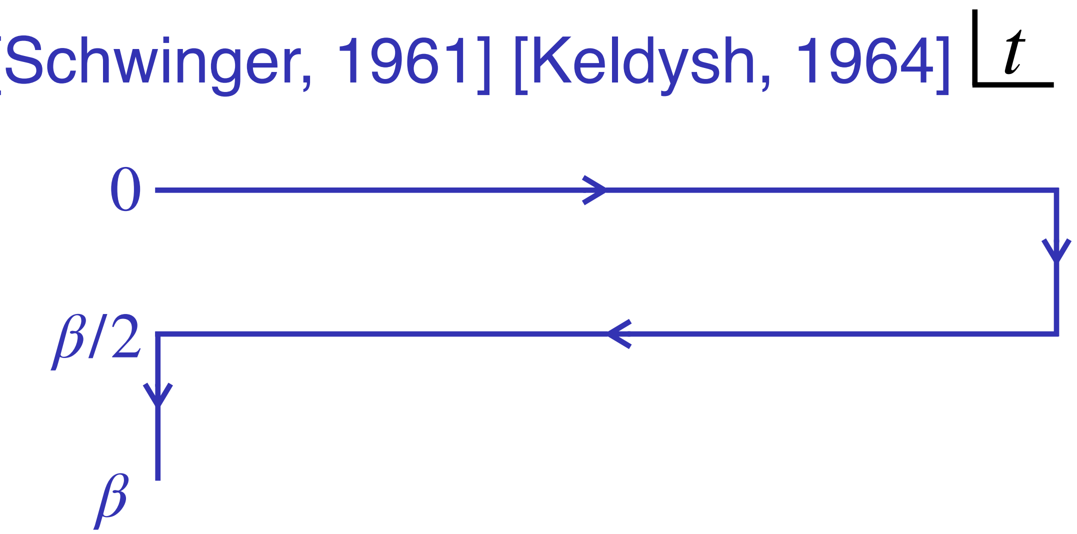
1/N Expansion is then well defined,the zero modes are projected out from linear fluctuations .

TFD at Large N

- Purification:



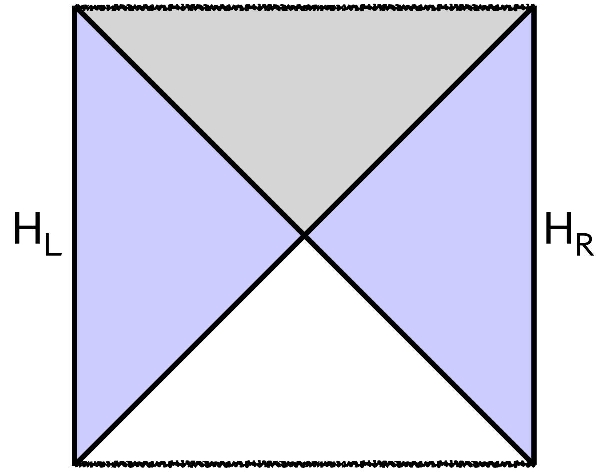
[Schwinger, 1961] [Keldysh, 1964]



$$|0(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n\rangle \otimes |\tilde{n}\rangle$$

$$\langle 0(\beta) | \mathrm{O} | 0(\beta) \rangle = \frac{1}{Z(\beta)} \mathrm{Tr}(e^{-\beta H} \mathrm{O})$$

- Thermo-field (modular) Hamiltonian: $\hat{H} = H - \tilde{H}$ $\hat{H} | 0(\beta) \rangle = 0$



O(N) TFD: Construction

- Two-sided black hole spacetime with L, R asymptotic AdS boundaries:

$$\mathrm{CFT_L} \otimes \mathrm{CFT_R}$$

- Direct product Gauging

$$\varphi_i \to O_{ij}\varphi_j \qquad \tilde{\varphi}_i \to \tilde{O}_{ij}\tilde{\varphi}_j$$

$$H_{\rm col}[\Pi, \Phi] \quad \Phi = \varphi^i\varphi^i \quad J_{ij}|0(\beta)\rangle = 0 \quad \tilde{J}_{ij}|0(\beta)\rangle = 0 \quad \tilde{\Phi} = \tilde{\varphi}^i\tilde{\varphi}^i \quad \tilde{H}_{\rm col}[\tilde{\Pi}, \tilde{\Phi}]$$

- There is a debate: $\mathrm{CFT_L} \otimes \mathrm{CFT_R}$ [Mathur] [Marolf and Wall] produces a dual connected spacetime of two-sided black hole [Maldacena, 2003].

- More generally [Van Raamsdonk], Entanglement ⇒ Wormhole spacetime:Einstein-Rosen bridge

Solving:for the Large N Thermofield

Diagonal gauging of the O$(N)\times$O(N) group [A.J. and Yoon, 2015]

$$(J^{ij}+\tilde{J}^{ij})\,|\,0(\beta)\rangle = 0$$

This ends up with 4 bi-local collective fields:

$$\Phi(\mathbf{x},\mathbf{y}) \equiv \begin{pmatrix} \Phi^{11} & \Phi^{12} \\ \Phi^{21} & \Phi^{22} \end{pmatrix}(\mathbf{x},\mathbf{y}) := \frac{1}{N}\begin{pmatrix} \varphi^i\varphi^i & \varphi^i\tilde{\varphi}^i \\ \tilde{\varphi}^i\varphi^i & \tilde{\varphi}^i\tilde{\varphi}^i \end{pmatrix}(\mathbf{x},\mathbf{y})$$

Mixed modes :Φ^{12} and Φ^{21} appear necessary for an extensive (complete) spectrum in the bulk

$$\hat{H}_{\mathrm{col}} = \frac{2}{N}\,\mathrm{Tr}[\Pi\star(\sigma_3\Phi)\star\Pi]$$

$$+\frac{N}{8}\,\mathrm{Tr}[\sigma_3\Phi^{-1}] + \frac{N}{2}\,\mathrm{Tr}[(-\nabla^2+m^2)\star(\sigma_3\Phi)] + \frac{Nc}{4}\left(\mathrm{Tr}[(\Phi^{11})^2] - \mathrm{Tr}[(\Phi^{22})^2]\right)$$

$$NV_{\mathrm{col}}$$

Thermal background

Thermal background is given by [A.J. and Yoon, 2015]]

$$\frac{\delta V_{\text{col}}}{\delta \Phi} = 0 \quad \Rightarrow \quad \Phi_f(\mathbf{x}, \mathbf{y}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_f(\mathbf{k})} \begin{pmatrix} \cosh f(\mathbf{k}) & \sinh f(\mathbf{k}) \\ \sinh f(\mathbf{k}) & \cosh f(\mathbf{k}) \end{pmatrix}$$

$$\omega_f(\mathbf{k})^2 = \mathbf{k}^2 + m^2 + \sigma_f$$

We have a one-parameter family of solutions labeled by f. The solutions are the static 2-pt functions at finite temperature:

$$f(\mathbf{k}) \equiv 2\theta(\mathbf{k}) = 2 \operatorname{arctanh} e^{-\beta\omega_f(\mathbf{k})/2} \qquad \Phi_f = \langle \Phi \rangle_\beta$$

- Two symmetries: $[\hat{H}, G] = [\hat{H}, H_+] = 0 \qquad [G, H_+] \neq 0$

TFD State: Wave functional

$1/N$ effects can be studied by expanding $\Phi = \Phi_f + \frac{1}{\sqrt{N}}\eta$

$$\hat{H}_2 = \frac{1}{2}\,\mathrm{Tr}[\pi^{\mathrm{T}} K \pi + \eta^{\mathrm{T}} V \eta]$$

At $\mathcal{O}(1)$ we have the TFD wave functional (Gaussian form):

$$\Psi_\beta[\eta] = \exp\left(-\frac{1}{2}\,\mathrm{Tr}[\eta^{\mathrm{T}} \mathcal{G}^{-1} \eta]\right)$$

- Elements of $\mathcal{G}$ are the corresponding $\mathcal{O}(1)$ (connected) correlation functions:

$$\mathcal{G}^{ab,cd} \sim \langle \eta^{ab} \eta^{cd} \rangle_\beta$$

- $\Psi_\beta[\eta]$ is annihilated by $\hat{H}_2$ as expected:

$$\hat{H}_2 \Psi_\beta[\eta] = 0$$

Emergence of Bulk fields

$$A_L \left\{ \begin{array}{ll} \eta_\theta^{11}(k_1,k_2) & \\ \text{All } |k_1|,|k_2| & \omega_{Bulk}(p) \geq |p|_{Bulk} \\ & \\ \eta_\theta^{12}(k_1,k_2) & \\ \eta_\theta^{21}(k_1,k_2) & \\ |k_1| \geq |k_2| & 0 \leq \omega_{Bulk}(p) \leq |p|_{Bulk} \end{array} \right\}$$

Extensive (Complete) Spectrum

$\omega_{Bulk}(p) \geq 0$

Similarly for A_R. We have a factorized algebra $\{A_L(\omega, p)\} \otimes \{A_R(\omega, p)\}$, with $\omega \in [0,\infty)$. These relate to "Generalized Free Fields" by Fourier transforms up to some normalization factors:

$$O_{L,R}(t,x) = \int d\omega dp\, \mathcal{N}(\omega,p)\, A_{L,R}(\omega,p)\, e^{i\omega t - ip\cdot x}$$

Zero modes and Symmetries

$$\hat{H}_2 = \frac{1}{2}\mathrm{Tr}[\pi^{\mathrm{T}} K \pi + \eta^{\mathrm{T}} V \eta]$$

We have zero modes $\mathrm{Tr}[K u_k] = 0$ $\quad \mathrm{Tr}[V u_k] = 0$

The symmetry operators instead have Large $\mathcal{O}(\sqrt{N})$ terms:

$$H_+ = \sqrt{N} H_+^1 + H_+^2 + \ldots \qquad G_f = \sqrt{N} G_f^1 + G_f^2 + \ldots$$

- The zero modes are in one-to-one correspondence with the leading order of the symmetry operators (sum over k is implicit)

$$H_{+,1} = \mathrm{Tr}[c_1 u^{\mathrm{T}} \eta] \quad \Longleftrightarrow \quad 0 = [\hat{H}_2, H_+^1] = \mathrm{Tr}[c_1 K u]$$

$$G_{f,1} = \mathrm{Tr}[c_2 v^{\mathrm{T}} \pi] \quad \Longleftrightarrow \quad 0 = [\hat{H}_2, G_f^1] = \mathrm{Tr}[c_2 V v]$$

- An infinite re-summation is needed to compute the symmetry transformation:
- Large N Matrix models also feature large operators of $\mathcal{O}(N)$
- And in numerical simulation a Single Zero mode was identified (X.Liu unpublished)

Collective coordinates and extended Hilbert space

A Short Summary

Extended Hilbert space: $\{\mathrm{A_L}\} \otimes \{\hat{q}, \hat{h}\} \otimes \{\mathrm{A_R}\}$ $\qquad [\hat{h}, \hat{q}] = i$

- Constraint: $$\hat{h} - \mathrm{H}_+[\Pi, \Phi]\,|\,0(\beta)\rangle = 0$$
- Gauge condition: $$\hat{q} - \frac{\mathrm{D}_+[\Pi, \Phi]}{\mathrm{H}_+[\Pi, \Phi]}\,|\,0(\beta)\rangle = 0$$

$\mathrm{D}_+ = \mathrm{D}_1 + \mathrm{D}_2$ is the dilatation operator.

Change of reference frame: by redefining the states and operators in the following way:

$$O'(\hat{q}) = e^{i\hat{q}H_+} O e^{-i\hat{q}H_+}$$

$$|\,\Psi'(\hat{q})\rangle = e^{i\hat{q}H_+}\,|\,\Psi\rangle$$

- Nonlinear:Systematic 1/N expansion

Witten's Treatment

Bulk algebra: $\{A_l\} \otimes \{A_r\}$

Central generator and its conjugate:

$$U = \frac{1}{N} H'_L \qquad \Pi = -i\frac{d}{dU}$$

Identification: $$\frac{1}{N} H'_R = U + \frac{1}{N\beta}\hat{h} \qquad \hat{h} = \beta \hat{H}$$

Re defined algebra: $$A_R = A_r \rtimes A_{U+\hat{h}/\beta N}$$

$$A_L = e^{i\Pi\hat{h}/\beta N} A_l e^{-i\Pi\hat{h}/\beta N} \rtimes A_U$$

This treatment is at O(1).

Problem of systematic 1/N Expansion : Exp (i f N U).

Conclusions

- (Goldstone) Symmetries in the Large N Hilbert Space[following the early work of Gervais et al]

- Concrete study case: The Large N TFD in O(N)

- Analogously: Matrix QM

- Jean- Loup Gervais , Eugene Cremmer and Costas Kounnas are greatly missed, their accomplishments and influence is here to stay.