CONTROVERSIAL TRACE ANOMALIES

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CHIRAL ANOMALY

Anomalies: classical symmetry broken at the quantum level

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \not{D} \psi$$

2 invariances:

$$\psi \to e^{i\alpha_V} \psi$$
$$\psi \to e^{i\alpha_A \gamma_5} \psi$$

$\xrightarrow{\text{Noether th.}}$

2 classically conserved currents:

$$\partial_\mu \left(V^\mu = \bar{\psi} \gamma^\mu \psi \right) = 0$$
$$\partial_\mu \left(A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi \right) = 0$$

quantum level:

$$\sim F_{\mu\nu} \tilde{F}^{\mu\nu}$$

× Rq: fermionic path integral measure is not invariant (FUJIKAWA)

$$Z = \int \underbrace{D\psi D\bar{\psi}}_{\text{quantum piece}} e^{iS}$$

ANOMALIES & NON-DECOUPLING

× An anomaly can have significant effects in the low energy EFT because it is not suppressed by any large energy cutoff scale

Ex: Chiral anomaly in QCD ($m_q = 0$)

- Effects unsuppressed by any $1/\Lambda_{QCD}$
- Chiral symmetry is only approximate ($m_q \neq 0$) but chiral anomaly gives the dominant contribution to $\pi^0 \to \gamma\gamma$ in SM
- It requires the explicit addition to the local effective action of a non-local term to account for its effects.

× The low energy measurement of $\pi^0 \to \gamma\gamma$ affords a clean non-trivial test of the underlying microscopic quantum theory of QCD
(3 colors of fractionally charged quarks)

CHIRAL ANOMALY & CP

$$\mathcal{L}_{QCD} = \bar{q}\left(i\not{D} - m_q e^{i\theta_{EW}}\right)q - \frac{1}{4}GG - \left(\theta_{QCD} - \theta_{EW}\right)G\tilde{G}$$

anomalous $q \to e^{i\alpha\gamma_5} q$

$\times$ CP ?

CPV

CPV

$$\underbrace{? \qquad \neq 0}_{< 10^{-10}}$$

(neutron EDM)

$\Longrightarrow$ conspiracy ?

<u>Strong CP problem</u>

PECCEI-QUINN SOLUTION

1/ New $U(1)_{PQ}$ (axial - global)

2/ Destroy it : - spontaneously broken $\to a$ (Goldstone axion)

- anomalous : $PQ(q_L) \neq PQ(q_R) \to \left(\theta + \frac{a}{f_a}\right) G\tilde{G}$

$= 0$ CP restored

3/ Non perturbative effects of QCD : $V_{eff}^{QCD}\Big(\underbrace{\theta + \frac{a}{f_a}}_{\text{minimum at } 0}, \dots\Big)$

AXIONS ARE BLIND TO ANOMALIES

J.Q. & C. SMITH, arXiv:1903.12559

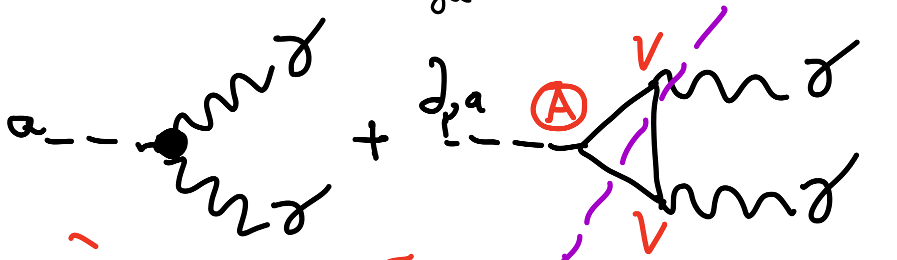
$$\mathcal{L}_{axion} \supset \phi \bar{\psi}_L \psi_R + h.c.$$

$\phi \sim \rho e^{i\frac{a}{f_a}}$ (polar rep.)

$\phi \sim \sigma + i a$ (linear rep.)

$$\mathcal{L}_{polar} \supset \# \frac{a}{f_a} G\tilde{G} + \frac{\partial_\mu a}{f_a}\left(\# \bar{\psi}\gamma^\mu \psi + \# \bar{\psi}\gamma^\mu\gamma^5\psi\right)$$

$$\mathcal{L}_{linear} \supset \frac{m_f}{f_a} a \bar{\psi} i \gamma^5 \psi$$



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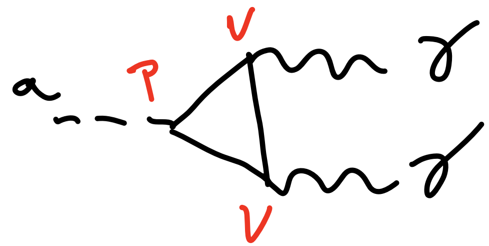
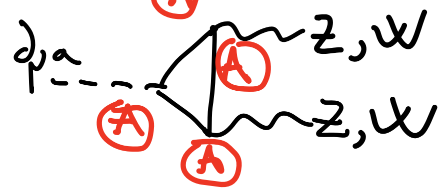


~~"axion couplings induced by the anomaly"~~

$\to 0$

$m_f \to 0$

$\neq 0$



$\neq 0$

× different features when adding S.S.B.

× important for ALP EFTs

× pheno implications ?

FROM CHIRAL TO TRACE ANOMALY

× In curved space an anomaly closely related to the chiral anomaly:
Trace anomaly.

× Additional infrared terms that do not decouple in the limit $L_{Pl} = \frac{1}{M_{Pl}} \to 0$
and should be added to the Einstein-Hilbert action to complete
the E F T of low energy gravity.

× Unlike local higher derivative terms in the effective action
the anomalous terms cannot be discarded in the low energy
and give sizable effects. (ex: Casimir effect)

SCALE, CONFORMAL, WEYL, TRACE

× Scale tf.: $x^\mu \to e^{\sigma} x^\mu$

× Weyl tf.: $g_{\mu\nu} \to e^{2\sigma(x)} g_{\mu\nu}$, $\psi(x) \to e^{-\Delta\sigma(x)} \psi(x)$

× $\delta_W S = \frac{\delta S}{\delta g_{\mu\nu}} \delta_W g_{\mu\nu} = \underset{\text{(EMT)}}{T^{\mu\nu}} g_{\mu\nu} = T_\mu{}^\mu$

$\boxed{\text{Classical Weyl inv.} \Rightarrow \text{Traceless EMT}}$

quantum correction: $\langle T_\mu{}^\mu \rangle = a E_4 - c W^2$

Trace anomaly $d=4$
(Capper, Duff 1975)
(perturbative QFT calculations)

$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ (Euler, Gauss-Bonnet density) (a-anomaly)

$W^2 = \_\_\_ - 2\_\_\_ + \frac{1}{3}\_\_\_$ (c-anomaly)

PONTRYAGIN DENSITY

$$\langle T_\mu{}^\mu \rangle = a E_4 - c W^2 + e\, \varepsilon^{\mu\nu\rho\sigma} R_{\alpha\beta\mu\nu} R^{\alpha\beta}{}_{\rho\sigma}$$ why omitted?

- $R\tilde{R}$: CPV → new source for SM, baryogenesis, gravitational waves...
- $iR\tilde{R}$: violates unitarity (calls for ν_R in SM?)

question : can the Pontryagin density be present in the trace anomaly?

→ YES : $R\tilde{R}$ satisfies WZ c.c.

$$[\delta_{w_1}, \delta_{w_2}] S = 0$$

anomaly obeys group law

Weyl rescalings form a group

question : can we compute the trace anomaly in simple systems?

→ Need γ^5 i.e chiral fermions

WEYL FERMION & GRAVITY

$$\mathcal{L} = i\,\bar{\Psi}_R\,\gamma^\mu(\partial_\mu + \omega_\mu)\,\Psi_L$$

spin connection

2014 ; BONORA et al. : $\langle T_\mu{}^\mu \rangle \supset i R\tilde{R}$

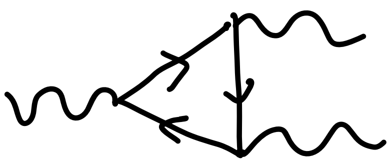
Several groups now found $e = i$ with several methods
$e = 0$ (Feynman diagrams, heat kernel, ...)

× Weyl fermions are subtle and probe spacetime in their own way.
In $d = 2 \pmod 4$ they give rise to gravitational (diffeo & Lorentz) anomalies

× Important to clarify the nature of this anomaly
(Rémy Larue, J.Q., Roman Zwicky : arXiv:2309.08670 & to appear)

FEYNMAN DIAGRAMS

× expansion $g_{\mu\nu}(x) \sim \eta_{\mu\nu} + h_{\mu\nu}(x)$

 ?

pb: NON invertible kinetic op.

$$\mathcal{L} = \bar{\Psi}_R \not{D} \Psi_L = \bar{\Psi} \underbrace{\not{D} P_L}_{} \Psi$$

maps L-handed to R-handed

non invertible in the space of Dirac fermion → no propagator in FD

× trick: remove P_L & decouple Ψ_R from gravity

$$D_\mu \Psi_R = \partial_\mu \Psi_R + \cancel{\omega_\mu \Psi_R}$$

⚠ break Lorentz invariance

× $\langle T_\mu{}^\mu \rangle_{P\text{-odd}} \supset i\, \varepsilon^{\mu\nu\rho\sigma} \left(\partial_\mu \partial_\rho h_\nu{}^z \partial_\sigma \partial_z h_\rho{}^\sigma \dots \right) + O(h^3)$

↓ "COVARIANT-IZATION"

$\langle T_\mu{}^\mu \rangle_{P\text{-odd}} \supset i R \tilde{R}$ with $D_\mu \langle T^{\mu\nu} \rangle_{P\text{-odd}} = 0$

Cel: neutrinos ⇒ SM unitarity problem

DET OF THE WEYL OPERATOR

× Dirac op. $i\not{D} P_L$ can not be inverted

× Weyl op. $D\psi_L = i\bar{\sigma}^\mu (\partial + \omega_L)_\mu \psi_L$

2-component Weyl fermion

× Effective action $W = -\log \det D$

ill-defined $D : \left(\frac{1}{2}, 0\right) \longrightarrow \left(0, \frac{1}{2}\right)$

Álvarez-Gaumé & Witten (1984)

phase is ambiguous

modulus is unaffected

× det is not an observable, however

$$\delta \log \det D = \mathrm{Tr}\, \delta D D^{-1}$$

$\delta D D^{-1} : \left(0, \frac{1}{2}\right) \longrightarrow \left(0, \frac{1}{2}\right)$ ✓

Leutwyler & Mallik 1985, 86

× 0 modes of D : another problem of definition for the det

× D^{-1} is singular but can be regularized

TRACE, DIFFEO & LORENTZ

* Definition of the anomalies:

$$\delta_\alpha W = -\mathrm{Tr}\, \delta_\alpha D\, D^{-1} = -\int d^4x\, e\, \alpha(x)\, \mathcal{A}$$

* Definition from the path integral

$$\delta_\alpha W = \frac{\int D\psi D\bar{\psi}\, (\delta_\alpha S)\, e^{-S}}{\int D\psi D\bar{\psi}\, e^{-S}} = -\langle \delta_\alpha S \rangle$$

$$\int d^4x\, e\, \sigma\, \mathcal{A}_{\mathrm{trace}} = -\delta_\sigma^W W = \int d^4x\, e\, \sigma\, e^a_\mu \langle T^\mu{}_a \rangle$$

$$\int d^4x\, e\, \xi_\mu\, \mathcal{A}^\mu_{\mathrm{diffeo}} = -\delta_\xi^d W = \int d^4x\, e\, \xi^\nu \langle \omega_\nu{}^{ab} T_{ab} - D^\mu T_{\mu\nu} \rangle$$

$$\int d^4x\, e\, \alpha_{ab}\, \mathcal{A}^{ab}_{\mathrm{Lorentz}} = -\delta_\alpha^L W = \int d^4x\, e\, \alpha_{ab} \langle T^{ab} \rangle$$

DIFFEO $\leftrightarrow$ LORENTZ

× $e T^{\mu}{}_{a} = \dfrac{\delta S}{\delta e^{a}{}_{\mu}}$ with fermions it is the vierbein (not the metric) the fundamental quantity

× EMT is not automatically symmetric $\longrightarrow$ Lorentz anomaly presents
$\longrightarrow$ additional piece in the diffeo anomaly

× diffeo anomaly $\xleftrightarrow{\text{local C.T.}}$ Lorentz anomaly
NON Polynomial $\Rightarrow$ a priori can't transfer one to each other
(we do the explicit check)

× Let's adapt 2 methods for Weyl fermions : – proper time reg. (P.I)
– Fujikawa

PROPER TIME REGULARISATION

× Need to regularise the singular op. D^{-1},

$$D^{-1}\Big|_\Lambda = \int_{\frac{1}{\Lambda^2}}^{\infty} dt\, D^\dagger e^{-tDD^\dagger}$$

$$\delta \log \det D\Big|_\Lambda = \mathrm{Tr}\ \delta D\, D^{-1} e^{-\frac{DD^\dagger}{\Lambda^2}}$$

× **Dirac fermion:** (well known case)

$$D \to i e^\mu{}_a \gamma^a D_\mu = i\not{D} \qquad (D^\dagger = D \ : \text{easy computation})$$

$$\delta^W_\sigma D = -\sigma D - \frac{1}{2}[D, \sigma]$$

$$\delta^W_\sigma W = \lim_{\Lambda\to\infty} \mathrm{Tr}\ \sigma\, e^{-\frac{(i\not{D})^2}{\Lambda^2}}$$

$$\xrightarrow{\text{CDE}} \mathcal{A}^{\text{Dirac}}_{\text{trace}} = \frac{1}{16\pi^2}\left(\frac{1}{72}R^2 - \frac{1}{45}R_{\mu\nu}^2 - \frac{7}{360}R_{\mu\nu\rho\sigma}^2 - \frac{1}{30}\Box R\right) \ \&$$

$$\boxed{\begin{array}{l}\mathcal{A}^{ab}_{\text{Lorentz}} = 0 \\ \mathcal{A}^{\mu}{}_{\text{diffeo}} = 0\end{array}}$$

× Weyl fermion :

$$D\tilde{\Psi}_2 = i\bar{\sigma}^\mu\left(\partial_\mu + \omega_\mu - \frac{\partial_\mu\sqrt{e}}{\sqrt{e}}\right)\tilde{\Psi}_2$$

$$\delta_\sigma^W D = -\sigma D - \frac{1}{2}[D,\sigma]$$

$$\delta_\xi^W D = -[D,\xi^\mu]\nabla_\mu - \xi^\mu[D,\nabla_\mu] - \frac{1}{2}[D,(D_\mu\xi^\mu)]$$

$$\delta_\alpha^L D = [D, \tfrac{1}{2}\alpha_{ab}\rho^{ab}] + \frac{1}{2}\alpha_{ab}(\rho^{ab} - \varsigma^{ab})D$$

$$\delta_\xi^d W = -\lim_{\Lambda\to\infty} \mathrm{Tr}\left[\gamma^5\left(\xi^\mu \underline{\nabla_\mu} + \frac{1}{2}(\underline{D_\mu}\,\xi^\mu)\right)e^{-\frac{(i\not{D})^2}{\Lambda^2}}\right] \overset{\text{CDE}}{=} 0$$

not manifestly covariant

heat kernel not straightforward

$$\delta_\alpha^L W = \lim_{\Lambda\to\infty} \mathrm{Tr}\,\frac{1}{2}\alpha_{ab}\Sigma^{ab}\gamma_5\, e^{-\frac{(i\not{D})^2}{\Lambda^2}} \overset{\text{CDE}}{=} 0$$

$$\delta_\sigma^W W = \frac{1}{2}\lim_{\Lambda\to\infty}\mathrm{Tr}\,\sigma\left(e^{-\frac{D^\dagger D}{\Lambda^2}} + e^{-\frac{DD^\dagger}{\Lambda^2}}\right)$$

$\Rightarrow$ $$\boxed{\begin{gathered}\mathcal{A}^{\text{Weyl}}_{\text{trace}} = \frac{1}{2}\mathcal{A}^{\text{Dirac}}_{\text{trace}} \\ \text{with } \mathcal{A}^\mu_{\text{diffeo}} = \mathcal{A}^{ab}_{\text{Lorentz}} = 0\end{gathered}}$$

No $R\tilde{R}$

× in agreement with the computation of the heat kernel coef. b_4 (rep. $(\frac{1}{2},0)$ of the Lorentz group)

→ we showed that the trace anomaly of a Weyl fermion is determined by : $\boxed{b_4(\tfrac{1}{2},0) + b_4(0,\tfrac{1}{2})}$

FUJIKAWA FOR WEYL FERMIONS

(known so far)

× Dirac fermions with a projector P_L : ill-defined path integral (non invertibility of $i\cancel{D}P_L$)

× anomaly arises from a non-trivial Jacobian

$$J[\alpha] = e^{-\int d^4x \, e \, \alpha(x) \mathcal{A}(x)}$$

can be written as a fraction of dets (well defined, as before)

$$= \frac{\det(D)}{\det(D - \delta_\alpha D)} = \frac{1}{\det(1\!\!1 - \delta_\alpha D D^{-1})} = e^{\boxed{\mathrm{Tr}\, \delta_\alpha D D^{-1}} + O(\alpha^2)}$$

↳ maps into the same Hilbert space

× We proceed to construct the path integral measure :

- The Weyl op, $D\Psi_L$, is not hermitian and does not have a well-defined eigenvalue pb
- $i\cancel{D}$ is ______ does have ____________

(Dirac counterpart and then project it in the left-right basis)

→ Same results as those obtained with the proper time regularisation.

TRACE ANOMALY A LA DUFF

(Rémy Larue, J.Q., Roman Zwicky: To appear)

$$W = \frac{1}{d-4} \int d^{\overset{d}{4}} x \sqrt{-g}\, \mathcal{L}^{\text{dim } 4}_{[d,\mu]}$$

divergent ← dim Reg, $d = 4-\varepsilon$

$$\langle T_{\alpha\beta} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\alpha\beta}} W$$

most generic form:

$$= f_{\alpha\beta} + \frac{1}{\varepsilon} g_{\alpha\beta} \left\{ a_1 R^2 + a_2 R_{\mu\nu}^2 + a_3 R_{\mu\nu\rho\sigma}^2 + a_4 \Box R + e_1 R\tilde{R} \right\}$$

+ "irrelevant"

× for theories not classically Weyl invariant:

$$\mathcal{A}_{\text{Weyl}} = g^{(4)}_{\mu\nu} \langle T^{\mu\nu} \rangle - \langle g^{(d)}_{\mu\nu} T^{\mu\nu} \rangle \underset{\varepsilon \to 0}{=} \underbrace{\left(g^{(4)}_{\mu\nu} - g^{(d)}_{\mu\nu} \right)}_{\mathcal{O}(\varepsilon)} \langle T^{\mu\nu} \rangle$$

(= 0 for CFTs)

$\frac{1}{\varepsilon}$: finite

SYMMETRY CONSTRAINTS ON THE TRACE ANOMALY

(not only Weyl fermions here...)

$$\mathcal{A}_{\text{Weyl}} \underset{\varepsilon \to 0}{=} a_1 R^2 + a_2 R_{\mu\nu}^2 + a_3 R_{\mu\nu\rho\sigma}^2 + a_4 \Box R + e_1 R\tilde{R}$$

× Diffeo anomaly : (NEW)

$$\mathcal{A}^{\mu}_{\text{diffeo}} = D_\nu \langle T^{\mu\nu} \rangle = 0 \Rightarrow \text{constraints on } \{a_i, e_1\}$$

- CP-even part : 7 constraints on the 10 parameters

$$\rightarrow \boxed{4a_1 + a_2 = -a_4} \quad \text{(known)}$$

- CP-odd part : $\rightarrow \boxed{e_1 = 0}$ NO $R\tilde{R}$

IF CFT : (Duff et al.) 1 extra constraint : $4a_1 + a_2 = a_1 - a_3$

CONCLUSION

× We investigated the trace anomaly of a free Weyl fermion in a curved space

$\longrightarrow$ controversy: existence of the Pontryagin density $R\tilde{R}$.

× Handling Weyl fermions is technically subtle and requires care as the Weyl det is ill-defined.

× We use the method of proper time regularisation (cf. Leutwyler) & develop the Fujikawa method for a 2-component Weyl fermion.

× Our results: $\mathcal{A}^{Weyl}_{trace} = \frac{1}{2}\mathcal{A}^{Dirac}_{trace} \Rightarrow$ No $R\tilde{R}$ (Hold in any even dim)

× We have concluded on the absence of $R\tilde{R}$ from symmetry argument on $T^{\mu\nu}$

× Our findings do not mean a $R\tilde{R}$-term could not play a role in fundamental physics. It can arise from sources other than Weyl fermions.

$\longrightarrow$ It may apear in conjunction with axions (connection with P & CP-violation)

TO EXPLORE !!

SPARE SLIDES

C - P - T DISCUSSION

$$C \circ R\tilde{R} = + R\tilde{R}$$
$$P \circ \text{—} = - \text{—}$$
$$T \circ \text{—} = - \text{—}$$

$$T \circ i = - i$$

$R\tilde{R}$ is P, CP-odd and CPT-even

$iR\tilde{R}$ is CPT violating & violates unitarity

d - DIMENSIONS

× So far, in $d=4$, no $R\tilde{R}$ in the trace anomaly for Weyl fermions

natural to ask: Could P & (or) CP-odd terms could be present in any even dimension?

× Our calculation is independent of the dimension

× $d=2 \pmod 4$: (Euclidean) Weyl rep are complex

$d=4 \pmod 4$: ———— real ⟹ allow Pauli-Villars reg.

mass term is symmetry preserving

no gravitational anomalies

× $d=2 \pmod 4$: P and CP-odd terms should not violate CPT. (i is dimension-dependent)

$d=2$: $\mathrm{Tr}(\gamma_\alpha \gamma_\beta \gamma_3) = 2\varepsilon_{\alpha\beta}$; $d=4$: $\mathrm{Tr}(\gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta \gamma_5) = 4i\,\varepsilon_{\alpha\beta\gamma\delta}$

However → $d=2 \pmod 4$ no "i" prefactor → no CPT

× $d=2 \pmod 4$: absence of parity-odd terms can be established without explicit computation

Bianchi identity : $\varepsilon^{\cdots\mu\nu\rho} R_{\alpha\mu\nu\rho} = 0$

$$\longrightarrow \varepsilon^{\alpha_1 \cdots \alpha_{2m}} R_{\alpha_1 \alpha_2 \cdots} \cdots R_{\alpha_{2m-1} \alpha_{2m} \cdots} \begin{cases} = 0 & d = 2 \pmod 4 \\ \neq 0 & d = 4 \pmod 4 \end{cases}$$

× $d = 4 \pmod 4$: a computation is required

INTEGRATING OUT - CDE - UOLEA

$$\mathcal{L}_{UV}[\phi,\Psi] = \mathcal{L}_0[\phi] + \bar{\Psi}\left(\not{P} - M - X[\phi]\right)\Psi$$

light heavy

$$\longrightarrow \quad S_{eff}^{1\text{-loop}} = -i \,\mathrm{Tr}\ln\left(\not{P} - M - X\right)$$

× Functional approach is powerful to compute Tr / Det

× CDE : Taylor expansion $\mathcal{L}_{eff}^{1\text{-loop}} = i \,\mathrm{tr} \sum \frac{1}{n} \int dq \left[\frac{-1}{q+M}\left(-P+M\right)\right]^n$

× Expand order by order : nice factorization $\left[\int dq\, f(q)\right] \times \mathcal{O}(P,X)$

Wilson coeff.

→ loops can be computed once for all

→ paradigm UOLEA :

$$\mathcal{L}_{eff}^{1\text{-loop}} \supset \mathrm{tr}\left\{ \cdots + \frac{1}{m^2}\left[\frac{-1}{12} XGG + \cdots\right] + \frac{1}{m^4}\left[\frac{1}{120}\left(P^2X^2\right)^2 + \cdots\right] + \cdots \right\}$$

scalars ✓ - 1/2 fermions ✓ - mixed heavy-light ✓ - chiral fermions ✓

× Dirac fermion:

× Lorentz anomaly

It remains to check:

$$\delta_\alpha^2 D = \frac{1}{2}\left[D, \alpha_{ab}\Sigma^{ab}\right]$$

$$\delta_\alpha^2 W = \lim_{\Lambda\to\infty} \mathrm{Tr}\, \frac{1}{2}\alpha_{ab}\Sigma^{ab}\left[D, D^{-1} e^{-\frac{D^2}{\Lambda^2}}\right] = 0$$

$$\Rightarrow \boxed{\mathcal{A}^{ab}_{Lorentz} = 0}$$

× Diffeo anomaly

$$\delta_\xi^d D = -\left[D, \xi^\mu\right]\nabla_\mu - \xi^\mu\left[D, \nabla_\mu\right] - \frac{1}{2}\left[D, (\nabla_\mu \xi^\mu)\right]$$

$$\delta_\xi^d W = 0 \quad \Rightarrow \boxed{\mathcal{A}^\mu_{diffeo} = 0}$$

× NB: we used rescaled fermionic variables

$$\tilde{\psi} = \sqrt{e}\,\psi, \qquad \tilde{\bar{\psi}} = \sqrt{e}\,\bar{\psi} \qquad (e = \det e^a{}_\mu)$$

with $D\psi D\bar{\psi}$ we would have obtained a spurious diffeo anomaly and the wrong trace anomaly.

CHIRAL ANOMALY A LA FUJIKAWA

Path integral measure for gauge theories with fermions is not invariant under the chiral transformation.

$$Z = \int D\psi D\bar\psi \, e^{-\int d^4x \, \bar\psi i\not{D}\psi}$$

↳ Hermitian op: $i\not{D}\varphi_m = \lambda_m \varphi_m$

× $[\gamma_5, \not{D}] \neq 0$: origin of the anomalous behavior

× Dirac basis:

$$\begin{cases} \psi'(x) = \sum_m a'_m \varphi_m(x) \\ \bar\psi'(x) = \sum_m \varphi_m^+(x)\, b'_m \end{cases}$$

$$D\psi D\bar\psi = \prod_m da_m \, db_m \xrightarrow{\psi \to e^{i\alpha\gamma_5}\psi} \left(\det e^{i\alpha\gamma_5}\right)^{-2} D\psi D\bar\psi$$

$$\mathcal{A} = \sum_m \varphi_m^+ \gamma^5 \varphi_m \quad \text{(ill-defined, conditionally convergent)}$$

$$= \lim_{M\to\infty} \sum_m \varphi_m^+ \gamma_5 \underbrace{e^{-\left(\frac{\lambda_m}{M}\right)^2}}_{\text{keep 0-modes}} \varphi_m$$

$$= \text{———————} e^{-\left(\frac{\not{D}}{M}\right)^2} \text{——}$$

$$= \lim_{M\to\infty} \mathrm{Tr} \int d^4k \, \gamma_5 \, e^{-ikx} e^{-\left(\frac{\not{D}}{M}\right)^2} e^{ikx} \quad \text{(plane wave basis)}$$

$$\boxed{\sim \mathrm{Tr}\, F\tilde{F}}$$

TRACE ANOMALY À LA FUJIKAWA

(Rémy Larue, J.Q., Roman Zwicky : arXiv:2309.08670)

$$Z = \int D\psi D\bar\psi \, e^{i\int d^4x \, \bar\psi i \cancel{D} \psi}$$

Weyl fermion : $\; \cancel{D} = \cancel{D}_L P_L + \cancel{D}_R P_R$

with $D_R = 0$

no need of invertibility,

spectator field

not hermitian, but $D^\dagger D$ & $DD^\dagger$ are

$D^\dagger D \, \varphi_n = \lambda_n^2 \varphi_n \; , \; DD^\dagger \phi_n = \lambda_n^2 \phi_n \; : \; \{\varphi_n\}, \{\phi_n\}$ 2 orthonormal eigenbasis

$$D\bar\psi' D\psi' = J_{\text{Weyl}} \, D\bar\psi D\psi$$

$$\mathcal{A}^{\text{Weyl}} \sim \lim_{\Lambda\to\infty} \sum_n \int d^4x \, f\left(\frac{\lambda_n}{\Lambda^2}\right) \left\{ \frac{\sigma}{2} \varphi_n^\dagger \varphi_n + \frac{\sigma}{2} \phi_n^\dagger \phi_n \right\}$$

$$\begin{cases} f\left(\frac{D^\dagger D}{\Lambda^2}\right) = \boxed{f\left(\frac{D_L^2}{\Lambda^2}\right) P_L} + f\left(\frac{D_R^2}{\Lambda^2}\right) P_R \\ f\left(\frac{DD^\dagger}{\Lambda^2}\right) = \boxed{\underline{\qquad} \, P_R} + \underline{\qquad} \, P_L \end{cases}$$

$$\sim \lim_{\Lambda\to\infty} \mathrm{Tr} \, \frac{\sigma}{2} \left[f\left(\frac{D_L^2}{\Lambda^2}\right) + f\left(\frac{D_R^2}{\Lambda^2}\right) \right]$$

$$\mathcal{A}^{\text{Diffeo Lorentz}} \sim \lim_{\Lambda\to\infty} \mathrm{Tr} \, (\ldots) \{ - \quad - \quad - \}$$

RESULTS

× **VL case :** $D_L = D_R$

$$\mathcal{A}^{\mathrm{Weyl}} \sim \lim_{\Lambda\to\infty} \mathrm{Tr}\, f\!\left(\frac{D^2}{\Lambda^2}\right) \sim -\frac{1}{72}R^2 + \frac{1}{45}R_{\mu\nu}^2 + \frac{7}{360}R_{\mu\nu\rho\sigma}^2 + \frac{1}{30}\Box R \quad ✓$$

$\mathcal{A}^{\mathrm{Diffeo}} = \mathcal{A}^{\mathrm{Lorentz}} = 0$ ✓

$f(x) = \frac{1}{1+x}$ + CDE in curved spacetime

(Rémy Larue, JQ. arXiv:2303.10203)

× **Weyl fermion :** $D_L = D,\ D_R = 0$

$\mathcal{A}^{\mathrm{Weyl}} \sim$ " $\frac{1}{2}$ VL " $\quad \mathcal{A}^{\mathrm{Diffeo}} = \mathcal{A}^{\mathrm{Lorentz}} = 0$ ✓ $\Rightarrow$ no $R\tilde{R}$

⚠ if arbitrary choice ex: regularise with $D^\dagger D$ only

$$\mathcal{A}^{\mathrm{Weyl}}_{D^\dagger D} \sim \lim_{\Lambda\to\infty} \mathrm{Tr}\, P_L f\!\left(\frac{D^2}{\Lambda^2}\right) \supset i R\tilde{R}$$

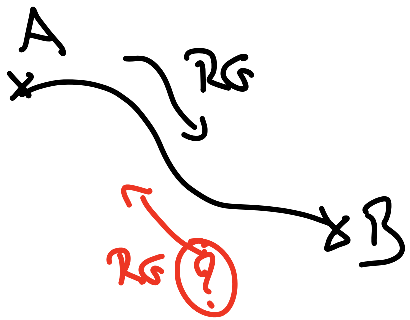
× key point : Spectrum of the Dirac op. & its hermiticity

→ choice of quadratic op. to regularise is crucial : must conserve all 0-modes

a AND c ANOMALY THEOREM

$$\langle T_\mu{}^\mu \rangle = a E_4 - c W^2$$

× idea: Reversibility of RG flux



× $D=2$: Answer is **NO**

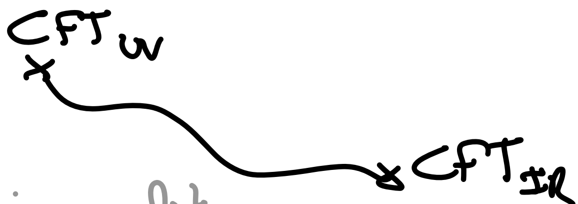
Zamolodchikov

$$\langle T_\mu{}^\mu \rangle_{2D} = -\frac{c}{24\pi} R \,, \quad \boxed{c>0}$$

- RG flux is irreversible
- effective measure of the # of d.o.f

× $D=4$: Cardy's conjecture $a \sim \int_{S^4} \langle T_\mu{}^\mu \rangle$ decreases as we flow

$$\Rightarrow \underline{a_{UV} > a_{IR}}$$



c-anomaly does not satisfy such an inequality