Spatial Curvature from Super-Hubble Cosmological Fluctuations

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Outline

Motivation

Correspondance between curvature and super-Hubble modes

Moments of the “observed” curvature

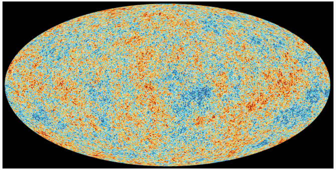
Probability distribution

Conclusion

Motivation

ΛCDM in a nutshell

- A spatially flat,
- statistically isotropic and uniform Universe,
- filled with ordinary matter,
- cold dark matter
- and dark energy



Credit: ESA CC BY-SA 3.0 IGO

ΛCDM contains 6 free parameters (and more derived parameters...)

- $h^2\Omega_\mathrm{b}$ density of Baryons
- $h^2\Omega_\mathrm{c}$ density of Cold Dark Matter
- τ reionization optical depth
- θ_MC observed angular size of the sound horizon at recombination
- A_s amplitude of curvature perturbations (at $k = 0.05\mathrm{Mpc}^{-1}$)
- n_s primordial spectral index

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Physics of the CMB & Astrophysics

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Inflation (or your favorite alternative)

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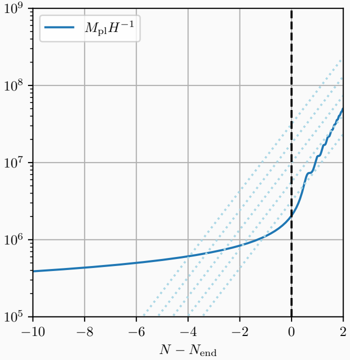

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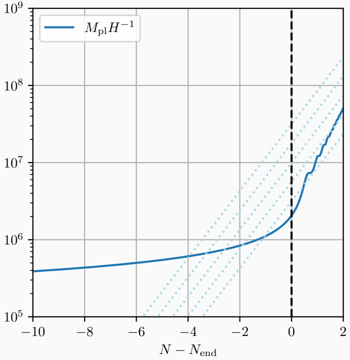

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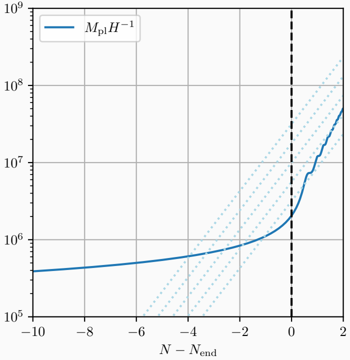

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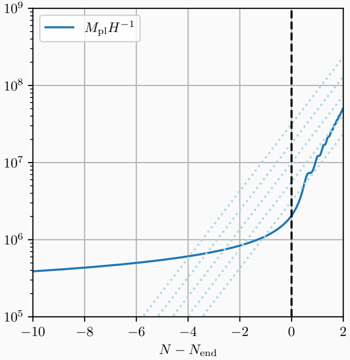
Question: Can super-Hubble modes be mistaken for non-zero space-time curvature of the background metric?



Correspondance between curvature and super-Hubble modes

Curvature in homogeneous FLRW spacetime

Assume a perfectly homogeneous curved FLRW spacetime

- Line-element reads

$$\mathrm{d}^2 s = -\,\mathrm{d}^2\tau + a^2(\tau)\frac{\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j}{\left(1+\frac{K}{4}\delta_{mn}x^m x^n\right)^2}\,,$$

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- Effect of spatial curvature on the 4-Ricci scalar
- Note that 4-Ricci scalar is a gauge-invariant quantity

Scalar inhomogeneities in flat FLRW spacetime

- We start with the inhomogeneous metric proposed in Refs.[1]

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a^2(\tau)e^{2\zeta(\tau,\boldsymbol{x})}\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j.$$

[1] Salopek and Bond 1990; Creminelli and Zaldarriaga 2004; Kolb et al. 2005; Lyth, Malik, and Sasaki 2005.

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- This metric can be viewed as an inhomogeneous realization of a flat FLRW spacetime having a space-dependent scale factor

$$b(\tau,\boldsymbol{x}) \equiv a(\tau)e^{\zeta(\tau,\boldsymbol{x})}$$

Motivations for this metric

- All inhomogeneities are contained in one scalar function ζ
- Most generic metric in absence of vector- and tensor-type inhomogeneities
- in the gauge where fixed time slices have uniform energy density
- and fixed spatial wordlines are comoving with matter

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Cosmology on the local Hubble patch

- We split the curvature perturbation

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- and the Hubble radius is the same for all observers

$$\tilde{H} = \frac{\dot{\tilde{a}}}{\tilde{a}} = \frac{\dot{a}}{a} = H$$

Stochastic curvature

- In the local Hubble patch, the 4- Ricci scalar becomes

$$R = 6\frac{\dot{\tilde{a}}^2}{\tilde{a}^2} + 6\frac{\ddot{\tilde{a}}}{\tilde{a}} + \boxed{\frac{6}{\tilde{a}^2}\left[-\frac{2}{3}\Delta\xi - \frac{1}{3}\left(\boldsymbol{\nabla}\xi\right)^2\right]} + \cdots .$$

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- Effect on the Ricci scalar
- One may be lead to identify

$$K = -\frac{2}{3}\Delta\xi - \frac{1}{3}\left(\boldsymbol{\nabla}\xi\right)^2 , \quad \Omega_{\mathrm{K}} = -\frac{K}{\tilde{a}^2\tilde{H}^2} = \frac{e^{-2\xi}}{a^2H^2}\left[\frac{2}{3}\Delta\xi + \frac{1}{3}\left(\boldsymbol{\nabla}\xi\right)^2\right]$$

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- Ω_{K} is promoted to a stochastic variable

Moments of the “observed” curvature

Statistics

- ζ: gaussian random field statistics and vanishing mean

$$\langle \zeta(\mathbf{k})\zeta(\mathbf{k}')\rangle = (2\pi)^3\,\delta(\mathbf{k}+\mathbf{k}')P_\zeta(k)$$

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$$\xi(\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\mathbf{k}\Theta(k_\sigma - k)\,\zeta(\mathbf{k})e^{i\mathbf{k}\cdot\boldsymbol{x}}$$

$$\Delta\xi = -\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Theta(k_\sigma - k)\,k^2\zeta(\mathbf{k})e^{i\mathbf{k}\cdot\boldsymbol{x}}$$

$$(\nabla\xi)^2 = -\int \frac{\mathrm{d}^3\boldsymbol{p}\mathrm{d}^3\boldsymbol{q}}{(2\pi)^6}\,\Theta(k_\sigma - p)\,\Theta(k_\sigma - q)\,\zeta(\boldsymbol{p})\,\boldsymbol{p}\cdot\boldsymbol{q}\,\zeta(\boldsymbol{q})e^{i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}}$$

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- Realizations of $\boldsymbol{\Xi} \equiv (\xi, \Delta\xi, \nabla\xi)$ determine

$$\Omega_\mathrm{K} = \frac{e^{-2\xi}}{a^2H^2}\left[\frac{2}{3}\Delta\xi + \frac{1}{3}\left(\boldsymbol{\nabla}\xi\right)^2\right]$$

Examples: ξ, the usual Gaussian variable

- Since ξ is gaussian, so its mean is 0

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- And its variance given by its power spectrum

$$\langle \xi^2 \rangle = \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{k}_1 \mathrm{d}^3\mathbf{k}_1 \Theta(k_\sigma - k_1)\, \Theta(k_\sigma - k_2) \left\langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \right\rangle e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\boldsymbol{x}}$$

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- All higher moments are expressed using W_{2n} the number of Wick's contractions between p pairs

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- The exponential can be expressed as a resummation of all the moments

$$\langle \exp(\xi) \rangle = \left\langle \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \right\rangle = \sum_{k=0}^{\infty} \frac{\left\langle \xi^2 \right\rangle^k}{2^k k!} = \exp\left(\frac{\left\langle \xi^2 \right\rangle}{2} \right)$$

Examples: $\Delta\xi$, the laplacian

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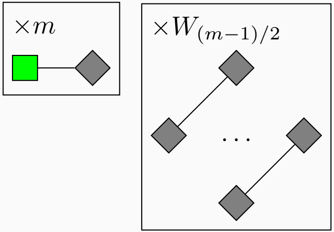
- with higher powers of ξ

$$\langle \Delta\xi\xi^{m=2p+1} \rangle =$$



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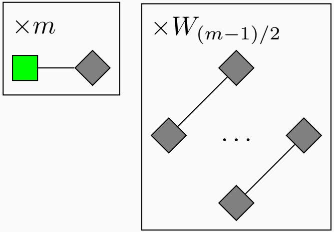

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Results for the first 4 moments

$$\langle \Omega_{\rm K} \rangle = -\frac{5}{a^2 H^2} \langle K \rangle \, e^{2\langle \xi^2 \rangle}$$

$$\langle \Omega_{\rm K}^2 \rangle = \frac{1}{a^4 H^4} \left(\langle K^2 \rangle + 80 \langle K \rangle^2 \right) e^{8\langle \xi^2 \rangle}$$

$$\langle \Omega_{\rm K}^3 \rangle = -\frac{\langle K \rangle}{a^6 H^6} \left(39 \langle K^2 \rangle + \frac{19430}{9} \langle K \rangle^2 \right) e^{18\langle \xi^2 \rangle}$$

$$\langle \Omega_{\rm K}^4 \rangle = \frac{1}{a^8 H^8} \left(3 \langle K^2 \rangle^2 + 1728 \langle K^2 \rangle \langle K \rangle^2 + \frac{736682}{9} \langle K \rangle^4 \right) e^{32\langle \xi^2 \rangle}$$

with

$$K = -\frac{2}{3} \Delta \xi - \frac{1}{3} \left(\boldsymbol{\nabla} \xi \right)^2$$

Results for the first 4 moments

For the first two moments

$$\langle \Omega_{\rm K} \rangle \simeq 10^{-9}$$
$$\sqrt{\langle \Omega_{\rm K}^2 \rangle - \langle \Omega_{\rm K} \rangle^2} \simeq 1.5 \times 10^{-5}$$

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Using the standardized moments $\tilde{\mu}_n$, the moments divided by the n^{th} power of the standard deviation

$$\tilde{\mu}_{n=2p} \simeq W_n \, e^{\left(2n^2 - 4n\right)\langle \xi^2 \rangle}$$

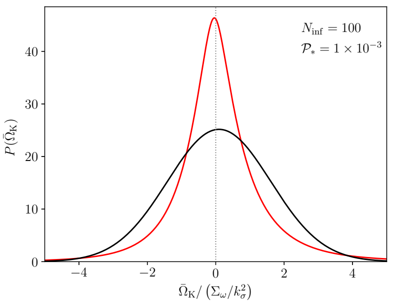
$$\tilde{\mu}_{n=2p+1} \simeq n W_{n-1} \left(1 + 4n\right) \frac{\sqrt{\mathcal{P}_*}}{2} \, e^{\left(2n^2 - 4n\right)\langle \xi^2 \rangle}.$$

Probability distribution

Idea of the computation

Ω_K can be seen as a non-functional over five stochastic Gaussian variables $\mathbf{\Xi} \equiv (\xi, \Delta\xi, \nabla\xi)$, with non-diagonal covariance matrix $\mathbf{\Sigma}$.

$$P(\bar{\Omega}_\mathrm{K}) = \int \frac{\mathrm{d}^5\mathbf{\Xi}}{(2\pi)^{5/2}} \delta\left(\bar{\Omega}_\mathrm{K} + \frac{K}{k_\sigma^2} e^{-2\xi}\right) \frac{e^{-\frac{1}{2}\mathbf{\Xi}^\mathrm{T}\mathbf{\Sigma}^{-1}\mathbf{\Xi}}}{\sqrt{\det \mathbf{\Sigma}}},$$



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- For long inflationary era, $\left\langle \xi^2 \right\rangle$ becomes large, therefore constraining the duration of the inflationary era

Thank you for your attention