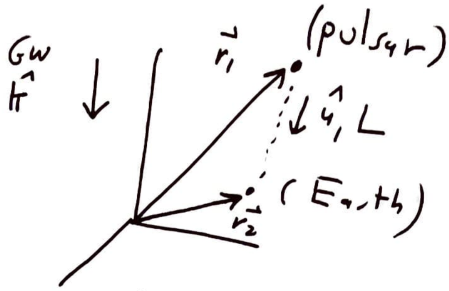
Timing residual response to GWs:



$$\Delta T(t_2) = \frac{1}{2c} u^i u^j \int_0^L \underbrace{ds}_{\text{arclength parameter}}\, h_{ij}(t(s), \vec{x}(s))$$

where

$$t(s) = t_1 + \frac{s}{c}$$

$$\vec{x}(s) = \vec{r}_1 + \hat{u}\, s$$

pulse starts at $\vec{r}_1$ at time t_1, and arrives at $\vec{r}_2$ at time t_2 in absence of GWs

$$\Rightarrow \vec{r}_2 = \vec{r}_1 + \hat{u} L$$

$$t_2 = t_1 + \frac{L}{c}$$

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plane wave expansion:

$$h_{ij}(t, \vec{x}) = \int_{-\infty}^{\infty} df \int d\hat{k} \sum_A \underbrace{h_A(f, \hat{k})}_{\text{Fourier coefficient}}\, \underbrace{e^A_{ij}(\hat{k})}_{\text{polarization tensors}}\, \underbrace{e^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}}_{\text{plane wave}}$$

(random fields for a GWB)

~~Simple case: Fixed Frequency~~

~~consider just single frequency for single plane wa~~

simple case: single plane wave with freq f, direction $\hat{k}$, and polarization A; and set $h_A(f, \hat{k}) = 1$

$$\rightarrow h_{ij}(t, \vec{x}) = e^A_{ij}(\hat{k})\, e^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

$$\rightarrow h_{ij}(t(s), \vec{x}(s)) = e^A_{ij}(\hat{k})\, e^{i2\pi f(t(s) - \hat{k}\cdot\vec{x}(s)/c)}$$

$$= e^A_{ij}(\hat{k})\, e^{i2\pi f[t_1 + \frac{s}{c} - \hat{k}\cdot(\vec{r}_1 + \hat{u}s)/c]}$$

$$= e^A_{ij}(\hat{k})\, e^{i2\pi f(t_1 - \hat{k}\cdot\vec{r}_1/c)}\, e^{i2\pi f \frac{s}{c}(1 - \hat{k}\cdot\hat{u})}$$

Thus,

$$\int_0^L ds\, e^{i2\pi f \frac{s}{c}(1-\hat{k}\cdot\hat{u})} = \frac{1}{\left(\frac{i2\pi f}{c}\right)(1-\hat{k}\cdot\hat{u})}\left[e^{i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}-1\right]$$

$\rightarrow$

$$\Delta T(t_2) = \frac{1}{2\cancel{c}}\frac{1}{\left(\frac{i2\pi f}{\cancel{c}}\right)}\frac{1}{(1-\hat{k}\cdot\hat{u})}u^i u^j e^A_{ij}(\hat{k})\, e^{i2\pi f(t_1-\hat{k}\cdot\frac{\vec{r}_1}{c})} \times\left[e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}-1\right]$$

$$= \frac{1}{2}\underbrace{u^i u^j e^A_{ij}(\hat{k})}_{\text{projection of GW polarization tensor on unit vector } \hat{u}}\underbrace{\left(\frac{1}{i2\pi f}\right)}_{\substack{=1 \text{ if we considered redshift measurements} \\ \frac{d\,\Delta T(t)}{dt} = \frac{\Delta\nu(t)}{\nu}}}\frac{1}{(1-\hat{k}\cdot\hat{u})}\left[\underbrace{e^{i2\pi f(t_2-\hat{k}\cdot\frac{\vec{r}_2}{c})}}_{\text{Earth term}} - \underbrace{e^{i2\pi f(t_1-\hat{k}\cdot\frac{\vec{r}_1}{c})}}_{\text{pulsar term}}\right]$$

$$= \frac{1}{2}u^i u^j e^A_{ij}(\hat{k})\left(\frac{1}{i2\pi f}\right)\frac{1}{(1-\hat{k}\cdot\hat{u})}e^{i2\pi f t_2}\cdot e^{-i2\pi f\hat{k}\cdot\vec{r}_2/c} \times\left[1-e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

$$\equiv \underbrace{e^{i2\pi f t_2}\overbrace{R^A(f,\hat{k})}^{\text{detector response function}}}_{\text{Fourier transform wrt time at } \vec{r}_2}$$

where

$$\boxed{R^A(f,\hat{k}) = \frac{1}{2}\frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})}\left(\frac{1}{i2\pi f}\right)\left[1-e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]\underbrace{e^{-i2\pi f\hat{k}\cdot\vec{r}_2/c}}_{\substack{=1 \text{ if} \\ \text{we set } \vec{r}_2=0}}}$$

GW wavelength: $\frac{c}{f} = \lambda_{gw} \rightarrow \frac{fL}{c} = \frac{L}{\lambda_{gw}}$

suppose: $L << \lambda_{gw}$ (long wavelength approximation)
e.g., for a L160 arm

$$\rightarrow R^A(f,\hat{k}) \approx \frac{1}{2}\frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})}\frac{1}{(i2\pi f)}\left[\cancel{1}-\left(\cancel{1}-i\frac{2\pi fL}{c}(1-\hat{k}\cdot\hat{u})\right)\right]$$

$$= \frac{1}{2}\frac{u^i u^j e^A_{ij}(\hat{k})}{\cancel{(1-\hat{k}\cdot\hat{u})}}\frac{1}{\cancel{(i2\pi f)}}\cancel{i2\pi f}\frac{L}{c}\cancel{(1-\hat{k}\cdot\hat{u})}$$

$$= \frac{1}{2}\left(\frac{L}{c}\right)u^i u^j e^A_{ij}(\hat{k})$$

For $L \gtrsim \lambda_{gw}$, cannot ignore the other factor, (LISA, PTAs)

Define: $F^A(\hat{k}) \equiv \frac{1}{2}\frac{u^i u^j e^A_{ij}(\hat{k})}{1-\hat{k}\cdot\hat{u}}$ (Earth-term-only response function for PTAs)

Then:

$$R^A(f,\hat{k}) = F^A(\hat{k})\underbrace{\left(\frac{1}{i2\pi f}\right)}_{\text{For PTAs this factor is absorbed in strain power spectrum}}\left[1-\underbrace{e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}}_{\text{pulsar term can be ignored when calculating cross-correlations between Earth and a pulsar, or between two different pulsars.}}\right]$$

For PTA's, $\hat{p} = -\hat{u}$:

$$R^A(f,\hat{k}) = F^A(\hat{k})\left[1 - e^{-i2\pi\frac{fL}{c}(1+\hat{k}\cdot\hat{p})}\right]\left(\frac{1}{i2\pi f}\right)$$

$$F^A(\hat{k}) = \frac{1}{2}\,\frac{p^i p^j e^A_{ij}(\hat{k})}{1+\hat{k}\cdot\hat{p}}$$

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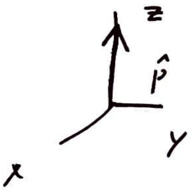
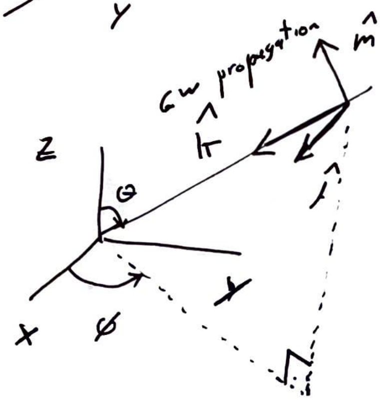
suppose $\hat{p} = \hat{z}$:

Q: what does $p^i p^j e^+_{ij}(\hat{k})$ and $p^i p^j e^\times_{ij}(\hat{k})$ equal?





$$\hat{k} = -\hat{r} = -[\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}]$$

$$\hat{l} = -\hat{\phi} = \sin\phi\,\hat{x} - \cos\phi\,\hat{y}$$

$$\hat{m} = -\hat{\theta} = -[\cos\theta\cos\phi + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}]$$

$$e^+_{ij}(\hat{k}) = l_i l_j - m_i m_j$$

$$e^\times_{ij}(\hat{k}) = l_i m_j + l_j m_i$$

$$\hat{z}^i\hat{z}^j e^+_{ij}(\hat{k}) = \cancel{(\hat{l}\cdot\hat{z})^2}^{\,0} - (\hat{m}\cdot\hat{z})^2 = \boxed{-\sin^2\theta}$$

can also write as $-(1-\cos\theta)(1+\cos\theta)$

$$\hat{z}^i\hat{z}^j e^\times_{ij}(\hat{k}) = 2(\cancel{\hat{l}\cdot\hat{z}}^{\,0})(\hat{m}\cdot\hat{z}) + $$

$$= \boxed{0}$$

$$1+\hat{k}\cdot\hat{p} = 1+\hat{k}\cdot\hat{z}$$

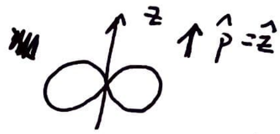
$$= \boxed{1-\cos\theta}$$

For long-wavelength approx: $(L << \lambda_{gw})$

$$R^A(f,\hat{k}) \approx \frac{1}{2}\left(\frac{L}{c}\right) p^i p^j e^A_{ij}(\hat{k})$$

$(\hat{p} = \hat{z}) \longrightarrow$ $R^+(f,\hat{k}) = -\frac{1}{2}\left(\frac{L}{c}\right)\sin^2\theta$, $\quad R^\times(f,\hat{k}) = 0$

$\rightarrow \sqrt{|R^+|^2 + |R^\times|^2}$ = "antenna pattern" $= \boxed{\frac{1}{2}\left(\frac{L}{c}\right)\sin^2\theta}$



rotate around z-axis (like a torus)

(same for $\hat{p} = -\hat{z}$)

For PTAs, Earth-term-only response:

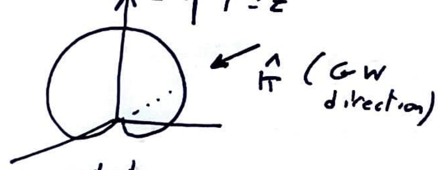
$\cancel{F^+(\hat{k}) = \frac{1}{2}\frac{p^i p^j e^+_{ij}(\hat{k})}{1+\hat{k}\cdot p}}$

$$F^A(\hat{k}) = \frac{1}{2}\frac{p^i p^j e^A_{ij}(\hat{k})}{1+\hat{k}\cdot\hat{p}}$$

$(\hat{p} = \hat{z}) \rightarrow$ $F^+(\hat{k}) = -\frac{1}{2}\frac{\cancel{(1-\cos\theta)}(1+\cos\theta)}{\cancel{(1-\cos\theta)}} = -\frac{1}{2}(1+\cos\theta)$

$F^\times(\hat{k}) = 0$

$\rightarrow \sqrt{|F^+|^2 + |F^\times|^2} = \boxed{\frac{1}{2}(1+\cos\theta)}$



rotate around z-axis

"upside-down apple"

Overlap Function / HD correlation:

"Overlap Function" is a transfer function between expected cross-power spectrum and GW strain power:

$$\underbrace{C_{ab}(f)}_{\substack{\text{cross-power spectrum} \\ \text{between measurements} \\ \text{made by two} \\ \text{detectors a and b}}} = \underbrace{\Gamma_{ab}(f)}_{\text{"overlap function"}} \underbrace{S_h(f)}_{\text{GW strain power}}$$

In term of detector response functions $R_a(f,\hat{k})$, $R_b(f,\hat{k})$

$$\rightarrow \boxed{\Gamma_{ab}(f) = \frac{1}{8\pi}\int d\hat{k} \sum_A R_a^A(f,\hat{k})\, R_b^A(f,\hat{k})^*}$$

= "sky-averaged" and "polarization-averaged" product of the response functions for detectors a and b.

- overlap functions are very general, not specific to PTAs
- can calculate for any pair of detectors, e.g., LIGO-Hanford with LIGO-Livingston, also for LISA, PTAs, ...
- For PTAs, $\Gamma_{ab}(f)$ is called "Hellings and Downs" correlations

Turns out that the freq dependence does not depend on the pulsars and can be absorbed into $S_h(f)$.

$$\rightarrow \boxed{\Gamma_{ab}} = \frac{1}{8\pi}\int d\hat{k} \sum_A F_a^A(\hat{k})\, F_b^A(\hat{k})\left[1 - e^{-i\frac{2\pi f L_a}{c}(1+\hat{p}_a\cdot\hat{k})}\right] \times \left[1 - e^{+i\frac{2\pi f L_b}{c}(1+\hat{p}_b\cdot\hat{k})}\right]$$

$$= \frac{1}{8\pi}\int d\hat{k} \sum_A F_a^A(\hat{k})\, F_b^A(\hat{k})(1+\delta_{ab})$$

pulsar terms only contribute when the two pulsars are the same

analytic expression using contour integration →

$$= \frac{1}{3}\left[\frac{1}{2} - \frac{1}{4}\left(\frac{1-\cos\gamma_{ab}}{2}\right) + \frac{3}{2}\left(\frac{1-\cos\gamma_{ab}}{2}\right)\ln\left(\frac{1-\cos\gamma_{ab}}{2}\right)\right](1+\delta_{ab})$$

optimal cross-correlation detection statistic:

Define: $S \equiv \sum_{ab} w_{ab}\, \rho_{ab}$ where $a<b$ (so distinct pulsar pairs) NANOGrav $\frac{67 \times 66}{2} = 2211$

w_{ab}: to be determined below

ρ_{ab}: interpulsar correlations

$\rho_{ab} \equiv \delta t_a^T\, Q_{ab}\, \delta t_b$

where Q_{ab} inverse-noise weights the correlation, normalized so that

$$\langle \rho_{ab} \rangle = A_{gw}^2\, \Gamma_{ab}$$

Γ_{ab}: HD coeffs

A_{gw}^2: squared amplitude of GWB

variance of ρ_{ab} in absence of spatial correlations:

$$\sigma_{ab,0}^2 \equiv \langle \rho_{ab}^2 \rangle_0 - \left(\cancel{\langle \rho_{ab} \rangle_0}^{\,0}\right)^2 \quad \text{where } \langle\,\rangle_0$$

$$= \langle \rho_{ab}^2 \rangle_0$$

$\langle\,\rangle_0$: in absence of spatial correlations

variance of S in absence of spatial correlations:

$$N^2 \equiv \langle S^2 \rangle_0 - \cancel{\langle S \rangle_0^2}^{\,0}$$

$$= \langle \sum_{ab} w_{ab}\, \rho_{ab} \sum_{cd} w_{cd}\, \rho_{cd} \rangle_0$$

$$= \sum_{ab} \sum_{cd} w_{ab}\, w_{cd}\, \langle \rho_{ab} \rho_{cd} \rangle_0$$

$$= \sum_{ab} \sum_{cd} w_{ab}\, w_{cd}\, \delta_{ac}\, \delta_{bd}\, \sigma_{ab,0}^2$$

no covariance between different pulsar pairs

$$= \sum_{ab} w_{ab}^2\, \sigma_{ab,0}^2$$

Determine weights w_{ab} such that $\dfrac{\langle S \rangle}{N}$ = maximum.

Equivalent to maximizing $\langle S \rangle^2 / N^2$:

$$\frac{\langle S \rangle^2}{N^2} = \frac{\left(\sum_{ab} w_{ab} \langle \rho_{ab} \rangle\right)^2}{\sum_{ab} w_{ab}^2\, \sigma_{ab,0}^2}$$

$$= \frac{(A_{gw}^2)^2 \left(\sum_{ab} w_{ab}\, \Gamma_{ab}\right)^2}{\sum_{ab} w_{ab}^2\, \sigma_{ab,0}^2}$$

Define ~~inner product~~ noise-weighted inner product:

$$(A,B) \equiv \sum_{ab} \frac{A_{ab} B_{ab}}{\sigma^2_{ab,0}}$$

Then

$$\frac{\langle S \rangle^2}{N^2} = \frac{(A^2_{gw})^2 \, (w\sigma^2, \Gamma)^2}{(w\sigma^2, w\sigma^2)}$$

RHS is maximized by taking $w\sigma^2 = q\Gamma$ (i.e., $w_{ab}\sigma^2_{ab,0} = q\Gamma_{ab}$) where q is some normalization constant

NOTE: $\dfrac{(\vec{A}\cdot\vec{B})^2}{\vec{A}\cdot\vec{A}} = \dfrac{|\vec{A}|^2|\vec{B}|^2\cos^2\theta_{AB}}{|\vec{A}|^2} = |\vec{B}|^2\cos^2\theta_{AB}$

is maximized by taking $\vec{A} \parallel \vec{B}$ so that $\theta_{AB} = 0$

Fix q by requiring that $N^2 = 1$:

$$\rightarrow \quad 1 = N^2 = (w\sigma^2, w\sigma^2) = q^2 \sum_{ab} \frac{\Gamma^2_{ab}}{\sigma^2_{ab,0}}$$

$$\rightarrow \quad \boxed{q = \frac{1}{\sqrt{\sum_{ab} \Gamma^2_{ab}/\sigma^2_{ab,0}}}} \quad \rightarrow \quad \boxed{w_{ab} = \frac{\Gamma_{ab}/\sigma^2_{ab,0}}{\sqrt{\sum_{ab} \Gamma^2_{ab}/\sigma^2_{ab,0}}}}$$

$$\rightarrow \quad \boxed{S = \frac{\sum_{ab} \Gamma_{ab}\rho_{ab}/\sigma^2_{ab,0}}{\sqrt{\sum_{ab} \Gamma^2_{ab}/\sigma^2_{ab,0}}}}$$

Zero mean unit variance in absence of spatial correlations, but not a gaussian distribution

this detection statistic is basically the dot product of ρ_{ab} with Γ_{ab} inverse-weighted by the noise $\sigma^2_{ab,0}$

→ compare observed value S_{obs} to null-distribution.
p-value ≡ prob. of obtaining $S > S_{obs}$

(9)

Optimal-binned estimator of HD correlation:

- Can do something similar for the optimal estimator of the HD correlation in an angular separation bin.
- Let j denote the bin ~~(e.g., ... 15 bins total)~~

NANOGrav used 15 bins
$\frac{2211}{15} \approx 150$ pairs/bin

NOTE: $ab \in j$ means the pulsar pair ab has angular separation $\gamma_{ab} \approx \gamma_j$.

- Define:

$$\boxed{\Gamma_{opt} \equiv \sum_{ab \in j} w_{ab}\, \rho_{ab}}$$

and determine weights w_{ab} such that

(i) $\langle \Gamma_{opt} \rangle = \Gamma_{bin}$ = HD correlation at γ_j

(ii) $\sigma^2_{opt} \equiv \langle \Gamma^2_{opt} \rangle - \langle \Gamma_{opt} \rangle^2$ has minimal value

such an estimator Γ_{opt} of Γ_{bin} is said to be a "minimal variance, unbiased estimator" of Γ_{bin}

Exercise: proceed as for optimal CC detection statistic to show

$$\boxed{\Gamma_{opt} = \frac{\Gamma_{bin}}{A^2_{gw}} \frac{\sum_{ab \in j} \sum_{cd \in j} \rho_{ab}\, C^{-1}_{ab,cd}\, \Gamma_{cd}}{\sum_{ab \in j} \sum_{cd \in j} \Gamma_{ab}\, C^{-1}_{ab,cd}\, \Gamma_{cd}}}$$

where $\langle \rho_{ab} \rangle = A^2_{gw} \Gamma_{ab}$

$\langle \rho_{ab} \rho_{cd} \rangle - \langle \rho_{ab} \rangle \langle \rho_{cd} \rangle \equiv C_{ab,cd}$ ← which has non-diagonal elements due to covariances induced by the GWB (cosmic variance)

$$\sigma^2_{opt} = \frac{\Gamma^2_{bin}}{A^4_{gw}} \frac{1}{\sum_{ab \in \gamma_j} \sum_{cd \in \gamma_j} \Gamma_{ab}\, C^{-1}_{ab,cd}\, \Gamma_{cd}}$$