Classical Observables from the Exponential Representation of the Gravitational S-Matrix

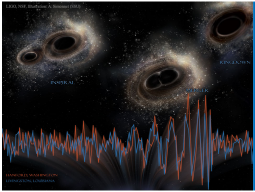
Pierre Vanhove

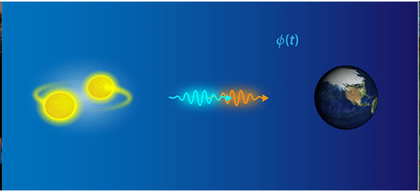
IPhT Saclay

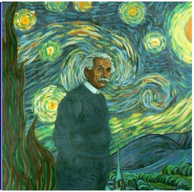
TUG, LPENS Paris, France

based on work done in collaboration with
Emil Bjerrum-Bohr, Poul Damgaard, Ludovic Planté, Elias Roos Hansen
[arXiv:2212.08957], [arXiv:2306.11454], [arXiv:2307.04746]

Einstein theory of gravity is the main paradigm for understanding the structure and dynamic of our observable Universe



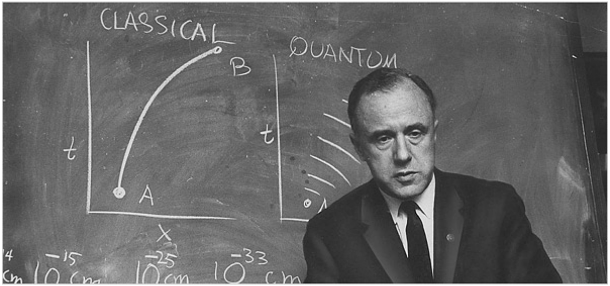




Black holes, compact stars and gravitational waves are amongst the most spectacular predictions of general relativity and natural probes of the fundamental principles of Einstein's theory and its extension, e.g.

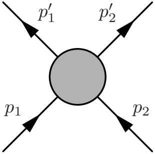
- The activation of scalar fields
- Gravitational leakage into large extra dimensions
- Variability of Newton's constant
- Propagation of gravitational waves
- gravitational Lorentz violation
- strong equivalence principle
- Higher-derivative corrections, …

Classical gravity from quantum scattering



One important **new** insight is that the **classical** gravitational two-body interactions (conservative and radiation) can be extracted from **quantum scattering amplitudes**

$$\mathcal{M}(\gamma, q^2) =$$



$$= \sum_{L=0}^{+\infty} G_N^{L+1} \mathcal{M}^{\mathrm{L-loop}}; \quad \gamma = \frac{p_1 \cdot p_2}{m_1 m_2} = \frac{1}{\sqrt{1-(\vec{v}_1 - \vec{v}_2)^2}}$$

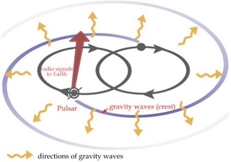
Classical gravity from quantum amplitudes

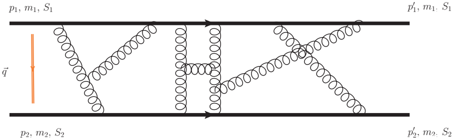
THE GENERATION OF GRAVITATIONAL WAVES.
IV. BREMSSTRAHLUNG*†‡

SÁNDOR J. KOVÁCS, JR.
W. K. Kellogg Radiation Laboratory, California Institute of Technology
AND
KIP S. THORNE
Center for Radiophysics and Space Research, Cornell University; and
W. K. Kellogg Radiation Laboratory, California Institute of Technology
Received 1977 October 21; accepted 1978 February 28

g) The Feynman-Diagram Approach

Any classical problem can be solved quantum-mechanically; and sometimes the quantum solution is easier than the classical. There is an extensive literature on the Feynman-diagram, quantum-mechanical treatment of gravitational bremsstrahlung radiation (e.g., Feynman 1961, 1963; Barker, Gupta, and Kaskas 1969; Barker and Gupta





We will then present a scheme using the modern quantum scattering amplitudes to provide an optimal framework for gravitational observable that will be the starting point of the wave-form analysis

Gravity effective field theories

The **effective field theories** approach is well suited for studying gravity at various (distance or energy) scales

We will be assuming :

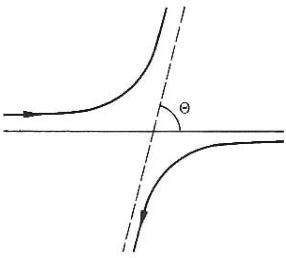
- Standard QFT (local, unitary, lorentz invariant, ...)
- The low-energy DOF: graviton, matter fields,...
- Standard symmetries: General relativity as we know it
- Work in D dimensions (*not* just $D = 4$ gravity)

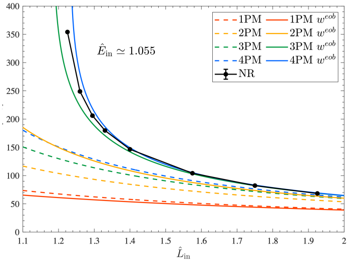
$$\mathcal{S}_{eff} = \mathcal{S}_{eff}^{\text{gravity}} + \mathcal{S}_{eff}^{\text{matter}}$$

$$\mathcal{S}_{eff}^{\text{gravity}} = \frac{1}{16\pi G_N} \int d^D x \sqrt{g} \mathcal{R} + \cdots$$

Matter fields couple to a unique metric $g_{\mu\nu}$: $\mathcal{S}_{eff}^{\text{matter}}(\psi_i, g_{\mu\nu})$

Scattering amplitudes





- Classical scattering: scattering angle χ : connection from scattering amplitude to EOB and comparison with numerical GR[1]
- A formalism valid from small relative velocities $\gamma \sim 1$ (Post-newtonian regime)[2] to the ultra-relativistic (ACV) regime $\gamma \to \infty$[3]
- The important problem of gravitational radiation: momentum kick and angular momentum loss[4]

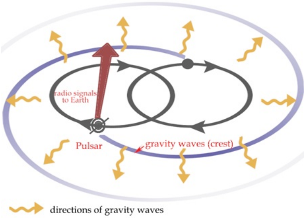
[1] [Rettegno, Pratten, Thomas, Schmidt, Damour]

[2] [Blanchet et al.; Bern et al.; Damour et al.]

[3] [di Vecchia, Heissenberg, Veneziano]

[4] [Mougiakakos, Riva, Vernizzi; Riva, Vernizzi, Wong; O'Connell et al.; ···]

Radiation reaction



The problem of **radiation reaction** has been one of the fundamental theoretical issues in general relativity. This is a needed contribution to match the binary-pulsar observations. But a consistent derivation of radiation-reaction effects has been missing for a long time

Several derivations combining diverse arguments have been used

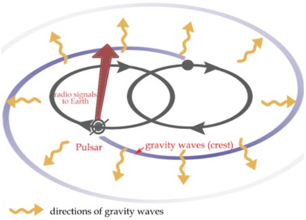
- Formal post-Newtonian expansions but plagued by infrared ambiguities from the matching nearzone gravitational field ("potential modes") and the wavezone gravitational field ("soft modes")
- High-energy scattering[5]
- Linear response to the angular momentum[6]
- Reverse unitarity and KMOC formalism[7]

[5] [di Vecchia, Russo, Heisenberg, Veneziano]

[6] [Damour; Veneziano, Vilkovisky; Manohar, Ridgway, Shen]

[7] [Parra-Martinez et al.; Mougiakakos, Riva, Vernizzi]

Radiation reaction



The problem of **radiation reaction** has been one of the fundamental theoretical issues in general relativity. This is a needed contribution to match the binary-pulsar observations. But a consistent derivation of radiation-reaction effects has been missing for a long time

- The radiation-reaction from the soft region of the amplitude (not in the potential region of[5])
- The radiation-reaction is needed for restoring a smooth continuity between the non-relativitic, relativistic and ultra-relativistic regimes[6]

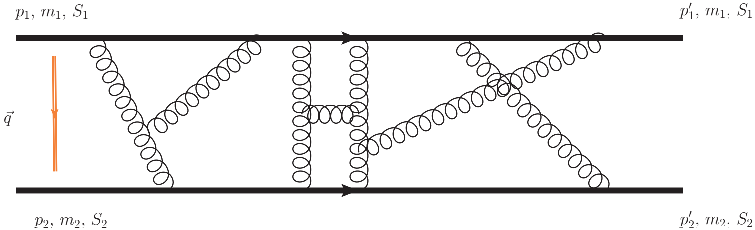
The evaluation of the complete classical scattering amplitude gives a clear-cut unified and unambiguous resolution of these issues[7]

[5] [Bern et al.]

[6] [Damour, Veneziano et al.]

[7] [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Classical physics from quantum loops



The *classical* limit $\hbar \to 0$ with $\underline{q} = q/\hbar$ fixed at each loop order

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0)$$

- A classical contribution of order $1/\hbar$ from all loop orders[8]
- **classical gravity physics contributions** are determined by the **unitarity** of the quantum scattering amplitudes

[8] [Iwasaki; Holstein, Donoghue; Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Доверяй, но проверяй : $\hbar$ counting

In QFT the propagator[9] has inverse $\hbar$ that the traditional counting disregards[10]

> To find the connection between L and the power of $\hbar$, we collect all factors $\hbar$. We leave aside the factor $\hbar$ that gives the mass term a correct dimension. In other words, the Klein-Gordon equation should read $[\partial_x^2 + (mc/\hbar)^2]\varphi = 0,$ indicating that the mass term is of quantum origin. This phenomenon is disregarded in the sequel. There are thus two origins of such factors. First the

At the $L+1$PM order, the two-body scattering amplitude scales with the masses as

$$\mathcal{M}_L(\gamma, q^2) = \frac{G_N^{L+1} m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \sum_{i=0}^{L} c_{L-i+2,i+2}(\gamma) m_1^{L-i} m_2^{i}$$

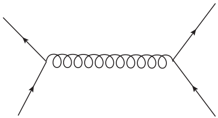
This piece will emerge from a L quantum amplitude as follows

$$\mathfrak{M}^L\Big|_{\text{classical}} \propto \frac{m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \hbar^{L-1} G_N^{L+1} \sum_i \left(\frac{m_1 c}{\hbar}\right)^{L-i} \left(\frac{m_2 c}{\hbar}\right)^{i} \propto \frac{\mathcal{M}_L(\gamma, \underline{q}^2)}{\hbar}$$

[9] [Iwasaki; Holstein, Donoghue; Bjerrum-Bohr, Damgaard, Planté, Vanhove]

[10] [Itzykson & Zuber "Quantum Field Theory", §6-2-1 page 288]

One graviton exchange : tree-level amplitude



$$\mathfrak{M}_0 = -16\pi G_N \hbar \frac{2(p_1\cdot p_2)^2 - m_1^2 m_2^2 - |\hbar \underline{\vec{q}}|^2 (p_1\cdot p_2)}{|\hbar \underline{\vec{q}}|^2}$$

The $\hbar$ expansion of the tree-level amplitude

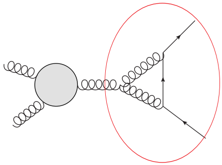
$$\mathfrak{M}_0 = \frac{\mathcal{M}_1^{(-1)}(p_1\cdot p_2)}{\hbar |\underline{q}|^2} + \hbar 4\pi G_N p_1\cdot p_2$$

The higher order in q^2 are quantum with powers of $\hbar$

The *classical* potential is obtained by taking the 3d Fourier transform
$E_i = \sqrt{p_i^2 + m_i^2}$

$$\mathcal{V}_1(p_1\cdot p_2, r) = \int \frac{d^3\underline{\vec{q}}}{(2\pi)^3} \frac{\mathcal{M}_1^{(-1)}(\vec{q})\, e^{i\underline{\vec{q}}\cdot\vec{r}}}{4E_1E_2} = \frac{G_N}{E_1E_2}\frac{m_1^2m_2^2 - 2(p_1\cdot p_2)^2}{r}$$

Classical physics from loops : the one-loop triangle



The Klein-Gordon equation reads

$$(\Box - \frac{m^2c^2}{\hbar^2})\phi = 0$$

The triangle with a massive leg $p_1^2 = p_2^2 = m^2$ reads

$$\int \frac{G_N\, d^4\ell}{(\ell+p_1)^2(\ell^2 - \frac{m^2c^2}{\hbar^2})(\ell-p_1')^2}\Big|_{\text{finite part}}$$

$$\sim \frac{G_N}{m^2}\left(\frac{\pi^2 mc}{\hbar|\underline{q}|} + \log(q^2)\right)$$

Fourier transformed with respect to the non-relativistic momentum transfert $|\vec{q}|$ leads to $2G_N m_1/(c^2 r)$ classical 2nd post-Minkowskian contribution

Exponentiation of the S-matrix

Using an exponential representation of the $\widehat{S}$ matrix[11]

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right)$$

doing the Dyson expansion with the conservative and radiation part

$$\hat{T} = G_N \sum_{L\geqslant 0} G_N^L \hat{T}_L + G_N^{\frac{1}{2}} \sum_{l\geqslant 0} G_N^L \hat{T}_L^{\rm rad}, \qquad \hat{N} = G_N \sum_{L\geqslant 0} G_N^L \hat{N}_L + G_N^{\frac{1}{2}} \sum_{l\geqslant 0} G_N^L \hat{N}_L^{\rm rad}$$

The classical radial action $\hat{N}^{\rm classical}$ does not have any $\hbar$. The higher power of $1/\hbar$ more singular than the classical **are needed for the consistency of the full quantum amplitude and the correct exponentiation of the amplitude**

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1}|\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar|\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0)$$

[11] [Damgaard, Planté, Vanhove]

Exponentiation of the S-matrix

Using an exponential representation of the $\widehat{S}$ matrix[12]

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right)$$

with the completeness relation that includes all the exchange of gravitons

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^D k_1}{(2\pi\hbar)^{D-1}} \delta^+((k_1)^2 - m_1^2) \frac{d^D k_2}{(2\pi\hbar)^D} \delta^+((k_2)^2 - m_2^2)$$
$$\times \frac{d^{D-1}\ell_1}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{\ell_1}} \cdots \frac{d^{D-1}\ell_n}{(2\pi\hbar)^{(D-1)}} \frac{1}{2E_{\ell_n}} \times |k_1, k_2; \ell_1, \ldots \ell_n\rangle\langle k_1, k_2; \ell_1, \ldots \ell_n|,$$

- The conservative part without exchange of virtual gravitons
- We have the exchange of the intermediate virtual gravitons
- The virtual gravitons will lead to the important radiation-reaction contributions $\hat{N}^{\text{rad}}$

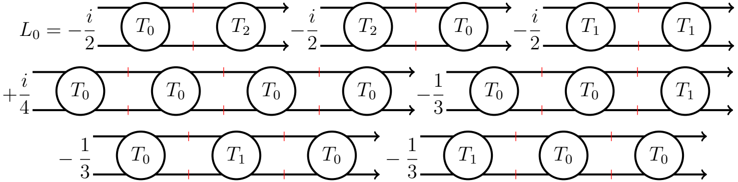
[12] [Damgaard, Planté, Vanhove]

Velocity cuts

The relation between $\hat{N}$ and $\hat{T}$ substracts the superclassical terms

$$\hat{N}_3 = \hat{T}_3 - \frac{i}{2\hbar}(\hat{N}_1^{\rm rad}\hat{N}_0^{\rm rad} + \hat{N}_0^{\rm rad}\hat{N}_1^{\rm rad}) - \frac{i}{2\hbar}\hat{T}_1^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_2 + \hat{T}_2\hat{T}_0)$$
$$- \frac{1}{12\hbar^2}[\hat{N}_0^{\rm rad},[\hat{N}_0^{\rm rad},\hat{N}_0]] - \frac{1}{3\hbar^2}(\hat{T}_0^2\hat{T}_1 + \hat{T}_0\hat{T}_1\hat{T}_0 + \hat{T}_1\hat{T}_0^2) + \frac{i}{4\hbar^3}\hat{T}_0^4\,.$$

No virtual graviton exchanges



The insertion of complete state induces velocity cuts on the massive line:

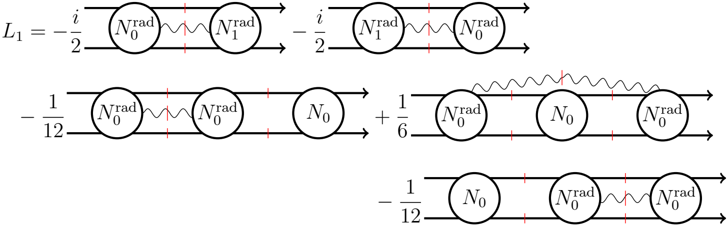
$$\delta^+((p_1+\hbar \underline{q})^2 - m_1^2) = \frac{\delta(2p_1\cdot \underline{q})}{\hbar} + O\left(\frac{1}{\hbar^2}\right)$$

Velocity cuts

The relation between $\hat{N}$ and $\hat{T}$ substracts the superclassical terms

$$\hat{N}_3 = \hat{T}_3 - \frac{i}{2\hbar}(\hat{N}_1^{\rm rad}\hat{N}_0^{\rm rad} + \hat{N}_0^{\rm rad}\hat{N}_1^{\rm rad}) - \frac{i}{2\hbar}\hat{T}_1^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_2 + \hat{T}_2\hat{T}_0)$$
$$- \frac{1}{12\hbar^2}[\hat{N}_0^{\rm rad}, [\hat{N}_0^{\rm rad}, \hat{N}_0]] - \frac{1}{3\hbar^2}(\hat{T}_0^2\hat{T}_1 + \hat{T}_0\hat{T}_1\hat{T}_0 + \hat{T}_1\hat{T}_0^2) + \frac{i}{4\hbar^3}\hat{T}_0^4 \,.$$

One virtual graviton exchanges



The insertion of complete state induces velocity cuts on the massive line:

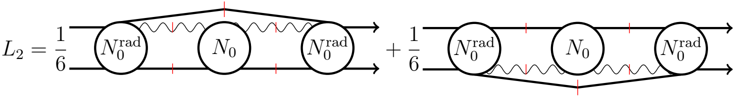
$$\delta^+((p_1 + \hbar \underline{q})^2 - m_1^2) = \frac{\delta(2p_1 \cdot \underline{q})}{\hbar} + O\left(\frac{1}{\hbar^2}\right)$$

Velocity cuts

The relation between $\hat{N}$ and $\hat{T}$ substracts the superclassical terms

$$\hat{N}_3 = \hat{T}_3 - \frac{i}{2\hbar}(\hat{N}_1^{\rm rad}\hat{N}_0^{\rm rad} + \hat{N}_0^{\rm rad}\hat{N}_1^{\rm rad}) - \frac{i}{2\hbar}\hat{T}_1^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_2 + \hat{T}_2\hat{T}_0)$$
$$- \frac{1}{12\hbar^2}[\hat{N}_0^{\rm rad}, [\hat{N}_0^{\rm rad}, \hat{N}_0]] - \frac{1}{3\hbar^2}(\hat{T}_0^2\hat{T}_1 + \hat{T}_0\hat{T}_1\hat{T}_0 + \hat{T}_1\hat{T}_0^2) + \frac{i}{4\hbar^3}\hat{T}_0^4 \,.$$

Two virtual gravitons exchanges



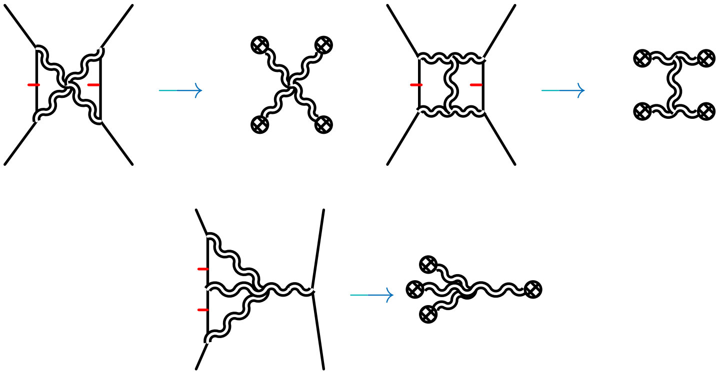
The insertion of complete state induces velocity cuts on the massive line:

$$\delta^+((p_1 + \hbar \underline{q})^2 - m_1^2) = \frac{\delta(2p_1 \cdot \underline{q})}{\hbar} + O\left(\frac{1}{\hbar^2}\right)$$

Velocity cuts and Classical amplitude

The matrix elements $\langle p_1, p_2|\hat{N}|p'_1, p'_2\rangle$ are *classical* with *exactly* L-delta functions on the massive propagators

There is a complete equivalence with the world-line formalism for the classical piece[13]



[13] [Bjerrum-Bohr, Damgaard, Planté, Vanhove; Damgaard, Hansen, Planté, Vanhove]

The $\hat{N}$ operator upto 1PM and 2PM

$$\hat{S} = \exp\left(\frac{i\hat{N}}{\hbar}\right); \qquad N(\gamma, q^2) := \langle p_1, p_2 | \hat{N} | p_1', p_2' \rangle$$

$$\tilde{N}(\gamma, J) \equiv \int \frac{d^2q}{(2\pi)^2} \frac{N(\gamma, q^2)}{4m_1m_2\sqrt{\gamma^2-1}} e^{i\frac{J\cdot q}{p_\infty}}; \qquad p_\infty = \frac{m_1m_2\sqrt{\gamma^2-1}}{\sqrt{m_1^2+m_2^2+2m_1m_2\gamma}}$$

$$\tilde{N}^{(1PM)}(\gamma, J) = \frac{Gm_1m_2(2\gamma^2-1)}{\sqrt{\gamma^2-1}}\Gamma(-\epsilon)J^{2\epsilon}$$

$$\tilde{N}^{(2PM)}(\gamma, J) = \frac{3\pi G^2m_1^2m_2^2(m_1+m_2)(5\gamma^2-1)}{4\sqrt{s}}\frac{1}{J}$$

The 1PM (tree-level) and 2PM (one-loop) contributions are the same as for a test mass in the Schwarzschild black hole of mass $M = m_1 + m_2$.

The $\hat{N}$ operator at 3PM

$$\tilde{N}^{(3PM)}(\gamma,J)=\frac{G^3m_1^3m_2^3\sqrt{\gamma^2-1}}{sJ^2}\left(\frac{s(64\gamma^6-120\gamma^4+60\gamma^2-5)}{3(\gamma^2-1)^2}\right.$$
$$-\frac{4}{3}m_1m_2\gamma(14\gamma^2+25)+\frac{4m_1m_2(3+12\gamma^2-4\gamma^4)\,\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}$$
$$\left.+\frac{2m_1m_2(2\gamma^2-1)^2}{\sqrt{\gamma^2-1}}\left(\frac{8-5\gamma^2}{3(\gamma^2-1)}+\frac{\gamma(-3+2\gamma^2)\,\mathrm{arccosh}(\gamma)}{(\gamma^2-1)^{\frac{3}{2}}}\right)\right)$$

At 3PM (two-loop) new phenomena arise

- The conservative part deviates from Schwarzschild as we have contributions which depends (linearly) on the relative mass[14] $\nu=\frac{m_1m_2}{m_1+m_2}$
- And the important Radiation-reaction terms[15]

14 [Damour; Bern et al.; di Vecchia et al.;Bjerrum-Bohr et al.]

15 [Damour; di Vecchia et al.;Bjerrum-Bohr et al.]

The $\hat{N}$ operator at 4PM

$$\tilde{N}^{(3)} = \tilde{N}^{(3)}_{PP+RR} + \tilde{N}^{(3)}_{PR} + \text{Compton}$$

A "conservative part" matching the result of [Bern et al.]

$$\tilde{N}^{(3)}_{PP+RR} = -\frac{G^4(m_1+m_2)^3 m_1^4 m_2^4 \pi(\gamma^2-1)}{8s^{\frac{3}{2}}J^3} \times \left(\mathcal{M}_4^p + \nu\left(4\mathcal{M}_4^t \log\left(\frac{\sqrt{\gamma^2-1}}{2}\right) + \mathcal{M}_4^{\pi^2} + \mathcal{M}_4^{\text{rem}}\right)\right)$$

The new result is the radiation from the soft region

$$\tilde{N}^{(3)}_{PR} = \frac{G^4(m_1+m_2)^3 m_1^4 m_2^4 \pi(\gamma^2-1)}{8s^{\frac{3}{2}}J^3} \frac{6\nu(2\gamma^2-1)(5\gamma^2-1)}{\sqrt{\gamma^2-1}} \times \left(\frac{16-10\gamma^2}{3(\gamma^2-1)} + \frac{2\gamma(-3+2\gamma^2)\,\text{arccosh}(\gamma)}{\gamma^2-1}\right)$$

Classical observables

The change in an observable is given by KMOC

$\langle\Delta\hat{O}\rangle := \langle \text{out}|\hat{O}|\text{out}\rangle - \langle \text{in}|\hat{O}|\text{in}\rangle$

$$\langle\Delta\hat{O}\rangle(p_1,p_2,b) = \int \frac{d^D(\hbar \underline{q})}{(2\pi)^{D-2}} \delta(2\hbar p_1\cdot\underline{q} - \hbar \underline{q}^2)\delta(2\hbar p_2\cdot\underline{q} + \hbar\underline{q}^2) e^{ib\cdot\underline{q}} \langle p_1' p_2'|\Delta O|p_1 p_2\rangle$$

can be expanded using the N-operator

$$\langle\Delta\hat{O}\rangle = \langle \text{in}|\hat{S}^\dagger\hat{O}\hat{S} - \hat{O}|\text{in}\rangle = \sum_{n\geqslant 1} \frac{(-i)^n}{\hbar^n n!} \langle \text{in}| \underbrace{[\hat{N},[\hat{N},\cdots,[\hat{N},\hat{O}]]]}_{n \text{ times}} |\text{in}\rangle$$

The $\hbar \to 0$ limit gives directly the classical contribution

$$\lim_{\hbar\to 0} \langle\Delta\hat{O}\rangle(p_1,p_2,b) = \Delta O^{\text{classical}}(p_1,p_2,b) + O(\hbar)$$

with the exponential representation all superclassical pieces cancel automatically[16]

[16] [Damgaard, Hansen, Planté, Vanhove]

The radial action

Applying the previous formalism to the momentum kick $\hat{O} = \hat{P}_1$

$$\Delta\tilde{P_1}^{\nu}(\gamma, b)|_{\rm cons} = -\frac{p_\infty b^\nu}{|b|}\sin\left(\frac{-\partial \tilde{N}(\gamma, J)}{\partial J}\right) + p_\infty^2 L^\nu\left(\cos\left(\frac{-\partial \tilde{N}(\gamma, J)}{\partial J}\right) - 1\right)$$

$$L^\mu \equiv \frac{(m_2^2 + m_1 m_2 \gamma)p_1^\mu - (m_1^2 + m_1 m_2\gamma)p_2^\mu}{m_1^2 m_2^2(\gamma^2 - 1)}; \quad \tilde{N}(\gamma, b) \equiv \int \frac{d^2q}{(2\pi)^2}\frac{N(\gamma, q^2)}{4m_1 m_2\sqrt{\gamma^2 - 1}}e^{ib\cdot q}$$

this shows that in the conservative sector the $\tilde{N}(\gamma, b)$ **is** the radial action used by [Landau, Lifshitz; Damour] for classical GR

$$\chi = -\frac{\partial \tilde{N}(\gamma, J)}{\partial J}$$

The momentum kick up to 4PM

Applying the previous formalism to the momentum kick $\hat{O} = \hat{P}_1$

$$\Delta\tilde{P}_1^{\nu,4PM} = G^4 \begin{pmatrix} \frac{p_1^\mu}{m_1} & \frac{p_2^\mu}{m_2} & \frac{b^\mu}{|b|} \end{pmatrix} \begin{pmatrix} p_\infty\left(-\frac{(\chi_{\rm cons}^{(0)})^4}{24} + \frac{(\chi_{\rm cons}^{(1)})^2}{2} + \chi_{\rm cons}^{(0)}\chi_{\rm cons}^{(2)}\right) \\ p_\infty\left(\frac{(\chi_{\rm cons}^{(0)})^4}{24} - \frac{(\chi_{\rm cons}^{(1)})^2}{2} - \chi_{\rm cons}^{(0)}\chi_{\rm cons}^{(2)}\right) - \frac{\tilde{P}_{1,1}^{u_2,(3)}}{2} + \frac{i}{6}\tilde{P}_{1,2}^{u_2,(3)} \\ p_\infty\left(\frac{(\chi_{\rm cons}^{(0)})^2\chi_{\rm cons}^{(1)}}{2} - \chi_{\rm cons}^{(3)} - \frac{\partial\tilde{L}_2(\gamma,J)}{\partial J}\right) + \frac{\mathcal{E}_1\chi_{\rm cons}^{(0)}\tilde{P}_{1,1}^{u_2,(2)}}{2} - \frac{\tilde{P}_{1,1}^{b,(3)}}{2} + \frac{i}{6}\tilde{P}_{1,2}^{b,(3)} \end{pmatrix}$$

- This agrees with the world-line derivation by [Dlapa, Kalin, Liu, Neef, Porto] and is confirmed by the results of [Jakobsen, Mogull, Sauer, Plefka, Steinhoff, Xu]
- Gives a diverging scattering angle when $\gamma \to \infty$ when applying the recoil formula by [Bini, Damour, Geralico]: $\chi = \arccos\left(\frac{(p_1+p_2)\cdot(p_3+p_4)}{|p_1+p_2||p_3+p_4|}\right)$

Outlook

We have presented a formalism for systematically computing the post-Minkowskian expansion for the coalescence of two massive objects

1. The exponential formalism takes care of the sub-classical pieces
2. The N operator is the radial action in the conservative sector and allows to compute the gravitation radiation
3. The formalism of velocity cuts streamlines the computation reducing the evaluation to a small set of master integrals (58 compared to around 600 in the world-line)
4. At the 4PM order there is a complete agreement between three different method : classical world-line (Porto et al.), world-line effective field theory (Plefka et al.), scattering amplitude (Damgaard et al.).
5. Still the 3PM and 4PM results pose a puzzle since the ultra-relativistic regime $\gamma \to \infty$ as the result does not match the result from the ACV approach
6. Still some confusion regarding the starting order (2PM? 3PM?) for angular momentum loss