MANITOU

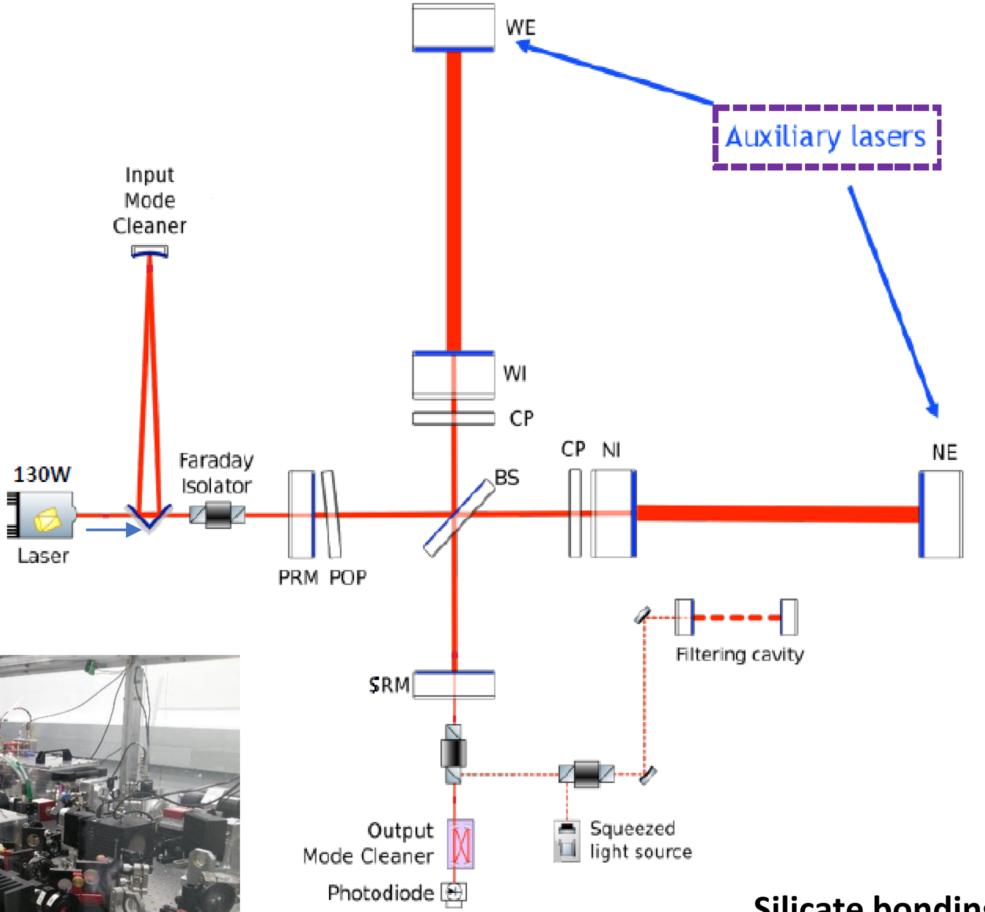
summer school

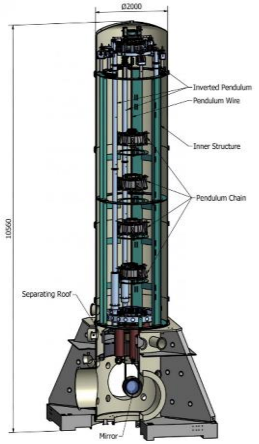
Ground based GW detectors

Walid Chaibi

French-Italian-(Dutch) ground based Gravitational wave detector

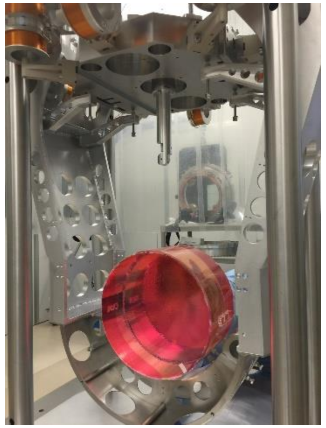
Cascina-Pisa-Italy

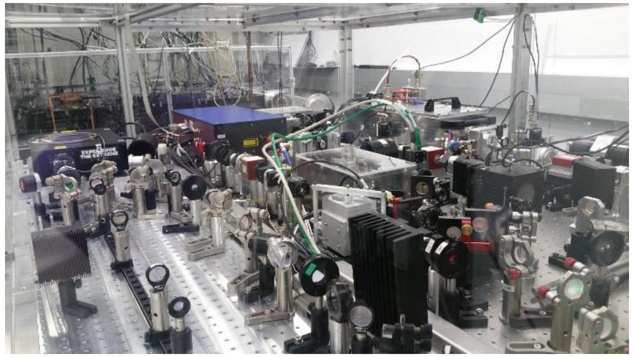




Super-attenuator

Monolithic suspension

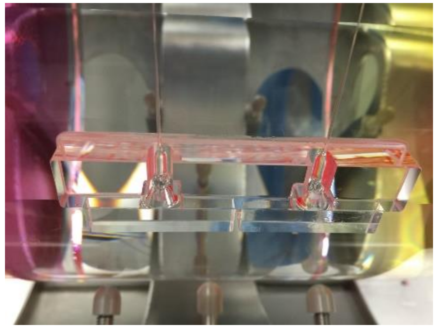




High power laser

- O4 : 80W
- O5 : 140W
- Post-O5 : 450W
- E-T : 700W

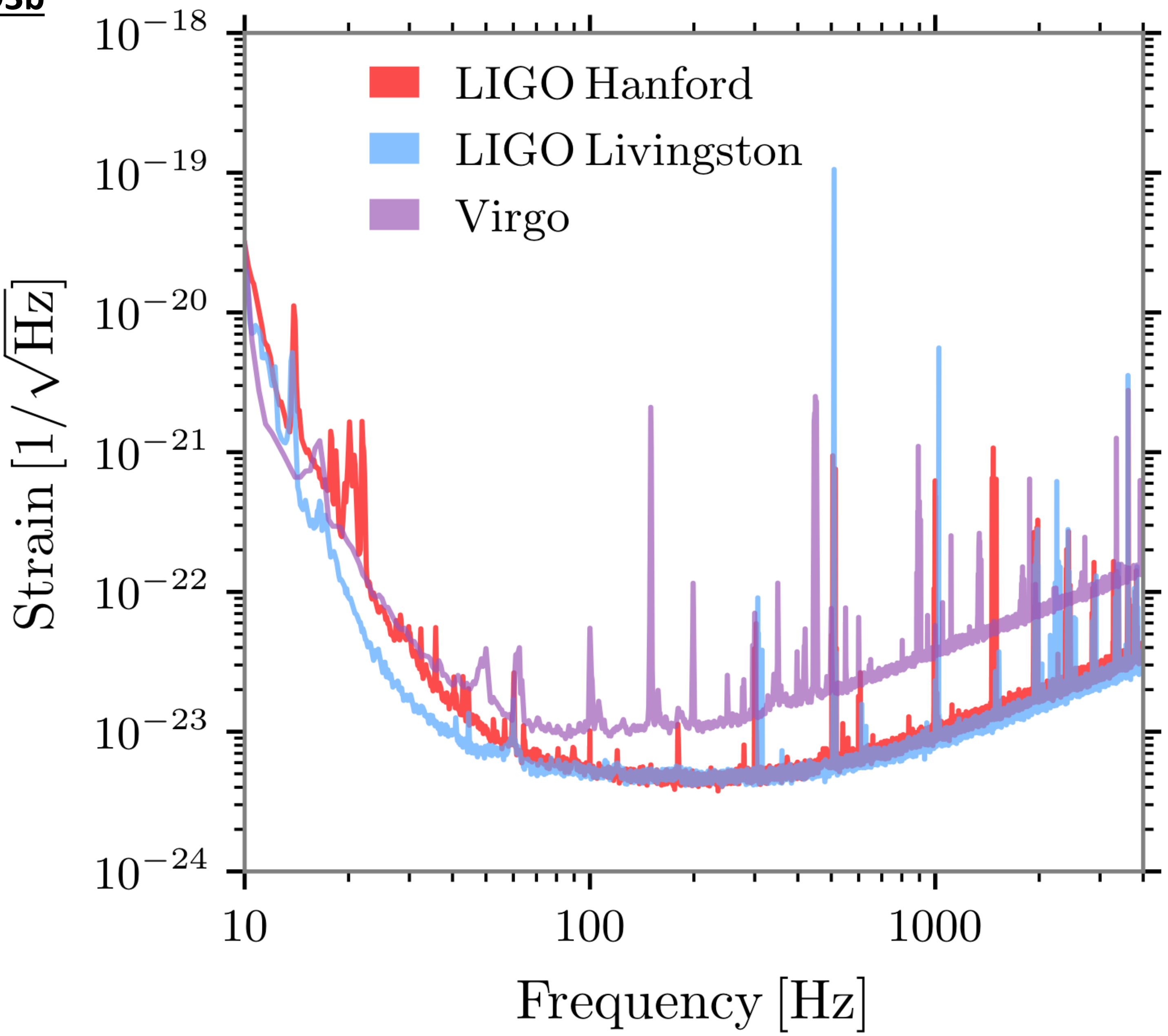
Silicate bonding



How does it work?

Sensitivity curves

Observing run: O3b



What does it mean?

Outline

Some math : The noise

Detection principle

Interferometer output and tuning :
- **The Michelson interferometer**
- **Sensitivity enhancement : The Fabry Perot cavity**
- **Shot noise limited detector**

Noise contribution:
- **Harmonic oscillator model**
- **Seismic noise**
- **Thermal noise**
- **Quantum noise**

Conclusion

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The noise: Definition

Noise : Random signal which is combined to a physical quantity to be measured

Specific case : additive noise

$$\underbrace{s}_{\text{signal}} = \underbrace{x}_{\text{Quantity to be measured}} + \underbrace{\varepsilon}_{\text{noise}}$$

$$\varepsilon \rightarrow p(\varepsilon)$$

Probability density

Specific case : time-dependent noise

$$s(t) = x(t) + \varepsilon(t)$$

$$\varepsilon(t) \rightarrow p_t(\varepsilon)$$

Time dependent probability density

Noise characterization

Mean value : $$m_\varepsilon(t) = \langle \varepsilon(t) \rangle = \int_{-\infty}^{+\infty} \varepsilon \times p_t(\varepsilon) d\varepsilon$$

$\langle \dots \rangle$ computed by repeating the experiments

Variance : $$m_{\varepsilon,2}(t) = \sigma_\varepsilon^2(t) = \left\langle \big(\varepsilon(t) - m_\varepsilon(t)\big)^2 \right\rangle = \int_{-\infty}^{+\infty} \big(\varepsilon(t) - m_\varepsilon(t)\big)^2 \times p_t(\varepsilon) d\varepsilon$$

$\downarrow$ .....

Moment of order *n*: $$m_{\varepsilon,n}(t) = \left\langle \big(\varepsilon(t) - m_\varepsilon(t)\big)^n \right\rangle = \int_{-\infty}^{+\infty} \big(\varepsilon(t) - m_\varepsilon(t)\big)^n \times p_t(\varepsilon) d\varepsilon$$

$\Rightarrow$ But this is not enough to characterize the whole process…

Autocorrelation

$$\Gamma_\varepsilon(t_1, t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_1 \times \varepsilon_2 \times \underbrace{p_{t_1,t_2}(\varepsilon_1, \varepsilon_2)}_{\text{joint probability density function}} d\varepsilon_1 d\varepsilon_2$$

In principle, we would need to know all autocorrelation functions of order *n*:

$$\Gamma_\varepsilon(t_1, t_2, \ldots, t_n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_1 \times \varepsilon_2 \times \cdots \times \varepsilon_n \times p_{t_1,t_2,\ldots,t_n}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n$$

In practice, the second order autocorrelation function already gives enough information

Stationary noise

Stationary noise : The noise characteristics do not change during time

$$\text{For all } T \qquad p_{t_1,t_2,\dots,t_n}(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n) = p_{t_1+T,t_2+T,\dots,t_n+T}(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)$$

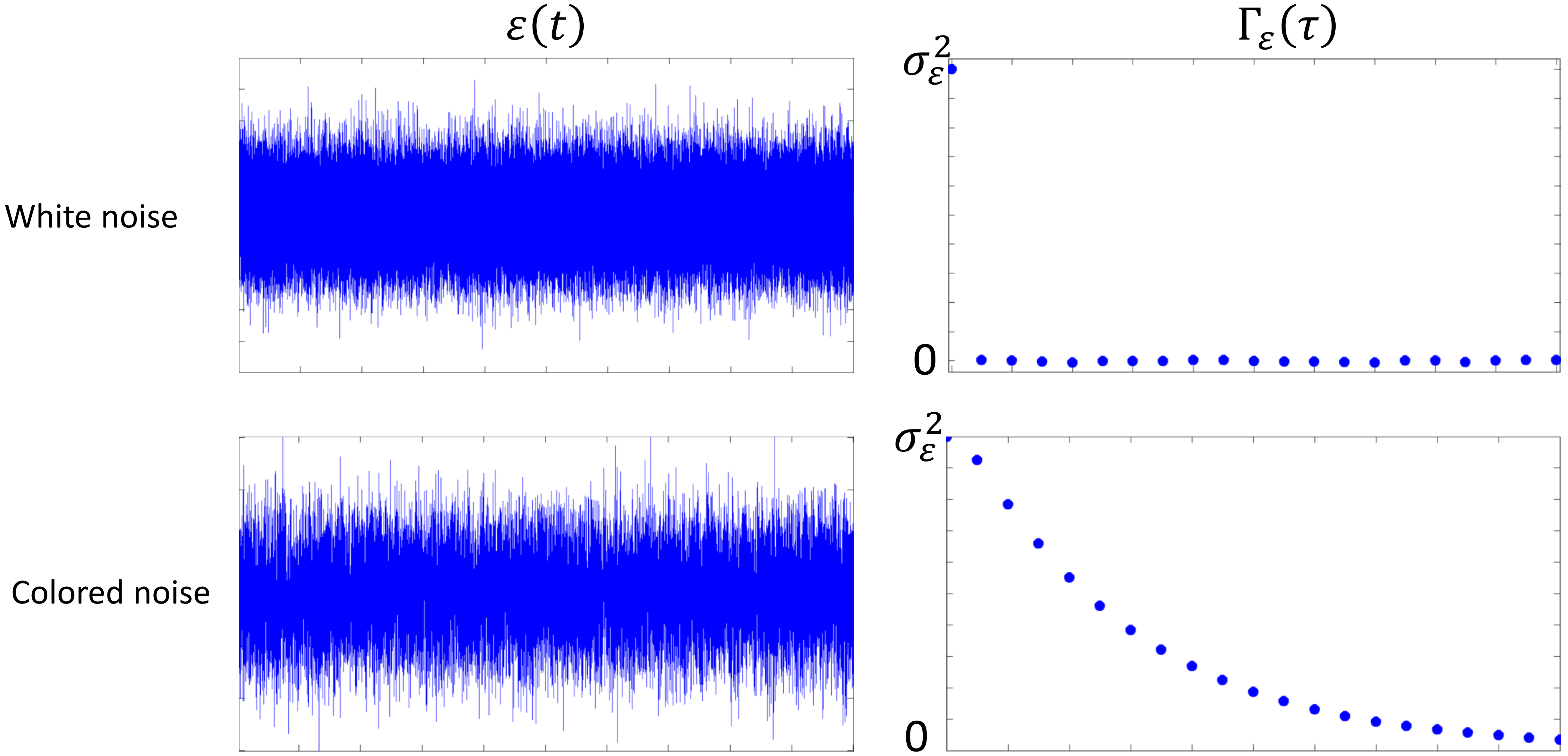
One interesting case: centered and 2nd order stationary noise: $p_t(\varepsilon) = p_{t+T}(\varepsilon)$ $\quad p_{t_1,t_2}(\varepsilon_1,\varepsilon_2) = p_{t_1+T,t_2+T}(\varepsilon_1,\varepsilon_2)$

Zero mean value $\quad \mu_\varepsilon(t) = \langle\varepsilon(t)\rangle = 0$

Standard deviation $\quad \sigma_\varepsilon^2(t) = \sigma_\varepsilon^2$

Autocorrelation $\quad \begin{cases} \Gamma_\varepsilon(t,t+\tau) = \Gamma_\varepsilon(0,\tau) = \Gamma_\varepsilon(\tau) \\ \sigma_\varepsilon^2 = \Gamma_\varepsilon(0) \geq \Gamma_\varepsilon(\tau) \end{cases}$

Stationary noise



Temporal average

for one experiment

$$m_f = \overline{f(\varepsilon)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f\big(\varepsilon(t)\big)dt$$

$$m_\varepsilon = \bar{\varepsilon} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varepsilon(t)dt$$

$$s_\varepsilon^2 = \overline{\varepsilon^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varepsilon^2(t)dt$$

$$C_\varepsilon(\tau) = \overline{\varepsilon(t)\varepsilon(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varepsilon(t)\varepsilon(t+\tau)dt$$

2nd order ergodicity : all experiments are equivalent

$$m_{\varepsilon_i} = m_{\varepsilon_j} = m_\varepsilon = \bar{\varepsilon}$$

$$s_{\varepsilon_i}^2 = s_{\varepsilon_j}^2 = s_\varepsilon^2 = \overline{\varepsilon^2}$$

$$C_{\varepsilon_i}(\tau) = C_{\varepsilon_j}(\tau) = C_\varepsilon(\tau) = \overline{\varepsilon(t)\varepsilon(t+\tau)}$$

Stationary and ergodic process

All considered process are centered, stationary and ergodic

$$m_\varepsilon(t) = \langle \varepsilon(t) \rangle = \int_{-\infty}^{+\infty} \varepsilon \times p_t(\varepsilon) d\varepsilon \qquad \mu_\varepsilon = \bar{\varepsilon} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varepsilon(t) dt$$

$$\overline{\langle f(\varepsilon_i(t)) \rangle} \overset{\text{linearity}}{=} \left\langle \overline{f(\varepsilon_i(t))} \right\rangle \overset{\text{ergodicity}}{=} \left\langle \overline{f(\varepsilon(t))} \right\rangle = \langle f_{\varepsilon,t} \rangle = f_{\varepsilon,t}$$

$$\overline{\langle f(\varepsilon_i(t)) \rangle} = \overline{f_{\varepsilon,e}(t)} \overset{\text{stationarity}}{=} \overline{f_{\varepsilon,e}} = f_{\varepsilon,e}$$

$$\Rightarrow \quad f_{\varepsilon,e} = f_{\varepsilon,t}$$

Averages on different experiments is equivalent to averages over time

$$m_\varepsilon = \mu_\varepsilon \; ; \; s_\varepsilon^2 = \sigma_\varepsilon^2 \; ; \; C_\varepsilon(\tau) = \Gamma_\varepsilon(\tau)$$

Harmonic analysis : The Fourier transform

Finite energy process: $$\mathcal{E} = \int_{-\infty}^{+\infty} |f(t)|^2 dt$$

its Fourier transform exists: $$\tilde{f}(\nu) = \int_{-\infty}^{+\infty} e^{-i2\pi\nu t} f(t) dt$$

Parseval equality: $$\mathcal{E} = \int_{-\infty}^{+\infty} \left|\tilde{f}(\nu)\right|^2 d\nu$$

$\left|\tilde{f}(\nu)\right|^2$ represents the energy density per spectral interval $d\nu$:
the **Energy Spectral Density (ESD)**

Harmonic analysis : The Fourier transform

In general, noise is not a finite energy process: $$\int_{-\infty}^{+\infty} |\varepsilon(t)|^2 dt = \infty$$

But…it has a finite mean power: $$\bar{P} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varepsilon(t)|^2 dt$$

If it exists, we're interested in the quantity: $$S_\varepsilon(\nu) = \lim_{T\to\infty} \frac{1}{2T} |\tilde{\varepsilon}_T(\nu)|^2 = \lim_{T\to\infty} \frac{1}{2T} \left| \int_{-T}^{T} e^{-i2\pi\nu t} \varepsilon(t) dt \right|^2$$

Summed over the frequency ν:

$$\int_{-\infty}^{+\infty} S_\varepsilon(\nu) d\nu = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} |\tilde{\varepsilon}_T(\nu)|^2 d\nu = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} |\varepsilon_T(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\varepsilon_T(t)|^2 dt = \bar{P}$$

➡ $S_\varepsilon(\nu)$represents the power density per spectral interval $d\nu$:
the **Power Spectral Density (PSD)**

Harmonic analysis : The Wiener-Khintchine Theorem

For a stationary and ergodic process (noise), the PSD exists and is given by the Fourier transform of the corresponding autocorrelation function:

$$S_\varepsilon(\nu) = \widetilde{\Gamma_\varepsilon}(\nu) = \int_{-\infty}^{+\infty} e^{-i2\pi\nu\tau}\Gamma_\varepsilon(\tau)d\tau$$

Very useful for analytic calculations, original definition used for measurements (FFT)

Unity…? example:

Voltage noise: $\delta V(t) \rightarrow \mathrm{V}$

$\Gamma_{\delta V}(\tau) \rightarrow \mathrm{V}^2$

PSD : $S_{\delta V}(\nu) \rightarrow \mathrm{V}^2/\mathrm{Hz}$

$$\delta V\ (\nu) = \sqrt{S_{\delta V}(\nu)} \rightarrow \mathrm{V}/\sqrt{\mathrm{Hz}}$$

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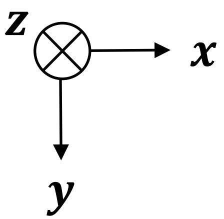
GW, plane wave



(+) polarization

(x) polarization

Space time metric in the TT gauge

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Minkowski

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & +h & 0 & 0 \\ 0 & 0 & -h & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

GW, (+) polarization

Light follows the geodesic $\quad g_{\mu\nu} dx^{\mu} dx^{\nu} = 0 \quad$ Einstein notation

$$dx^{\mu} = (cdt, dx, dy, dz) \qquad c^2 dt^2 + dx^2(-1+h) + dy^2(-1-h) - dz^2 = 0$$

Effect on a photon

Photon propagating along $x \rightarrow dy = 0\,; dz = 0$

$$dx = \pm c\left(1 + \frac{1}{2}h\right)dt$$

Photon propagating along $y \rightarrow dx = 0\,; dz = 0$

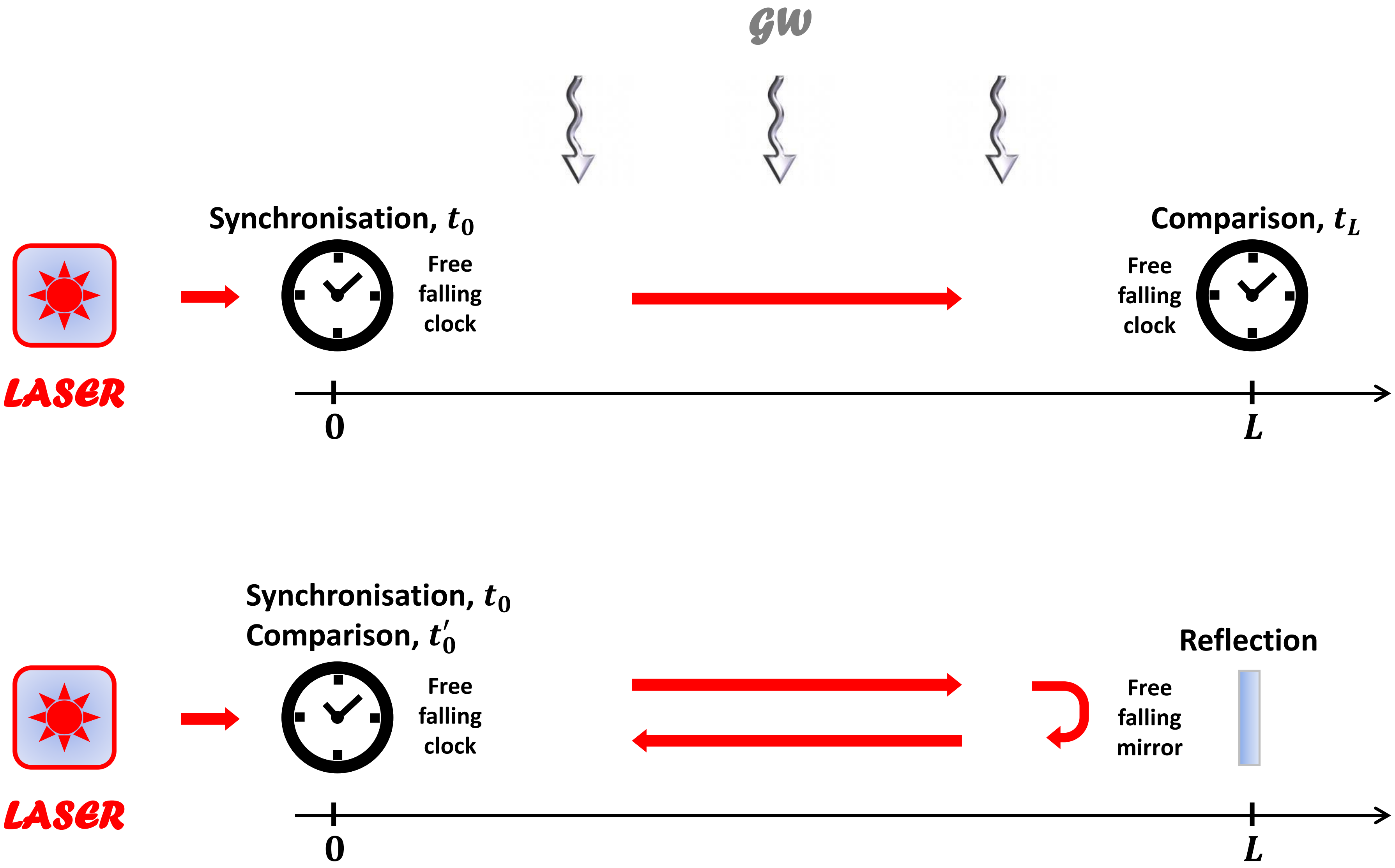
$$dy = \pm c\left(1 - \frac{1}{2}h\right)dt$$

Towards $x > 0$, from 0 to L

$$\int_0^L dx = c\int_{t_0}^{t_L}\left(1 + \frac{1}{2}h(t)\right)dt = c\int_{t_0}^{t_L} dt + \frac{1}{2}\int_{t_0}^{t_L} h(t)dt$$

$$t_L = t_0 + \frac{L}{c}\left(1 - \frac{1}{2}h(t)\right) \quad \text{if} \quad t[h] \gg \frac{L}{c}$$

Detection by time delay measurement



$$t_L = t_0 + \frac{L}{c}\left(1 - \frac{1}{2}h(t)\right)$$

$$t_0' = t_0 + \frac{2L}{c}\left(1 - \frac{1}{2}h(t)\right)$$

What is a clock?

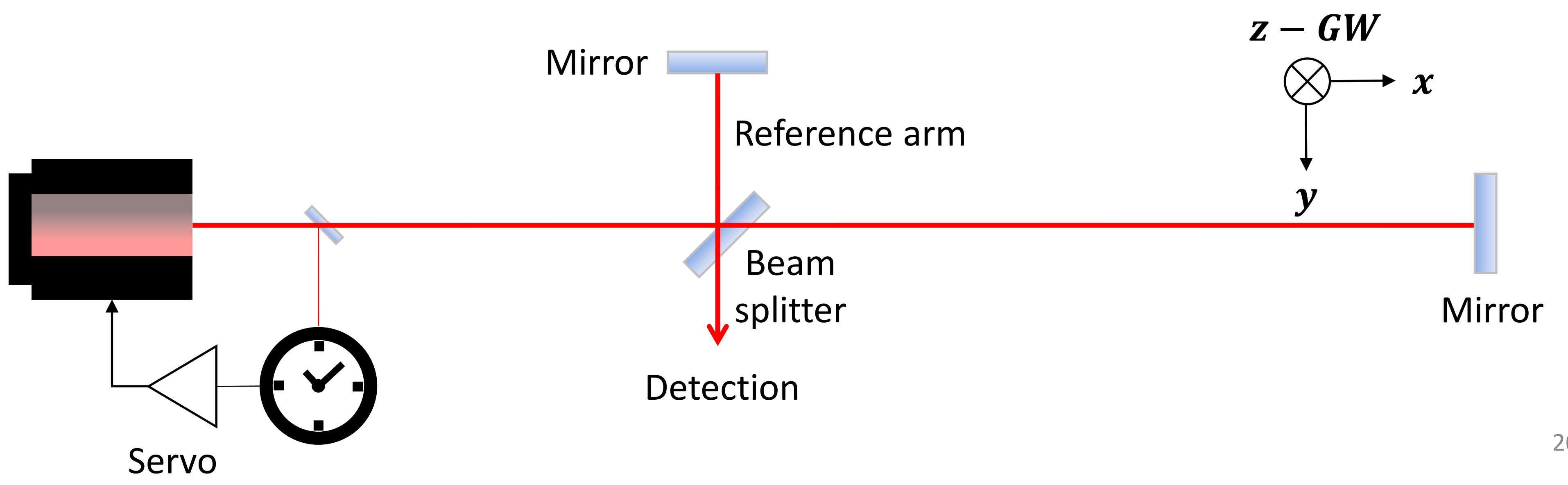
Oscillator synchronized on a reference (atomic transition, optical cavity)

Oscillator $$E(t) = E_0\big(1 + a(t)\big) \times e^{-i\left(2\pi\nu_0 t + \varphi(t)\right)}$$ ➡ **Single frequency laser**

Amplitude (E_0), **Amplitude noise** ($a(t)$), **Phase** ($2\pi\nu_0 t$), **Phase noise** ($\varphi(t)$)

Time propagation measurement ➡ Phase measurement ➡ Interference

Synchronization ➡ Locking techniques (servo loops)



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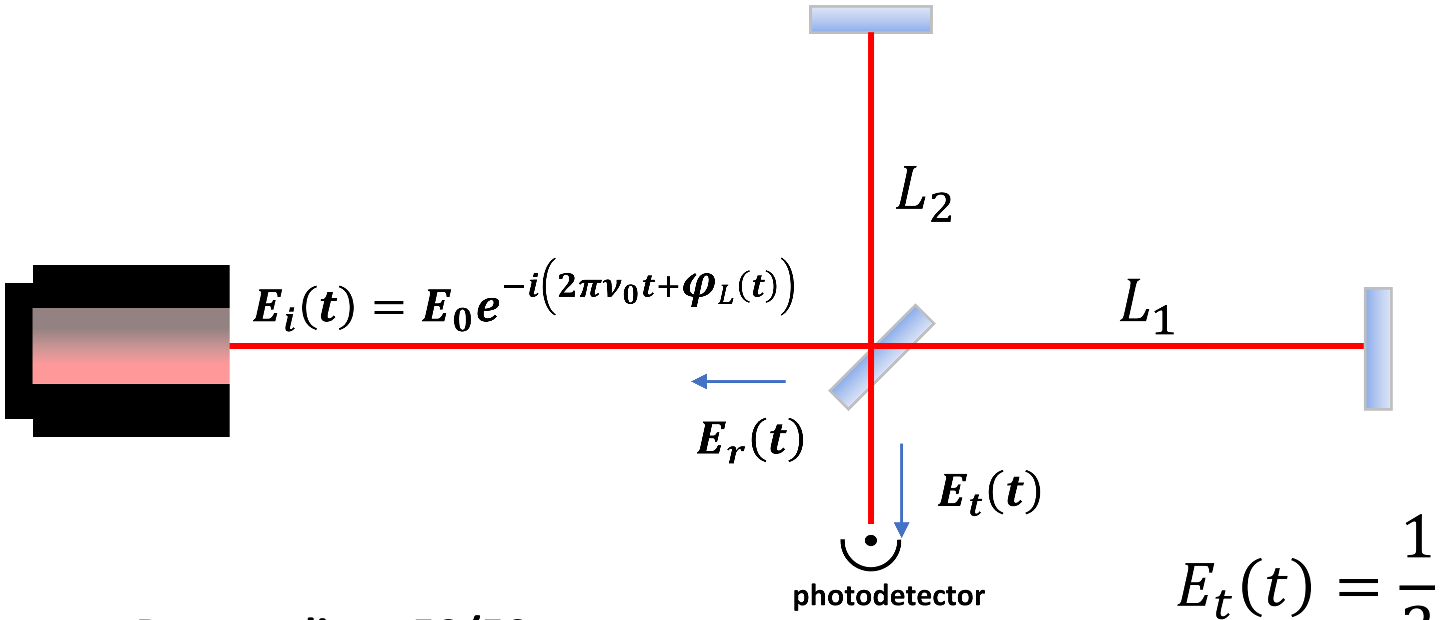
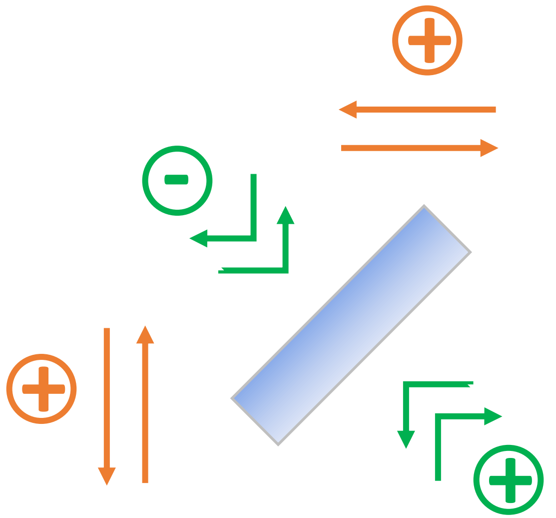
Michelson interferometer



Beam splitter 50/50



$$E_t(t) = \frac{1}{2} E_0 e^{-i2\pi\nu_0 t}\left(e^{i\phi_1} - e^{i\phi_2}\right)$$

$$E_r(t) = \frac{1}{2} E_0 e^{-i2\pi\nu_0 t}\left(e^{i\phi_1} + e^{i\phi_2}\right)$$

$$\phi_1 = \frac{4\pi\nu_0}{c} L_1\left(1 - \frac{h(t)}{2}\right) - \varphi_L\left(t - \frac{2L_1}{c}\right)$$

$$\phi_2 = \frac{4\pi\nu_0}{c} L_2\left(1 + \frac{h(t)}{2}\right) - \varphi_L\left(t - \frac{2L_2}{c}\right)$$

Michelson interferometer

$$\phi = \frac{\phi_2 + \phi_1}{2} \quad ; \quad \Delta\phi = \frac{\phi_2 - \phi_1}{2} \quad ; \quad L = \frac{L_2 + L_1}{2} \quad ; \quad \Delta L = \frac{L_2 - L_1}{2}$$

$$E_t(t) = -iE_0 e^{-i2\pi\nu_0 t} \times e^{i\phi} \sin(\Delta\phi)$$

$$E_r(t) = E_0 e^{-i2\pi\nu_0 t} \times e^{i\phi} \cos(\Delta\phi)$$

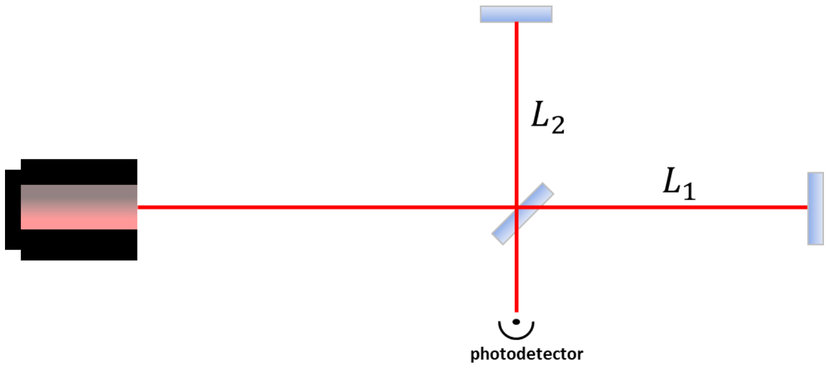
<u>Frequency noise</u>

$$\delta\nu(t) = \frac{1}{2\pi}\frac{d\varphi_L(t)}{dt}$$

$$\phi = \frac{1}{2}\left(\frac{8\pi\nu_0 L}{c} + \frac{4\pi\nu_0\Delta L}{c}h(t) - 2\varphi_L(t) + \frac{8\pi\delta\nu(t)L}{c}\right)$$

$$\Delta\phi = \frac{1}{2}\left(\frac{8\pi\nu_0\Delta L}{c} + \frac{4\pi\nu_0 L}{c}h(t) - \frac{2\pi\delta\nu(t)\Delta L}{c}\right)$$

Michelson interferometer : Dark fringe



$$P_t(t) = |E_t(t)|^2 = P_0 \times \sin^2\left(4\pi\frac{\Delta L}{\lambda} + 2\pi\frac{Lh(t)}{\lambda} - \pi\frac{\delta\nu(t)}{\nu_0}\frac{\Delta L}{\lambda}\right)$$

Signal $\propto \frac{Lh(t)}{\lambda}$: increase the arm length of the interferometer

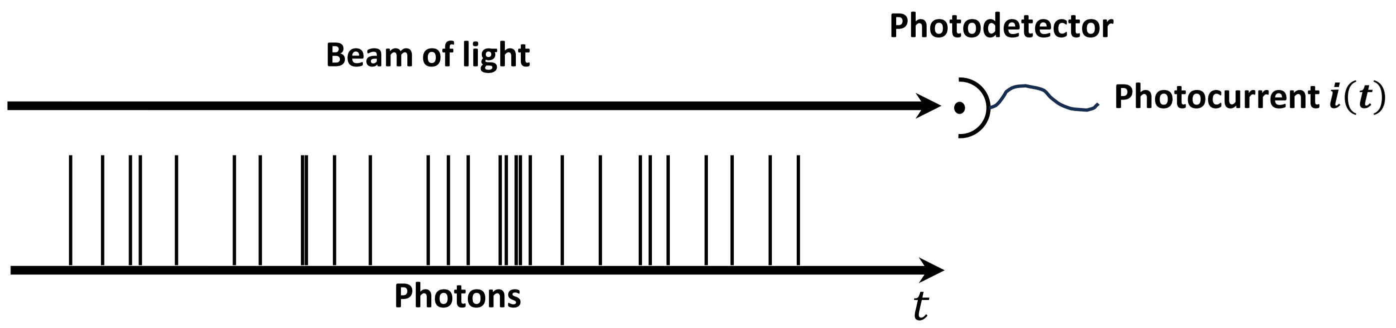
Contamination by the frequency noise : reduced for $\Delta L \to 0$, dark fringe

Dark fringe $\Delta L = 0, P_t \propto \left(\frac{L\times h}{\lambda}\right)^2$: need an offset $\Delta L_{\mathrm{offset}}$

How to choose the offset?

Detection noise

The shot noise



Number of photons $N(t)$ during Δt follows a Poissonian law

Mean value $\bar{N}$

Variance $\sigma_N^2 = \bar{N}$

Autocorrelation $\Gamma(\tau) = \sigma_N^2\, \delta(\tau)$

Power: $P_{\Delta t}(t) = \frac{h_P \nu\, N(t)}{\Delta t}$; Variance: $\sigma_{P,\Delta t}^2 = \frac{h_P \nu_0 P_0}{\Delta t} = \int_{1/\Delta t} h_P \nu_0 P_0\, df$; PSD: $S_{\mathrm{shot}}(f) = h_P \nu_0 P$

$P = 10\ \mathrm{mW}$; $\sqrt{S_{\mathrm{shot}}(f)} \simeq 4 \times 10^{-11}\, \frac{\mathrm{W}}{\sqrt{\mathrm{Hz}}}$ @ $\lambda = 1064\ \mathrm{nm}$, $\mathrm{RIN}(f) \simeq \frac{\sqrt{S_{\mathrm{shot}}(f)}}{P} = 4 \times 10^{-9}\ \mathrm{Hz}^{-1/2}$

The photodetector dark noise

$$\sqrt{S_{\mathrm{D}}(f)} \simeq 200\ \frac{\mathrm{nV}}{\sqrt{\mathrm{Hz}}}\ ;\ \ \rho_D = \frac{\sqrt{S_{\mathrm{D}}(f)}}{\sqrt{S_{\mathrm{shot}}(f)}} \simeq 0.1$$

Signal to noise ratio: SNR

$$P_t(t) = |E_t(t)|^2 = P_0 \times \sin^2\left(4\pi\frac{\Delta L}{\lambda} + 2\pi\frac{Lh(t)}{\lambda}\right)$$

Δ

On the photodetector $\Delta, \frac{Lh(t)}{\lambda} \ll 1$

$$P_{PD}(t) = 16\pi^2\Delta^2\ P_0 + 16\pi^2 P_0 \Delta \frac{Lh(t)}{\lambda} + \delta P_{\mathrm{shot}}(16\pi^2\Delta^2\ P_0, t) + \delta P_{\mathrm{D}}(t)$$

$$S_{P_{PD}}(f) = 256\pi^4\Delta^4 P_0 \delta(f) + 256\pi^4\Delta^2 P_0^2 \frac{L^2}{\lambda^2} S_h(f) + h_P\nu_0 16\pi^2\Delta^2 P_0 + S_{\delta P_{\mathrm{D}}}(f)$$

SNR $$\rho^2(f) = \frac{256\pi^4\Delta^2 P_0^2 \frac{L^2}{\lambda^2} S_h(f)}{h_P\nu_0 16\pi^2\Delta^2 P_0 + S_{\delta P_{\mathrm{D}}}(f)}$$

$$\Rightarrow \quad \rho^2(f) = \frac{16\pi^2 P_0}{h_P\nu_0}\frac{L^2}{\lambda^2} S_h(f)$$

Does not depend on the offset!

Δ choice :

- not to be limited by the dark noise
- What technically possible $16\pi^2\Delta^2\ P_0 < 100\ \mathrm{mW}$

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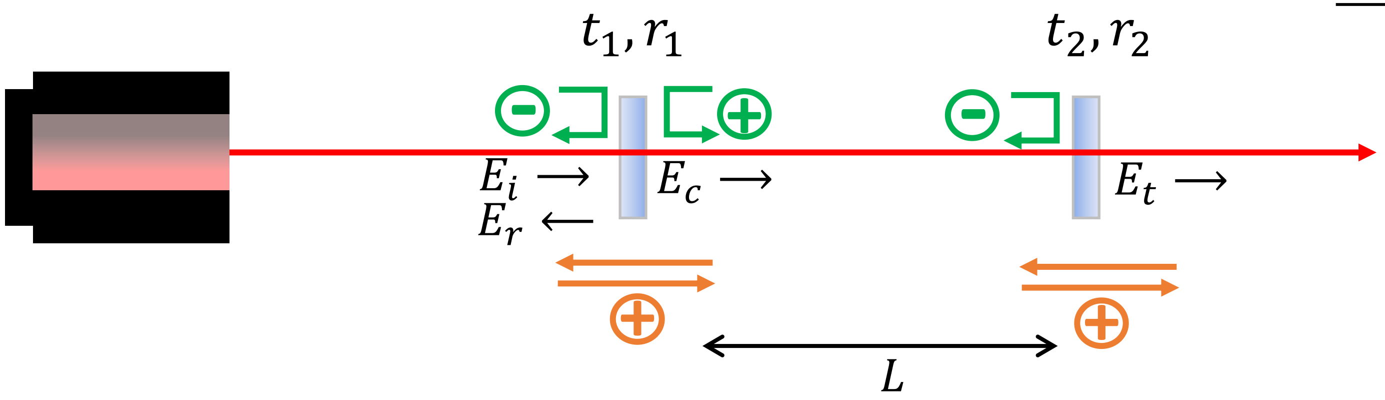
Enhancement of the interferometer sensitivity: Optical Cavities

2 mirrors cavity



Perfect mirrors : energy conservation

$$t_1^2 + r_1^2 = 1$$

$$t_2^2 + r_2^2 = 1$$

Cavity equation

$$\begin{cases} E_c(t) = t_1\, E_i(t) - r_2 r_1\, E_c\left(t - \dfrac{2L}{c}\right) \\ E_t(t) = t_2 E_c\left(t - \dfrac{L}{c}\right) \\ E_r(t) = -r_1\, E_i(t) - r_2 t_1 E_c\left(t - \dfrac{L}{c}\right) \\ E_{(i,r,c,t)}(t) = E_{(i,c,r,t)0}(t) e^{-i2\pi\nu_0 t} \end{cases}$$

Steady state

$E_{(i,c,r,t)0}(\cancel{t})$

$$\begin{cases} E_{c0} = t_1\, E_{i0} - r_2 r_1 e^{i\frac{4\pi\nu_0 L}{c}}\, E_{c0} \\ E_{t0} = t_2 e^{i\frac{2\pi\nu_0 L}{c}} E_{c0} \\ E_{r0} = -r_1\, E_{i0} - r_2 t_1 E_{c0} e^{i\frac{4\pi\nu_0 L}{c}} \end{cases}$$

Intracavity wave

$\phi = \frac{4\pi\nu_0 L}{c}$: Round trip propagation phase

$$E_{c0} = \frac{t_1}{1+r_2 r_1 e^{i\phi}} E_{i0} = S(\phi)\ E_{i0}$$

$S(\phi)$: Enhancement factor

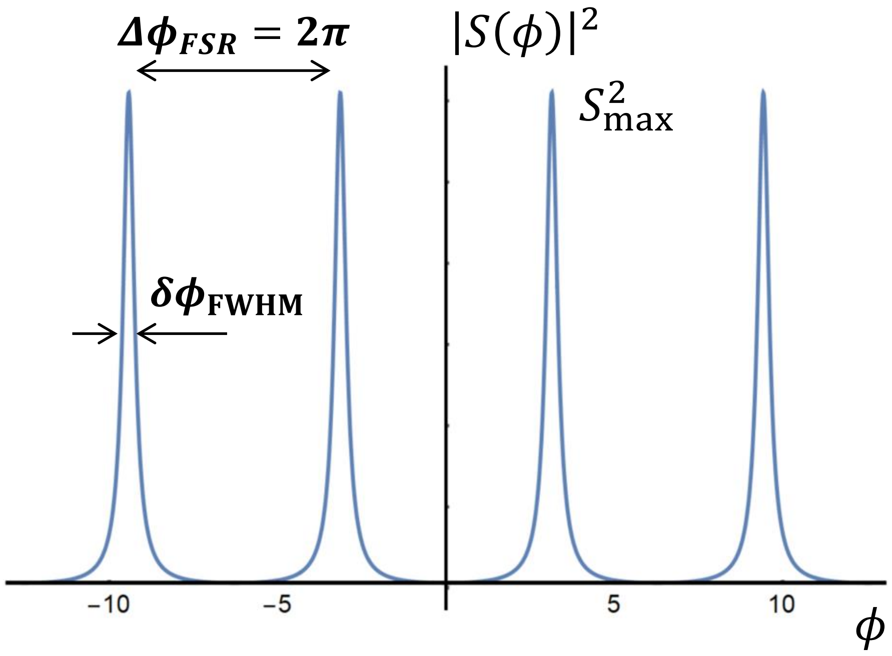
$$|S(\phi)|^2 = \frac{S_{\max}^2}{1 + \frac{4F^2}{\pi^2}\cos^2\left(\frac{\phi}{2}\right)}$$

$$S_{\max}^2 = \frac{t_1^2}{(1 - r_2 r_1)^2} \simeq \frac{1}{t_1^2} \gg 1$$

$$F = \frac{\pi\sqrt{r_1 r_2}}{1 - r_2 r_1} \simeq \frac{1}{t_1^2} \gg 1, F = 10 \to 10^6$$

$$\Delta\phi_{FSR} = 2\pi\ ;\ \Delta\nu_{FSR} = \frac{c}{2L}\ ;\ \Delta L_{FSR} = \frac{\lambda}{2}$$

$$\delta\phi_{\mathrm{FWHM}} = \frac{2\pi}{F}\ ;\ \delta\nu_{\mathrm{FWHM}} = \frac{\Delta\nu_{FSR}}{F}\ ;\ \delta L_{\mathrm{FWHM}} = \frac{\lambda}{2F}$$



Resonance condition

$$\boldsymbol{\phi = \pi + 2l\pi}$$

Gaussian beams

Propagation phase Gouy phase

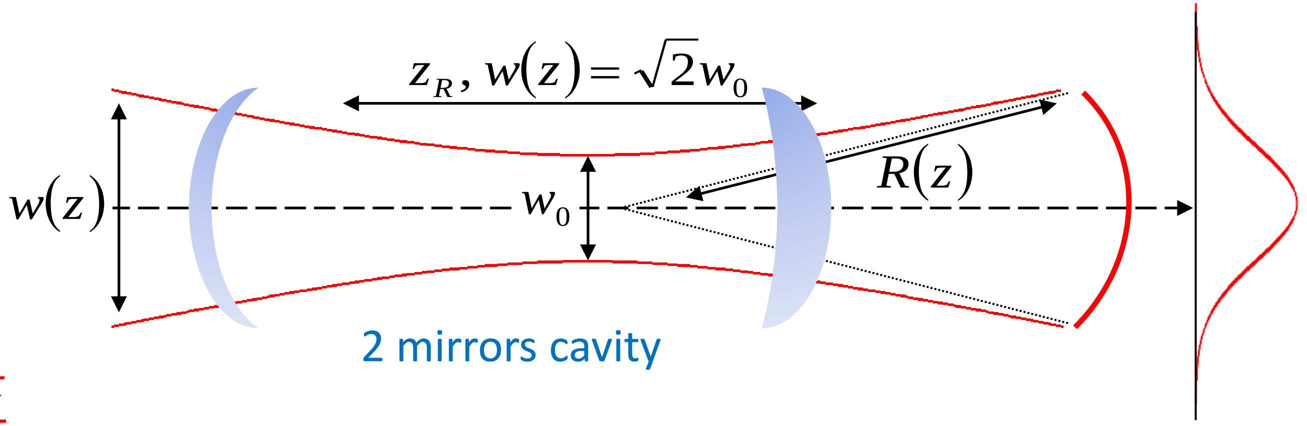
$$E(x,y,z,t) = E_0 \underbrace{\frac{1}{w(z)}}_{\text{Beam radius}} \times e^{i\left(\frac{2\pi}{\lambda}z + \psi_G(z)\right)} \times \underbrace{e^{-i\frac{2\pi}{\lambda}\frac{x^2+y^2}{2R(z)}}}_{\text{Spherical wave front}} \times \underbrace{e^{-\frac{x^2+y^2}{w^2(z)}}}_{\text{Gaussian profile}}$$

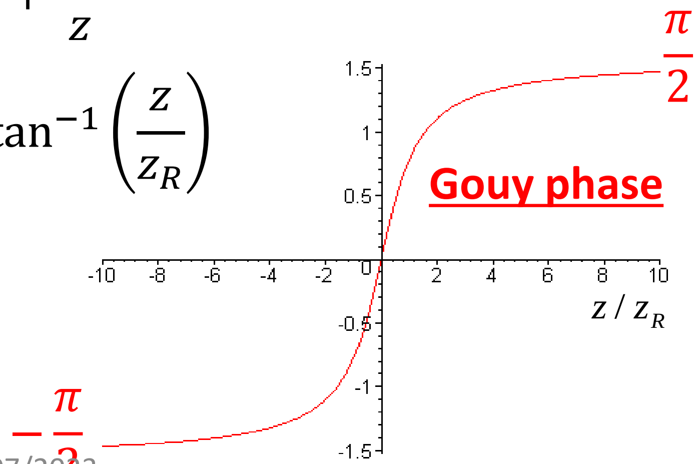
$z_R = \frac{\pi w_0^2}{\lambda}$: Rayleigh range

$$w(z) = w_0\sqrt{1+\frac{z^2}{z_R^2}}$$

$$R(z) = z + \frac{z_R^2}{z}$$

$$\psi_G(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$$





Resonance condition

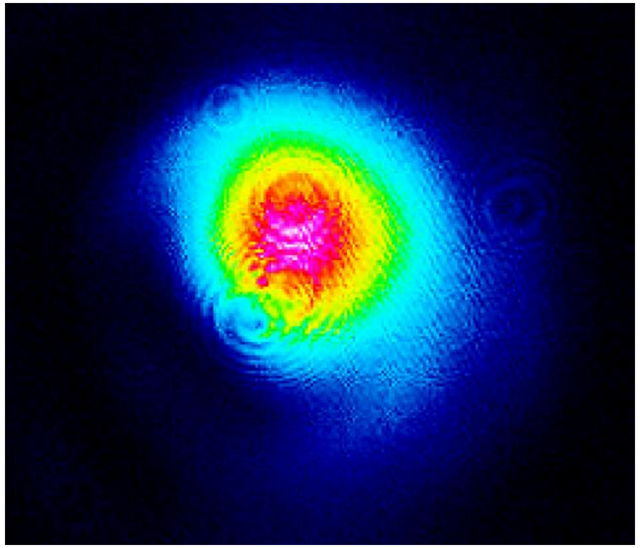
$$\boldsymbol{\phi_{res} + 2\,\Delta\psi_G = \pi + 2l\pi}$$

High order modes

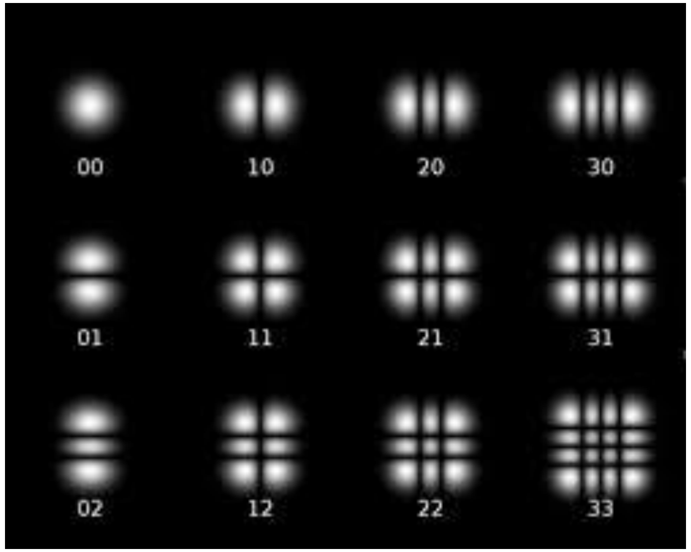
Gaussian mode : fundamental mode of a set of high order mode ; example : Hermite-Gauss modes

$$E_{n,m}(x,y,z,t) = E_{n,m,0}\frac{1}{w(z)} \times e^{i\left(\frac{2\pi}{\lambda}z+(n+m+1)\psi_G(z)\right)} \times H_m\left(\frac{\sqrt{2}x}{w(2)}\right) H_n\left(\frac{\sqrt{2}y}{w(2)}\right) e^{-i\,\frac{2\pi}{\lambda}\frac{x^2+y^2}{2R(z)}} \times e^{-\frac{x^2+y^2}{w^2(z)}}$$

H_j: Hermite polynomial of order j



decomposition



Resonance condition

$$\boldsymbol{\phi_{n,m,res} + 2(m+n+1)\,\Delta\psi_G = \pi + 2l\pi}$$

depends on the mode order

Cavity transmission

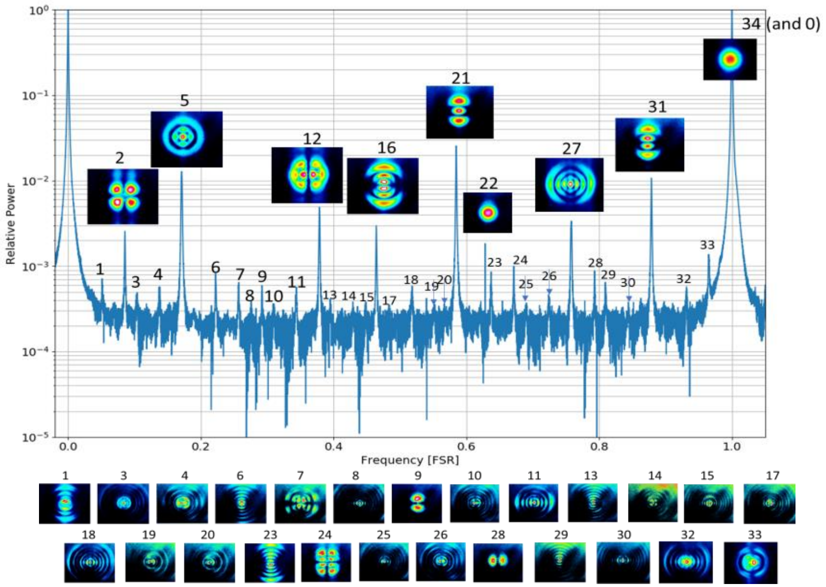
Single mode injected → Transmission : $|T(\phi)|^2 = \left|\frac{E_t}{E_i}\right|^2 = \frac{T_{\max}^2}{1+\frac{4F^2}{\pi^2}\cos^2\left(\frac{\phi}{2}\right)}$ Resonance → Maximum transmission

$$r_2 = r_1\ ;\ t_2 = t_1 \longrightarrow T_{\max}^2 = 1$$

Totally transmissive cavity

Real beam: $E_i = \sum_{m,n} c_{m,n,i} \times E_{m,n,i}$ → Resonance on the fundamental mode

$$\begin{cases} |E_t|^2_{0,0} = \left|c_{0,0,i}\right|^2 \\ |E_t|^2_{m\neq 0,n\neq 0} \propto \frac{\left|c_{m,n,i}\right|^2}{F^2} \simeq 0 \end{cases}$$



→ **Mode cleaner cavity**

Cavity reflectivity

$$R(\phi) = \frac{E_r}{E_i}$$

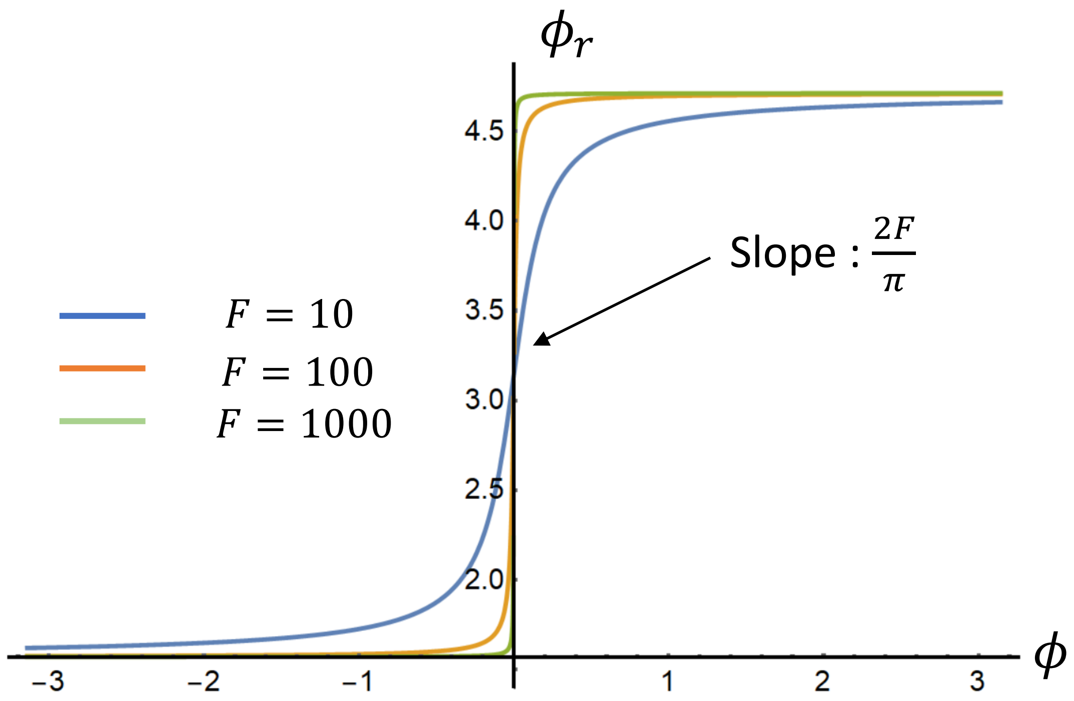
Specific configuration : $r_2 = 1\ ;\ t_2 = 0$ ➡ $T = 0\ ;\ |R(\phi)|^2 = 1$

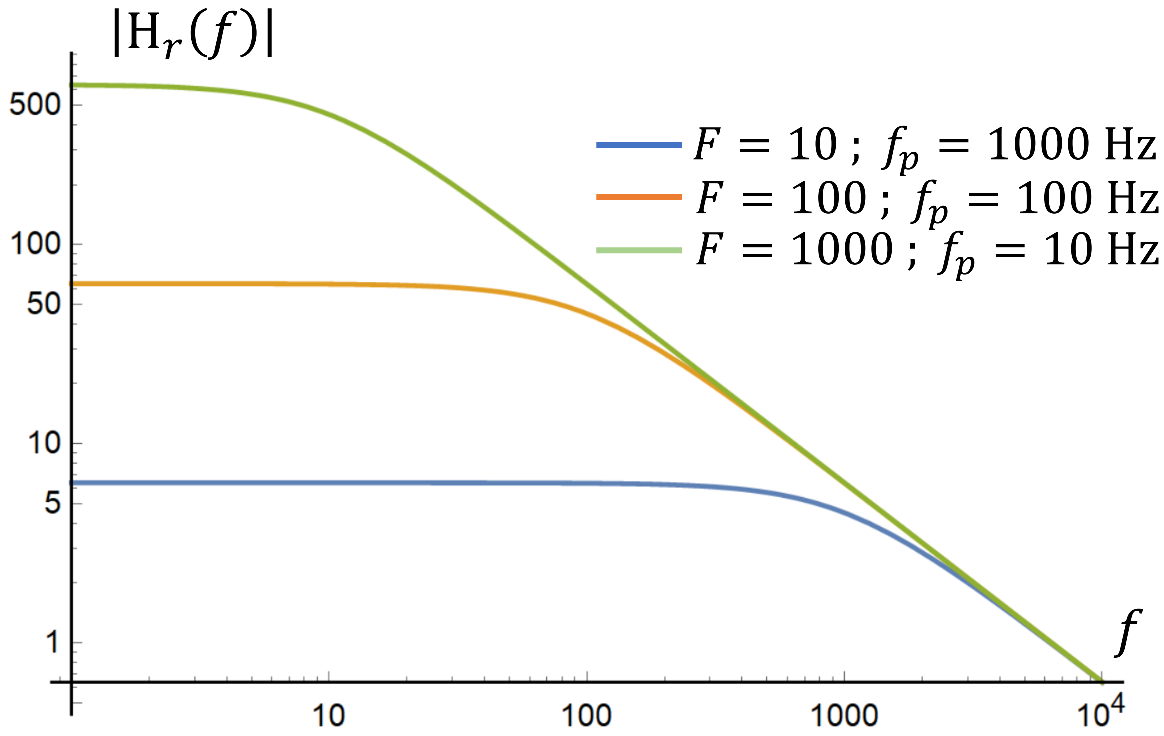
What about the phase?

$$\phi_r = \mathrm{Arg}\big(R(\phi)\big) = \pi + \tan^{-1}\left(\frac{2F}{\pi}\phi\right)$$

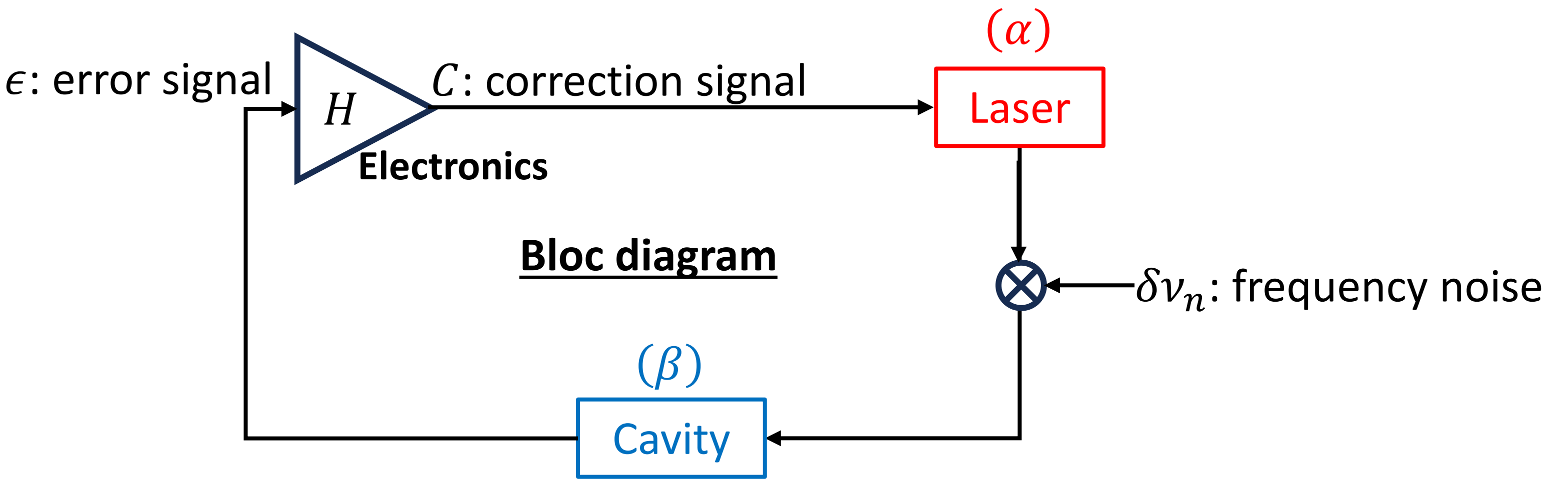
$$H_r(f) = \frac{\phi_r}{\phi}(f) \simeq \frac{2F/\pi}{1+i\frac{f}{f_p}}\ ;\ f_p = \frac{\delta\nu_{FWHM}}{2}\text{: cavity pole}$$

1st order filter





Cavity as a frequency reference : servo loop



Equations:

$C = H \times \epsilon$

$\delta\nu = \alpha \times C + \delta\nu_n$

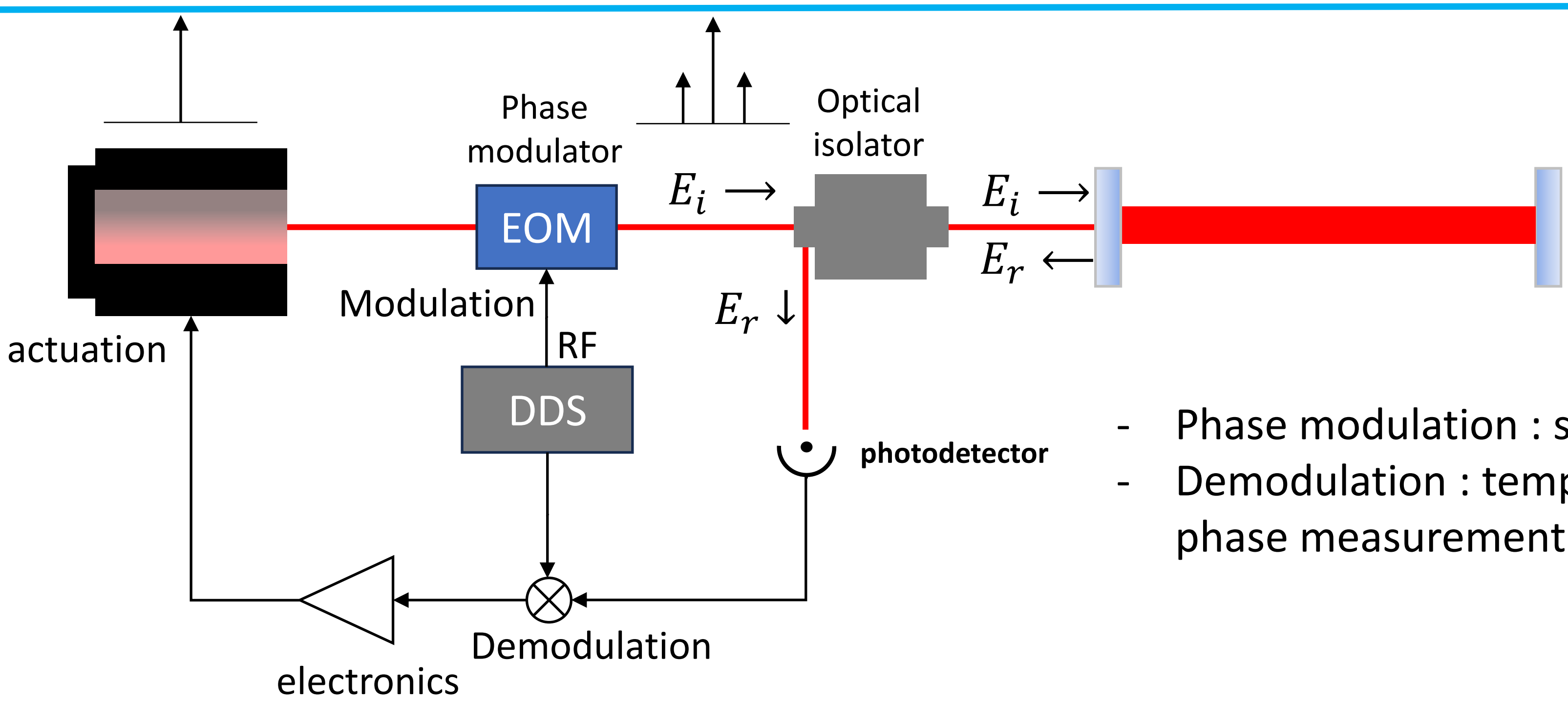
$\epsilon = \beta \times \delta\nu = \beta \times \delta\nu_n + \alpha\beta H \times \epsilon$

$$\Rightarrow \quad \epsilon = \frac{\beta \times \delta\nu_n}{1 - \alpha\beta H}$$

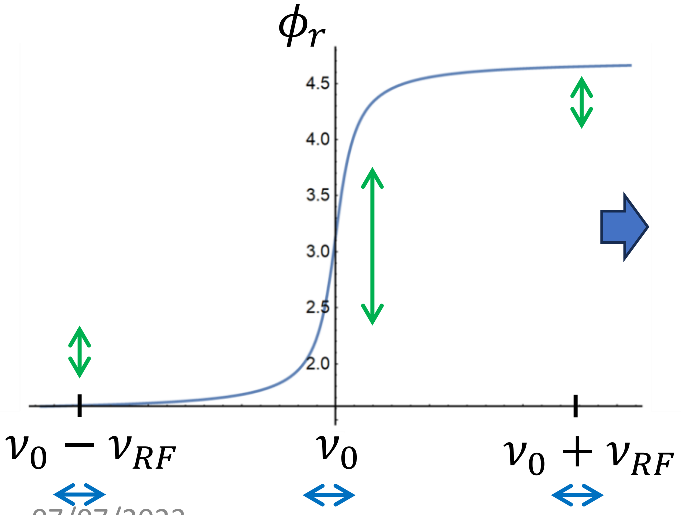
$$\alpha\beta H \gg 1 \quad \Rightarrow \quad \epsilon \simeq \frac{\delta\nu_n}{\alpha H} \longrightarrow 0$$

Problem : How to generate the error signal?

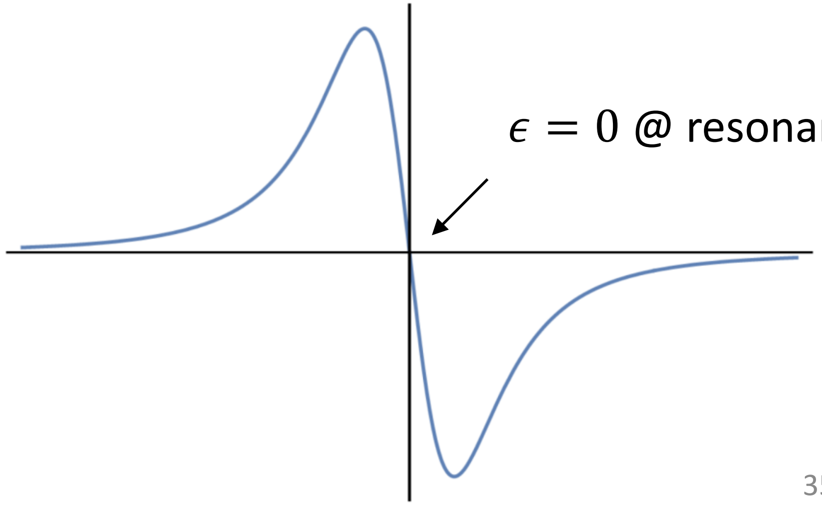
Pound Drever Hall (PDH) tecnique



- Phase modulation : sidebands generation
- Demodulation : temporal interference for phase measurement



Sidebands : phase reference



Outline

Some math : The noise

Detection principle

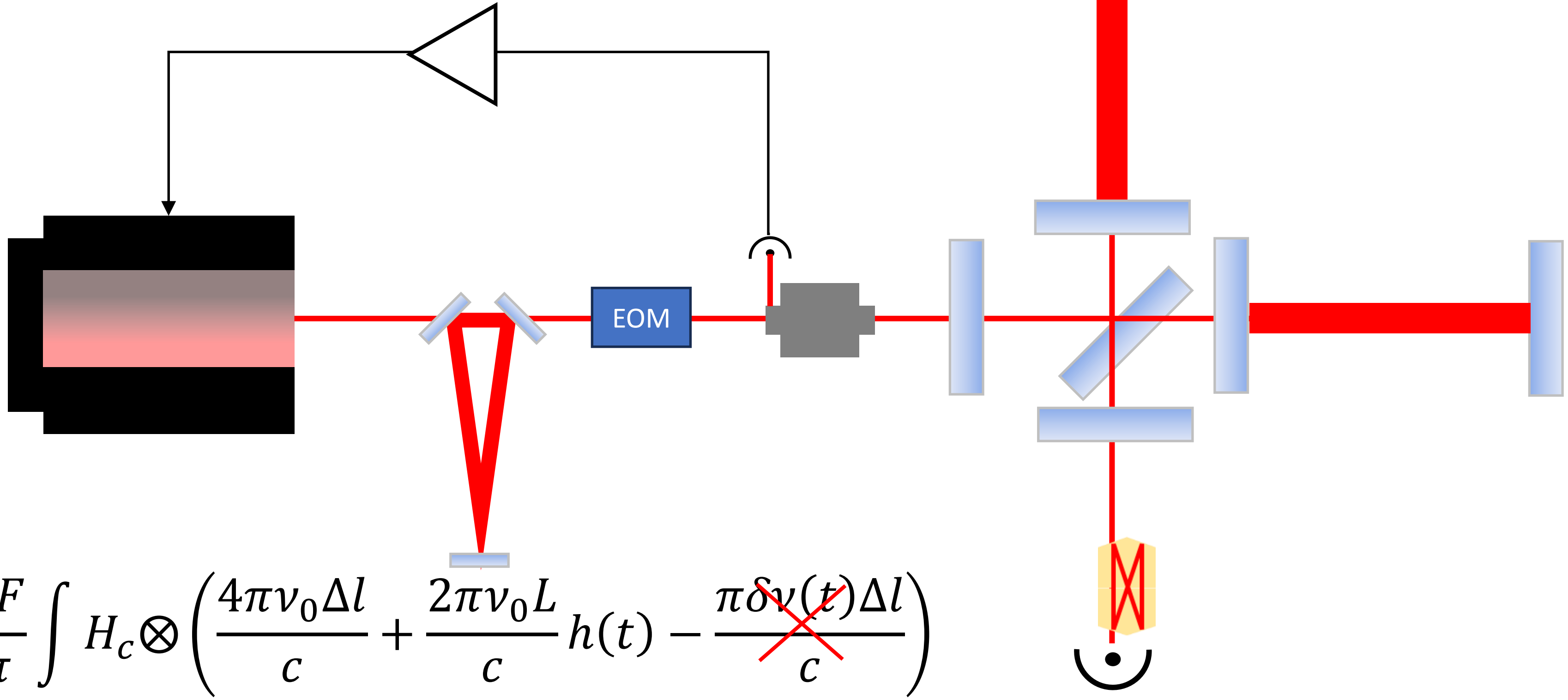
Interferometer output and tuning :
- **The Michelson interferometer**
- **Sensitivity enhancement : The Fabry Perot cavity**
- **Shot noise limited detector**

Noises contribution:
- **Harmonic oscillator model**
- **Seismic noise**
- **Thermal noise**
- **Quantum noise**

Conclusion

Shot noise limited detector

$$\phi = \frac{4\pi\nu_0 L}{c} + \propto \cancel{\Delta l h(t)} + \propto \left(-\cancel{\varphi_L(t)} + \frac{4\pi\delta\nu(t)L}{c}\right)$$



$$\Delta\phi = \frac{2F}{\pi}\int H_c \otimes \left(\frac{4\pi\nu_0\Delta l}{c} + \frac{2\pi\nu_0 L}{c}h(t) - \cancel{\frac{\pi\delta\nu(t)\Delta l}{c}}\right)$$

$P_0 = 25\ \mathrm{W}\,;\ G = 50\,;\ P_{PD} = 10\ \mathrm{mW}\,;\ F = 400\,;\ h = 10^{-23}\mathrm{Hz}^{-1/2}@100\mathrm{Hz}\,;\ \delta\nu < \mu\mathrm{HzHz}^{-1/2}$

Cavity resonance condition

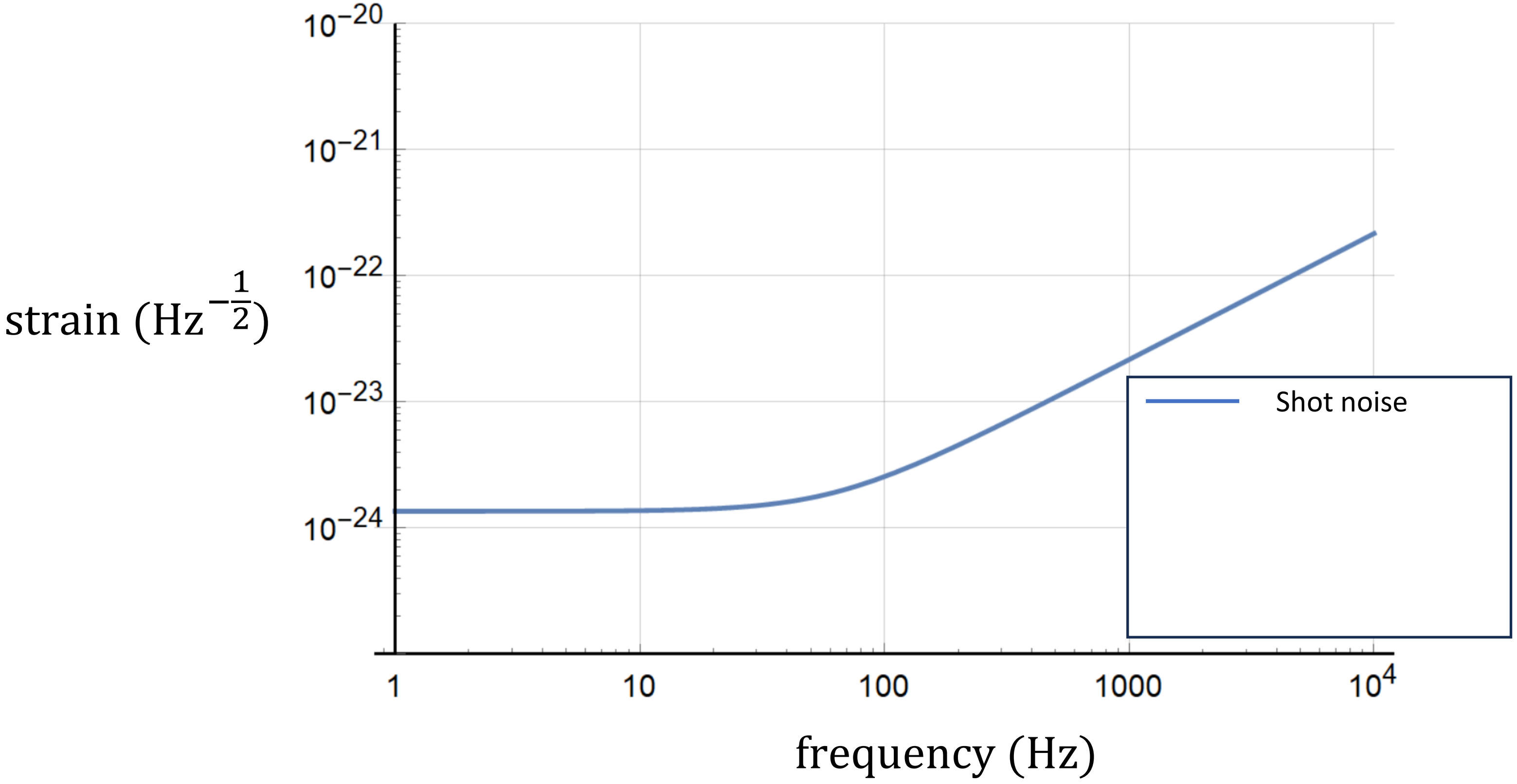
$$\frac{4\pi\nu_0 L}{c} = \pi + 2l\pi$$

$$\frac{\delta L}{L} \simeq \frac{\delta\nu}{\nu_0} \simeq 3\times 10^{-21}\mathrm{Hz}^{-1/2}$$

Shot noise limited detector : sensitivity curve

$$\rho^2_{\text{shot}}(f) = \frac{64F^2GP_0}{h_P\nu_0}\frac{L^2}{\lambda^2}\left|\frac{1}{1+i\frac{f}{f_p}}\right|^2 S_h(f)$$

Sensitivity defined by $\rho^2_{\text{shot}}(f) = 1$ $\Rightarrow$ $S_{h,\text{shot}}(f) = \dfrac{h_P\nu_0}{64F^2GP_0\frac{L^2}{\lambda^2}}\left(1+\dfrac{f^2}{f_p^2}\right)$



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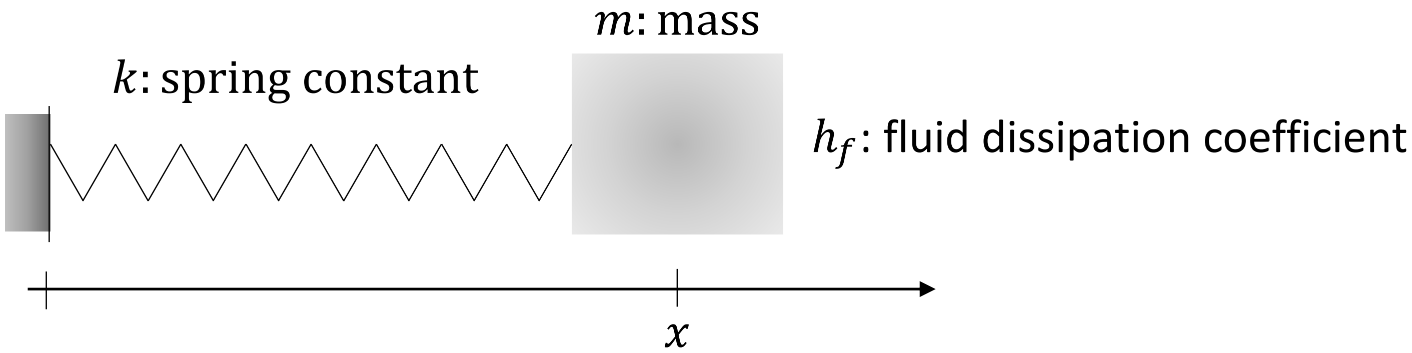
Harmonic oscillator



Newton 2nd law: $$m\frac{d^2x}{dt^2} = F_0 - k(x - x_0) - h_f\frac{dx}{dt}$$

$$\chi(f) = \frac{x(f)}{F_0(f)/m + \omega_0^2 x_0(f)} = \frac{1/m}{\omega_0^2 - \omega^2 + i\frac{\omega\omega_0}{Q}}$$: mechanical susceptibility

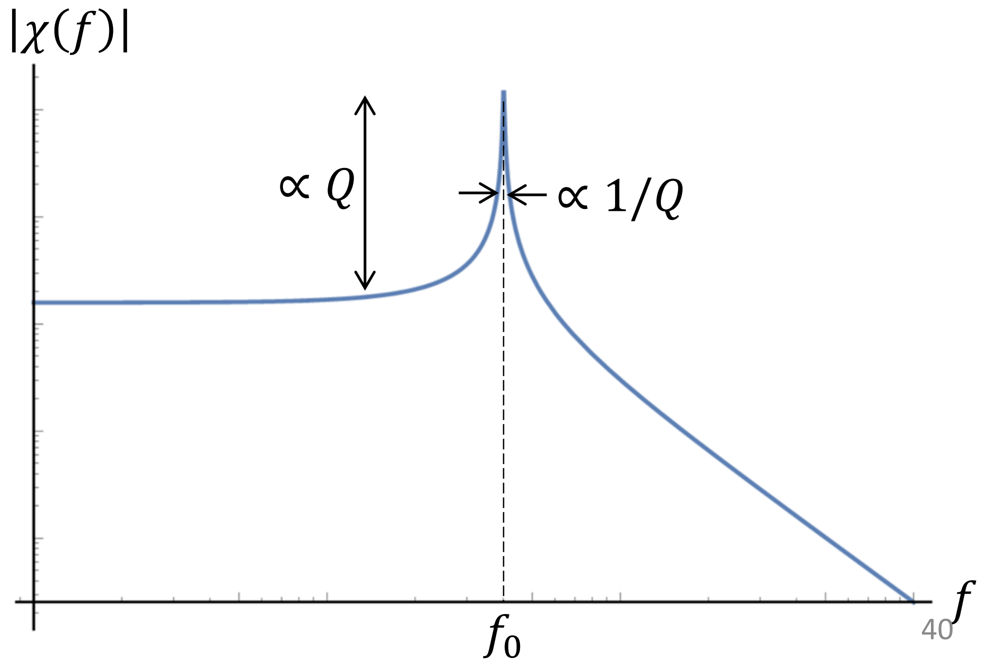
$\omega = 2\pi f$: angular frequency

$\omega_0 = 2\pi f_0 = \sqrt{\frac{k}{m}}$, f_0: resonance frequency

$Q = \omega_0 \frac{m}{h_f}$: mechanical quality factor

$p = m\frac{dx}{dt}$: momentum

$\mathcal{E} = \frac{p^2}{2m} + m\frac{\omega_0^2 x^2}{2}$: Energy



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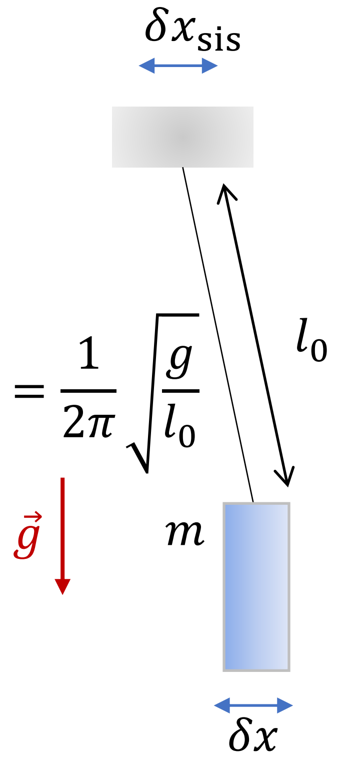
Seismic noise

Free falling mirrors ➡ Isolated from the external environment : seismic noise

Seismic noise : white acceleration noise for $f \geq 1$ Hz $\quad \delta a(f) \simeq 4 \times 10^{-6}\ \mathrm{ms^{-2}/\sqrt{Hz}}$

$$\Rightarrow \delta x_{\mathrm{sis}}(f) \simeq 10^{-7} \left(\frac{1\mathrm{Hz}}{f}\right)^2 \mathrm{m/\sqrt{Hz}}$$



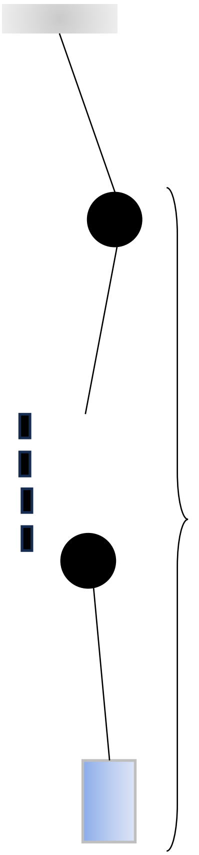
$$f_0 = \frac{1}{2\pi}\sqrt{\frac{g}{l_0}}$$

$l_0 = 70$ cm ; $f_0 = 0.6$ Hz $\qquad \delta x(f) = \chi(f) \times \omega_0^2\, \delta x_{\mathrm{sis}}(f)$

$f \gg f_0 \qquad \delta x(f) \simeq \frac{f_0^2}{f^2} \delta x_{\mathrm{sis}}(f) \simeq 10^{-7}(1\ \mathrm{Hz}) \times f_0^2 \times \left(\frac{1}{f}\right)^{2+2} \mathrm{m/\sqrt{Hz}}$

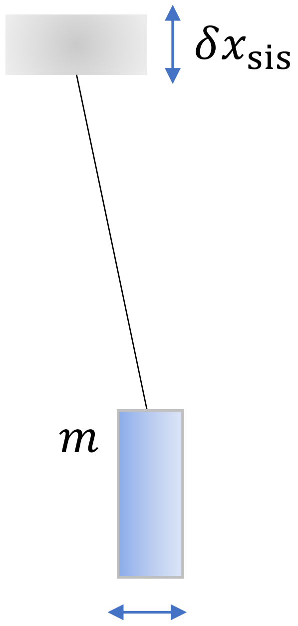
for 2 pendulums $\qquad \delta x(f) \simeq 10^{-7}(1\ \mathrm{Hz}) \times f_0^4 \times \left(\frac{1}{f}\right)^{2+4} \mathrm{m/\sqrt{Hz}}$

Seismic noise : Superattenuators



n pendulums

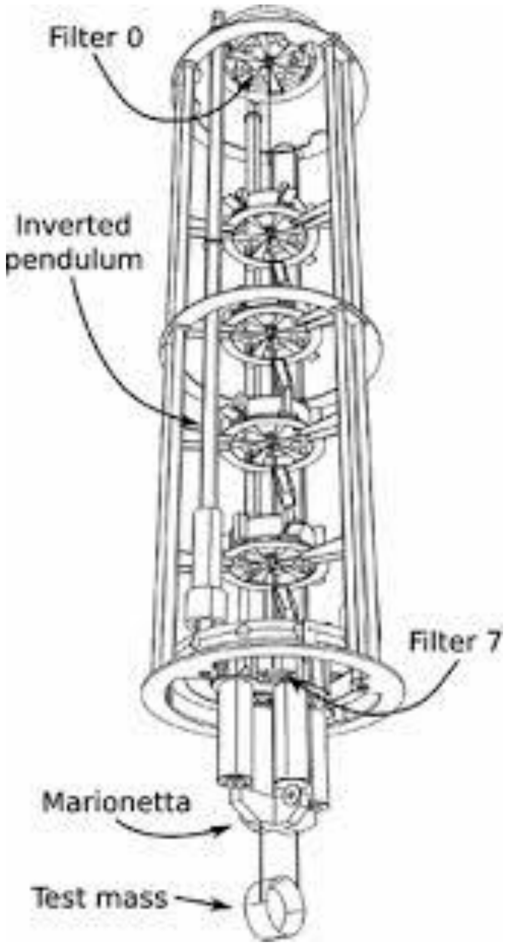
Vertical to horizontal coupling



$$\delta x = \alpha_{\rm coup} \times \delta x_{\rm sis}$$

$$\alpha_{\rm coup} \simeq 1\%$$

Need of vertical attenuator

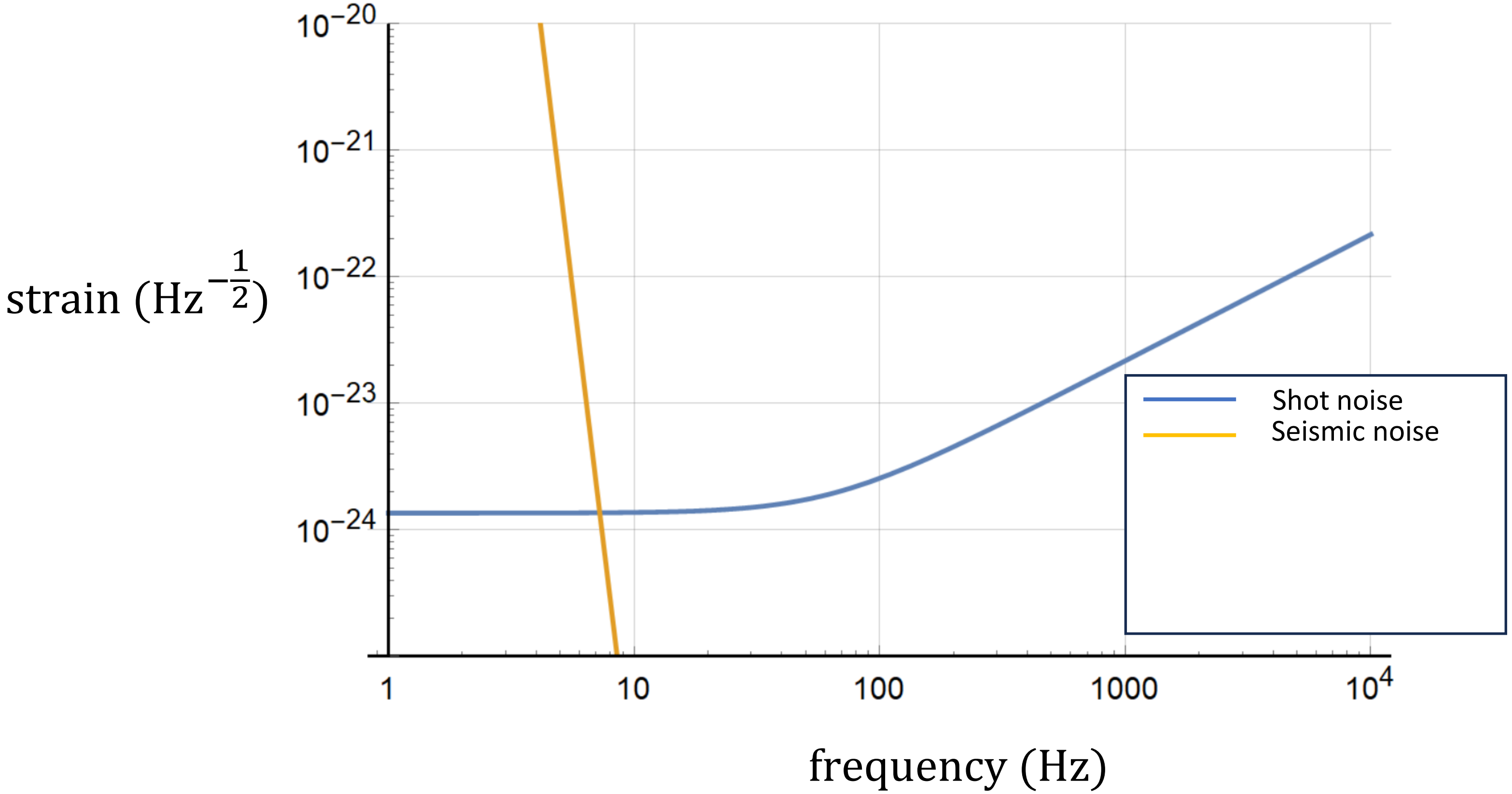


$$\delta x(f) \simeq 10^{-7}(1\ \mathrm{Hz}) \times f_0^{2n} \times \left(\frac{1}{f}\right)^{2n+2}\ \mathrm{m}/\sqrt{\mathrm{Hz}}$$

$$\delta x(f) \simeq \sqrt{\left[\delta x_{\rm sis,h}(f) \times \left(\frac{f_0}{f}\right)^{14}\right]^2 + \left[\delta x_{{\rm sis},v}(f) \times \left(\frac{f_0}{f}\right)^{10} \times \alpha_{\rm coup}^5\right]^2}$$

Seismic noise : sensitivity curve

$$S_{sism}(f) \simeq \frac{1}{L^2}\left[\delta x_{\mathrm{sis,h}}(f) \times \left(\frac{f_0}{f}\right)^{14}\right]^2 + \left[\delta x_{\mathrm{sis},v}(f) \times \left(\frac{f_0}{f}\right)^{10} \times \alpha_{\mathrm{dis}}^5\right]^2$$



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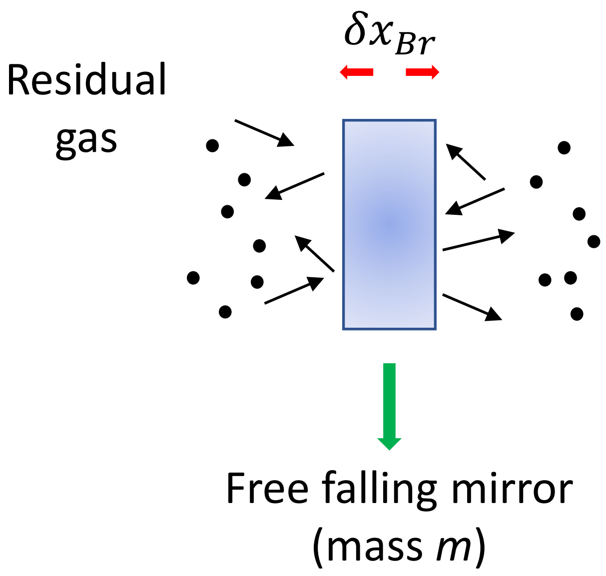
Thermal noise



Brownian Motion

Langevin theory 1908

$$m\frac{d^2x}{dt^2} = m\frac{dv}{dt} = -m\gamma v + F_L(t)$$

➡ Stochastic differential equation

$F_L(t)$: Stochastic Langevin force, results from the momentum transfer of the surrounding particles

Centered process : $\langle F_L(t)\rangle = 0$

Markovian process : $\Gamma_{F_L}(\tau) = \sigma_{F_L}^2\delta(\tau)$

γ and F_L have the same physical origin, but what is the relationship between F_L and γ?

Langevin equation resolution

$$\boldsymbol{\frac{dv}{dt} + \gamma v = F_L(t)}$$

Homogeneous equation

$$\frac{dv}{dt} + \gamma v = 0 \quad \Rightarrow \quad v(t) = A \times e^{-t/\tau}$$

General solution

$$v(t) = A(t) \times e^{-t/\tau} \quad \Rightarrow \quad \frac{dA}{dt} = \frac{e^{\gamma t} \times F_L(t)}{m}$$

$$\begin{cases} A(t) = \dfrac{1}{m}\displaystyle\int_0^t e^{\gamma t'}\, F_L(t')dt' + A_0 \\[2ex] v(t) = v_0 e^{-\gamma t} + \dfrac{1}{m}\displaystyle\int_0^t e^{-\gamma(t-t')}\, F_L(t')dt' \\[2ex] F_L(t=0) = 0\,; v(t) = v_0 \end{cases}$$

Langevin force

Mean value: $\langle v\rangle(t) = v_0 e^{-\gamma t} + \frac{1}{m}\int_0^t e^{-\gamma(t-t')}\langle F_L(t')\rangle dt' = v_0 e^{-\gamma t}$

Variance: $\sigma_v^2(t) = \left\langle \left(\frac{1}{m}\int_0^t e^{-\gamma(t-t')} F_L(t')dt'\right)^2 \right\rangle = \frac{1}{m^2}\int_0^t dt' \int_0^{t'} dt'' \langle F_L(t')F_L(t'')\rangle e^{-\gamma(t-t')} e^{-\gamma(t-t'')}$

$$\Rightarrow \qquad \sigma_v^2(t) = \left\langle \left(\frac{1}{m}\int_0^t e^{-\gamma(t-t')} F_L(t')dt'\right)^2 \right\rangle = \frac{\sigma_{F_L}^2}{4\gamma m^2}\left(1 - e^{-2\gamma t}\right)$$

Mirror in thermodynamic equilibrium with a bath at temperature T

$$\langle U_k\rangle(t) = \frac{1}{2}m\langle v^2\rangle(t) \xrightarrow[t \longrightarrow \infty]{} \langle U_k\rangle(t) = \frac{1}{2}m\sigma_v^2(t \longrightarrow \infty) = \frac{k_B T}{2}$$

$$\sigma_{F_L}^2 = 4m\gamma k_B T \qquad S_{F_L}(f) = 4m\gamma k_B T$$

k_B: Boltzman constant ; T: temperature

mechanical susceptibility : $\chi(f) = \frac{1/m}{-\omega^2 + i\omega\gamma} = \frac{1}{m}\frac{-\omega - i\gamma}{\omega(\omega^2+\gamma^2)}$ $\Rightarrow$ $S_x(f) = |\chi(f)|^2 S_{F_L}(f) = \frac{4k_B T}{\omega}\left|\mathcal{Im}\left(\chi(f)\right)\right|$

Generalized by Callen & Welton(1952) : Fluctuation-Dissipation Theorem

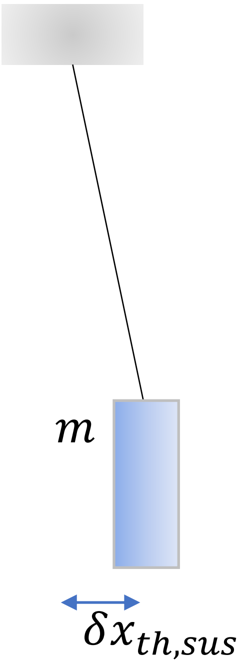
Dissipation $\Longrightarrow$ Fluctuation :
- Any mechanical system
- Any dissipation mechanism

Suspension thermal noise

Internal dissipation : $k \longrightarrow k(1+i\phi_l)$ ϕ_l: loss angle, frequency independent

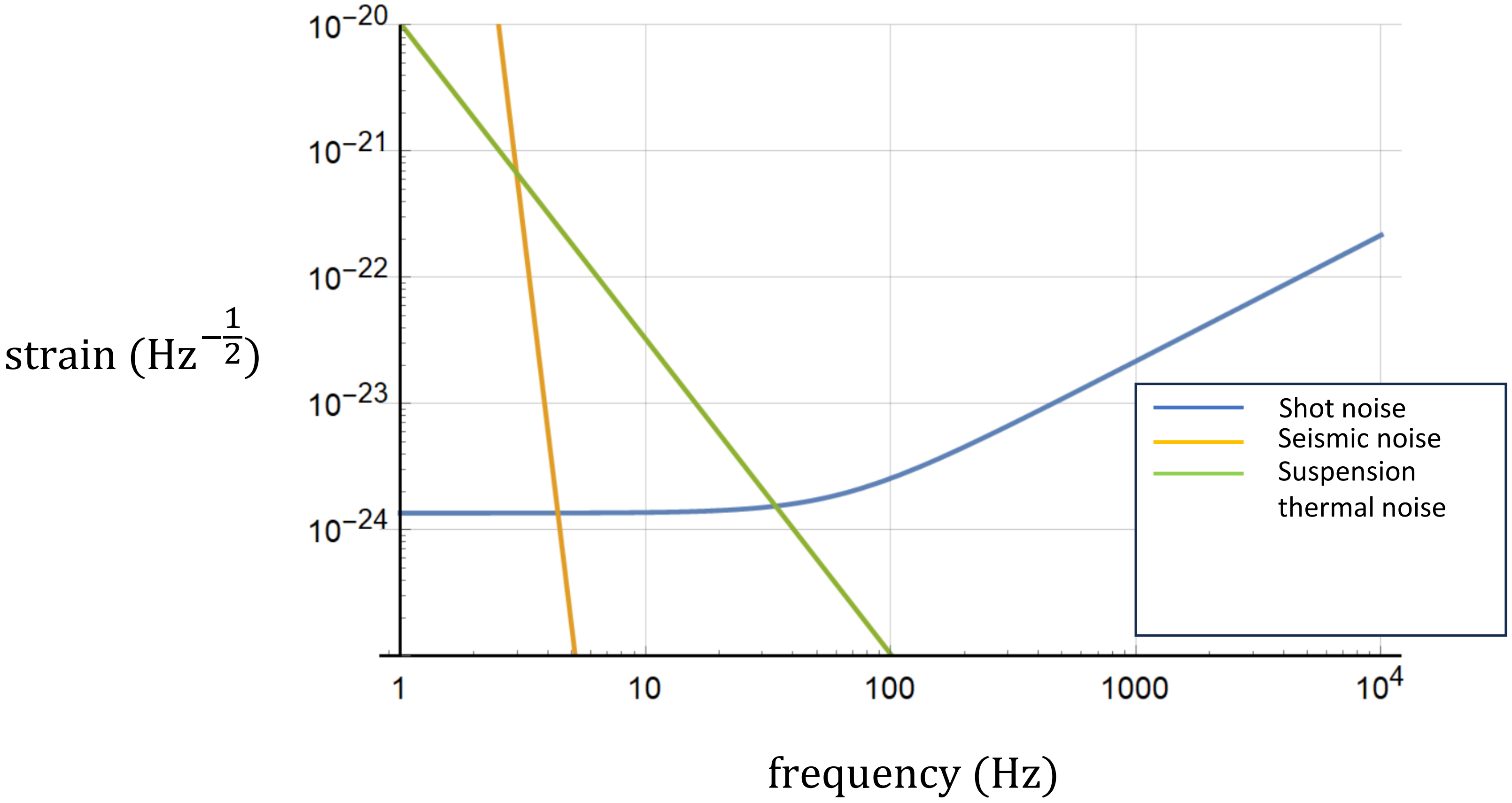
$$\Rightarrow \quad \chi_{sus}(f) = \frac{1/m}{\omega_0^2 - \omega^2 + i\phi_l\omega_0^2} = \frac{1}{m}\frac{\omega_0^2 - \omega^2 - i\phi_l\omega_0^2}{(\omega_0^2-\omega^2)^2 + \phi_l^2\omega_0^4}$$

$$\left|\mathcal{Im}\left(\chi_{sus}(f)\right)\right| = \frac{1}{m}\frac{\phi_l\omega_0^2}{(\omega_0^2-\omega^2)^2+\phi_l^2\omega_0^4} \xrightarrow[\omega \gg \omega_0]{} \left|\mathcal{Im}\left(\chi_{sus}(f)\right)\right| = \frac{1}{m}\frac{\omega_0^2}{\omega^4}\,\phi_{l,sus}$$



Suspension thermal noise : sensitivity curve

$$S_{th,sus}(f) \simeq \frac{1}{L^2}\underbrace{(4}_{\text{mirrors}} \times \underbrace{4)}_{\text{wires}} \frac{4k_B T\, f_0^2}{m(2\pi)^3 f^5}\phi_{sus} \qquad m = 40\ \mathrm{kg}\,;\ \phi_{sus} = 10^{-10}$$



Mirror thermal noise

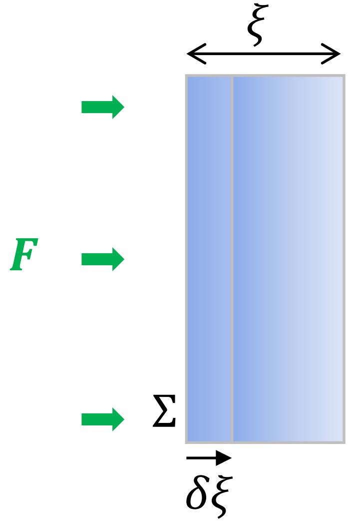
Multiple resonances system $\quad \chi_{mir}(f) = \sum_j \frac{1/m_j}{\omega_j^2 - \omega^2 + i\phi_{l,mir}\omega_j^2} = \sum_j \frac{1}{m_j} \frac{\omega_j^2 - \omega^2 - i\phi_{l,mir}\omega_j^2}{\left(\omega_j^2 - \omega^2\right)^2 + \phi_{l,mir}^2\omega_j^4}$

First resonance 1 kHz $\Rightarrow$ $\left|\mathcal{Im}\left(\chi_{mir}(f)\right)\right| \simeq \phi_{l,mir} \sum_j \frac{1}{m_j\omega_j^2}$ For $\omega \ll \omega_1 < \omega_2 \ldots$

And… $\quad |\chi_{mir}(f)| \simeq \sum_j \frac{1}{m_j\omega_j^2}$ $\Rightarrow$ $|\mathcal{Im}(\chi_{mir})| = \phi_{l,mir}|\chi_{mir}|$

Interaction energy for a low frequency applied force: $\mathcal{E} = F \times x_{mir} = |\chi_{mir}(f)|F^2$ $\Rightarrow$ $|\mathcal{Im}(\chi_{mir})| = \phi_{l,mir}|\chi_{mir}| = \phi_{l,mir}\frac{\mathcal{E}}{F^2}$

We just need to compute the elastic energy when we apply a constant force



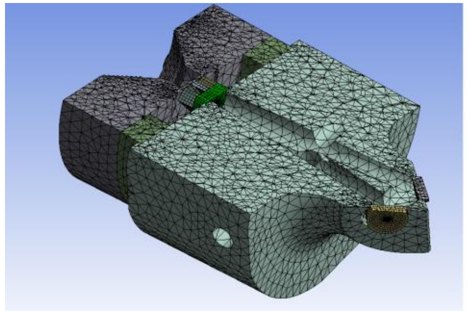
Y: Young modulus

$$\frac{\delta\xi}{\xi} = \frac{F/\Sigma}{Y}$$

Elastic energy

$$\mathcal{E} = \frac{1}{2} F \times \delta\xi$$

$\Rightarrow$ $$\frac{\mathcal{E}}{F^2} = \frac{1}{2}\frac{\xi}{Y \times \Sigma}$$

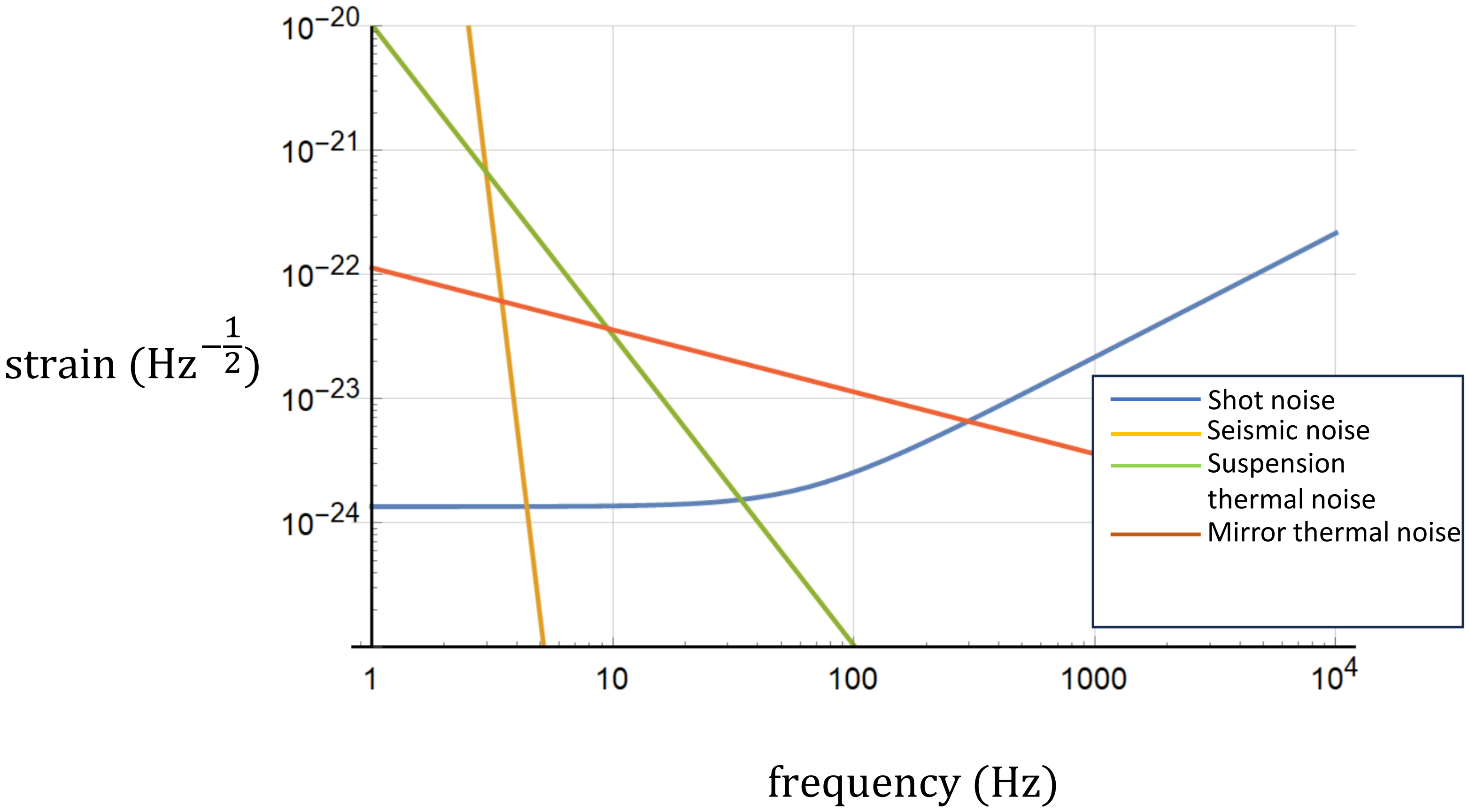


Mirror thermal noise : Sensitivity curve

$$S_{th,sus}(f) \simeq \frac{1}{L^2} \times 4 \times \frac{k_B T}{\pi f} \frac{\xi}{Y \pi r_{mir}^2} \phi_{mir}$$

mirrors

$\xi = 20$ cm ; $r_{mir} = 20$ cm ; $Y = 72$ GPa ; $\phi_{mir} = 10^{-6}$



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Quantum harmonic oscillator

$$\mathcal{E} = \frac{p^2}{2m} + m\frac{\omega_0^2 x^2}{2}$$

x and p : conjuagte canonical variables

quantification $\Rightarrow$

$$\begin{cases} [\hat{x}, \hat{p}] = i\hbar = i\,\dfrac{h_P}{2\,\pi} \\ \hat{H} = \dfrac{\hat{p}^2}{2m} + m\dfrac{\omega_0^2 \hat{x}^2}{2} \end{cases}$$

Annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}}\hat{x} + \frac{i}{\sqrt{\hbar m \omega_0}}\hat{p}\right)$$

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega_0}{\hbar}}\hat{x} - \frac{i}{\sqrt{\hbar m \omega_0}}\hat{p}\right)$$

Hamiltonian

$$\hat{H} = \hbar\omega_0\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

Non-zero energy
fundamental state

Classical electromagnetic radiation

Single mode real electric field ; plane wave ; in vacuum

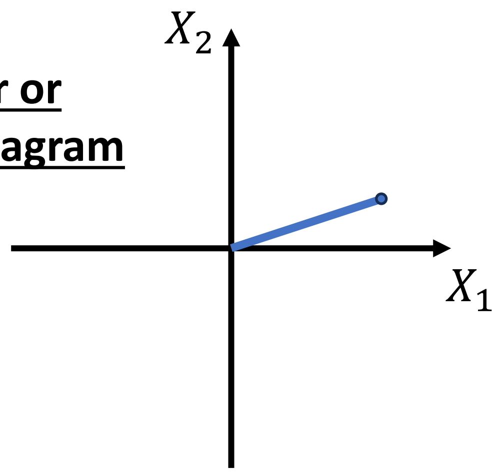
$$\mathfrak{E}(t) = \mathfrak{E}_0\left(\alpha(t)e^{-i2\pi\nu t} + \alpha^*(t)e^{+i2\pi\nu t}\right) \qquad \alpha(t) = \alpha_0(t)e^{-i\varphi(t)}$$

$$\begin{cases} X_1(t) = \alpha(t) + \alpha^*(t) \\ X_2(t) = -i\left(\alpha(t) - \alpha^*(t)\right) \end{cases} \Rightarrow \quad \mathfrak{E}(t) = \mathfrak{E}_0\left(X_1(t)\cos(2\pi\nu t) + X_2(t)\sin(2\pi\nu t)\right)$$

Single mode energy

$$\mathcal{E} = \frac{h_P\nu}{4}(X_1^2 + X_2^2)$$

Phasor or Fresnel diagram



X_1, X_2: field quadratures

Electromagnetic radiation quantification

Electromagnetic radiation quantification

$$\mathcal{E} = \frac{h_P \nu}{4}(X_1^2 + X_2^2)$$

X_1 and X_2 : conjuagte canonical variables

quantification $\Rightarrow$

$$\begin{cases} [\hat{X}_1, \hat{X}_2] = i\hbar \\ \hat{H} = \frac{h_P \nu}{4}(\hat{X}_1^2 + \hat{X}_2^2) \end{cases}$$

Using annihilation and creation operators

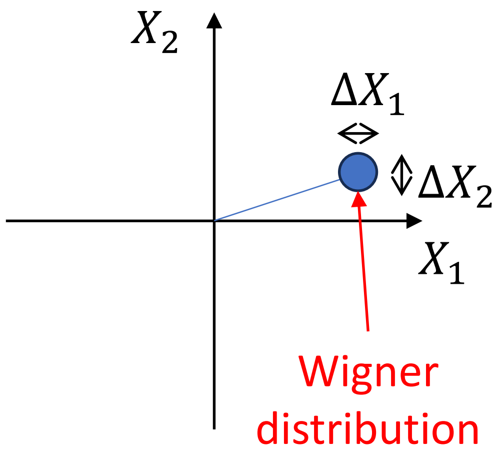
$$\hat{H} = \left(\hat{a}^+\hat{a} + \frac{1}{2}\right) = h_P \nu \left(\hat{N}_j + \frac{1}{2}\right)$$

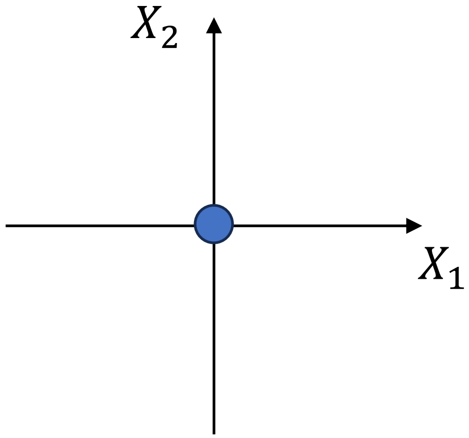
Zero photons energy : vacuum is not empty

Photon energy

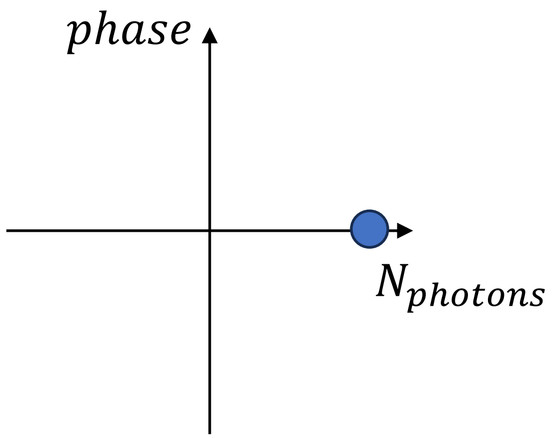
Number of photons

Wigner distribution

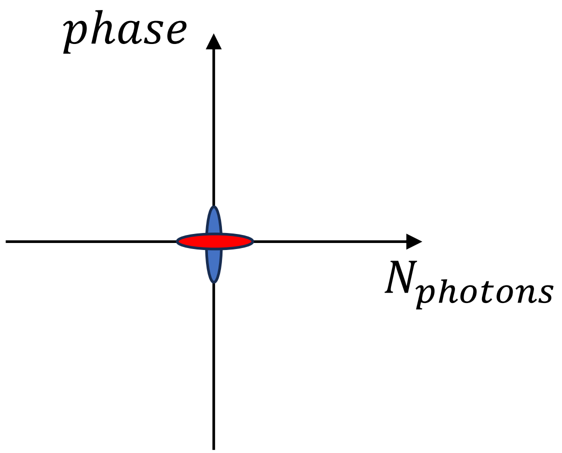




<u>Vacuum state</u>



<u>Coherent state ≈ single mode laser</u>

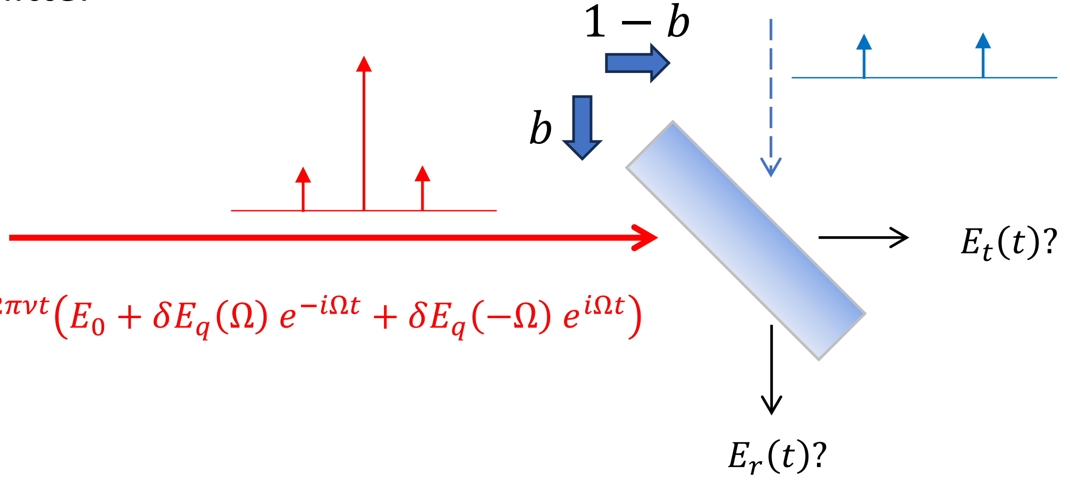


<u>Squeezed vacuum state</u>

Quantum noise as sidebands

The beam splitter

$$E_v(t) = e^{-i2\pi\nu t}\left(\delta E_v(\Omega)\, e^{-i\Omega t} + \delta E_v(-\Omega)\, e^{i\Omega t}\right)$$



$$E(t) = e^{-i2\pi\nu t}\left(E_0 + \delta E_q(\Omega)\, e^{-i\Omega t} + \delta E_q(-\Omega)\, e^{i\Omega t}\right)$$

$$E_r(t) = \sqrt{b}\, E(t) + \sqrt{1-b}\, E_v(t)$$

$$P_r(t) = b\, P_0 + \underbrace{\sqrt{b}E_0\left(\sqrt{b}\,\delta E_q^*(-\Omega) + \sqrt{1-b}\,\delta E_v^*(-\Omega) + \sqrt{b}\,\delta E_q(\Omega) + \sqrt{1-b}\,\delta E_v(\Omega)\right)}_{\delta P(\Omega)} e^{-i\Omega t} + c.c$$

$$S_{\delta P}(\Omega) = \langle \delta P(\Omega)\delta P(\Omega)^*\rangle = bP_0\Big(b\underbrace{\langle \delta E_q^*(-\Omega)\delta E_q(-\Omega)\rangle}_{\frac{h_P\nu}{2}} + b\underbrace{\langle \delta E_q(\Omega)\delta E_q^*(\Omega)\rangle}_{\frac{h_P\nu}{2}} + (1-b)\underbrace{\langle \delta E_v^*(-\Omega)\delta E_v(-\Omega)\rangle}_{\frac{h_P\nu}{2}} + (1-b)\underbrace{\langle \delta E_v(\Omega)\delta E_v^*(\Omega)\rangle}_{\frac{h_P\nu}{2}}\Big)$$

$$S_{\delta P}(\Omega) = h_P\nu b P_0 = h_P\nu P_r \;\Rightarrow\; \text{Shot noise}$$

Radiation pressure noise

Radiation pressure on a mirror: transfer of photon momentum

$$F_{rad,p} = \frac{dp_{tot}}{dt} = \frac{dN_{photons}}{dt} \times \underset{\substack{\uparrow \\ \text{Total reflection}}}{2} \times p_{photon} = \frac{2P_{inc}}{c}$$

$$\Rightarrow \quad S_{\delta F_{rad,p}} = \frac{4}{c^2} \times S_{\delta P}$$

$$S_{\delta x_{rad,p}} = \underset{\substack{\uparrow \\ \text{Mirrors motion are anti-correlated}}}{4} \times S_{\delta F_{rad,p}} \times |\chi(f)|^2 = \frac{16}{c^2} \times S_{\delta P} \times |\chi(f)|^2$$

$$|\chi(f)| \simeq \frac{1}{m\,\omega^2}$$

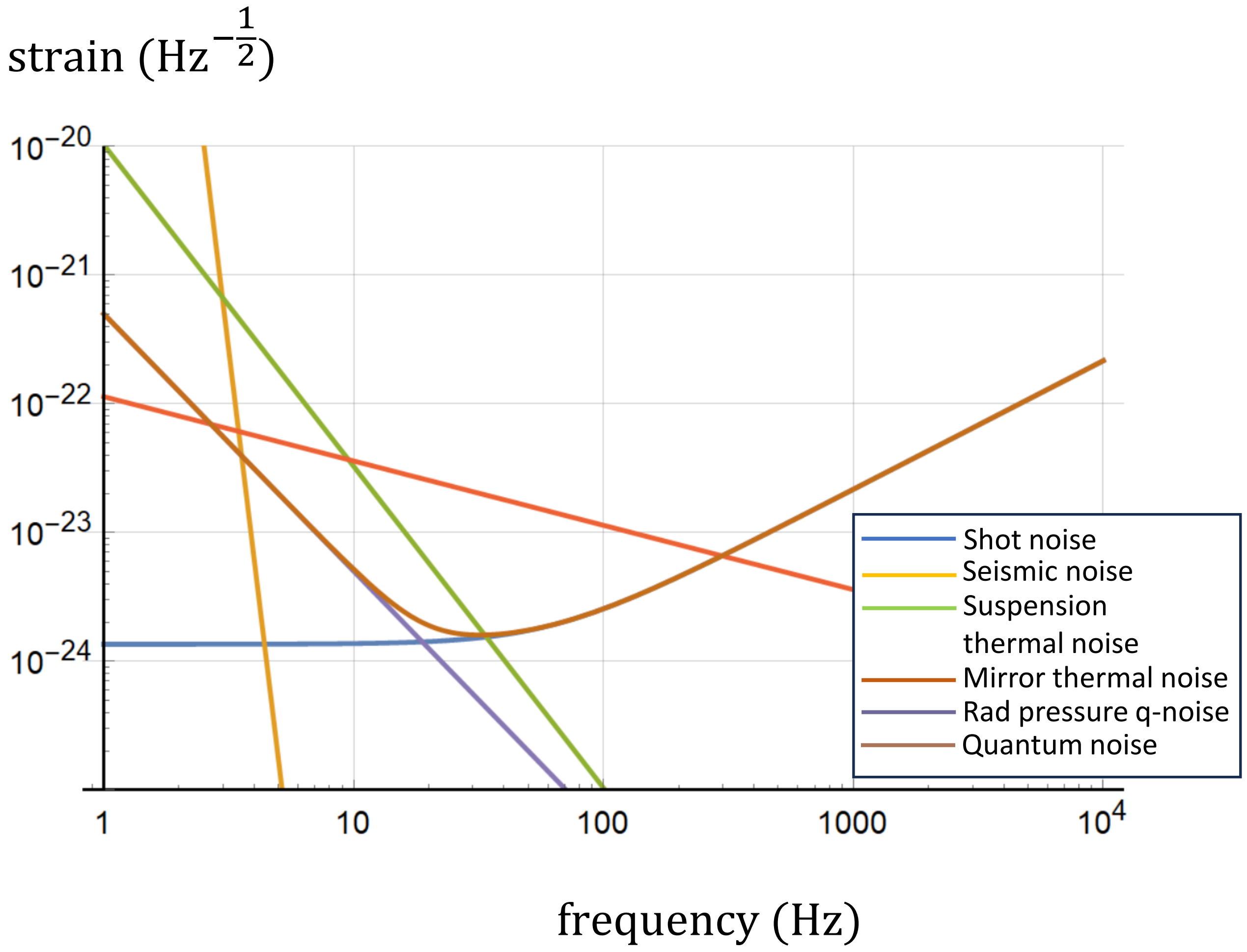
@ frequencies higher than the pendulum resonance

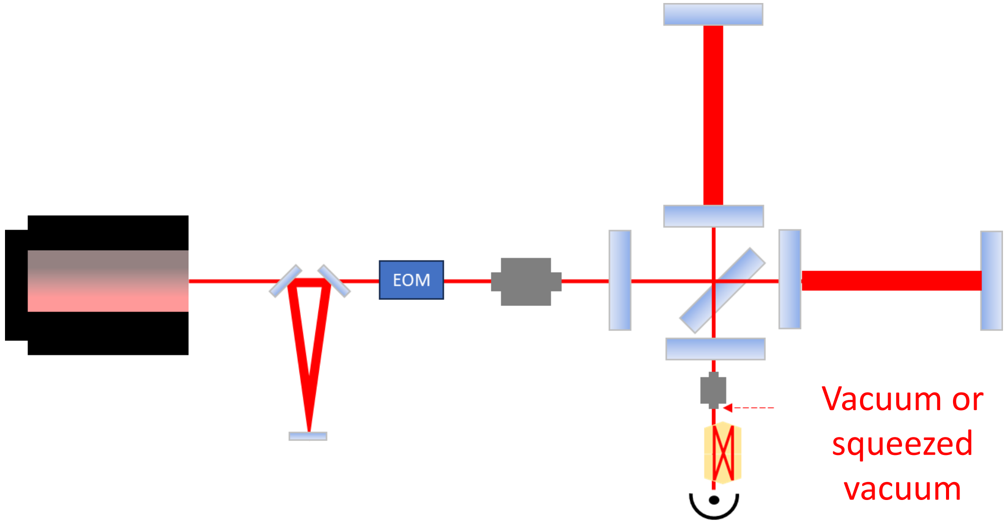
$$S_{\delta P}(f) \simeq h\nu \frac{2F}{\pi} \frac{P_0}{2}$$

@ frequencies lower than the cavity pole

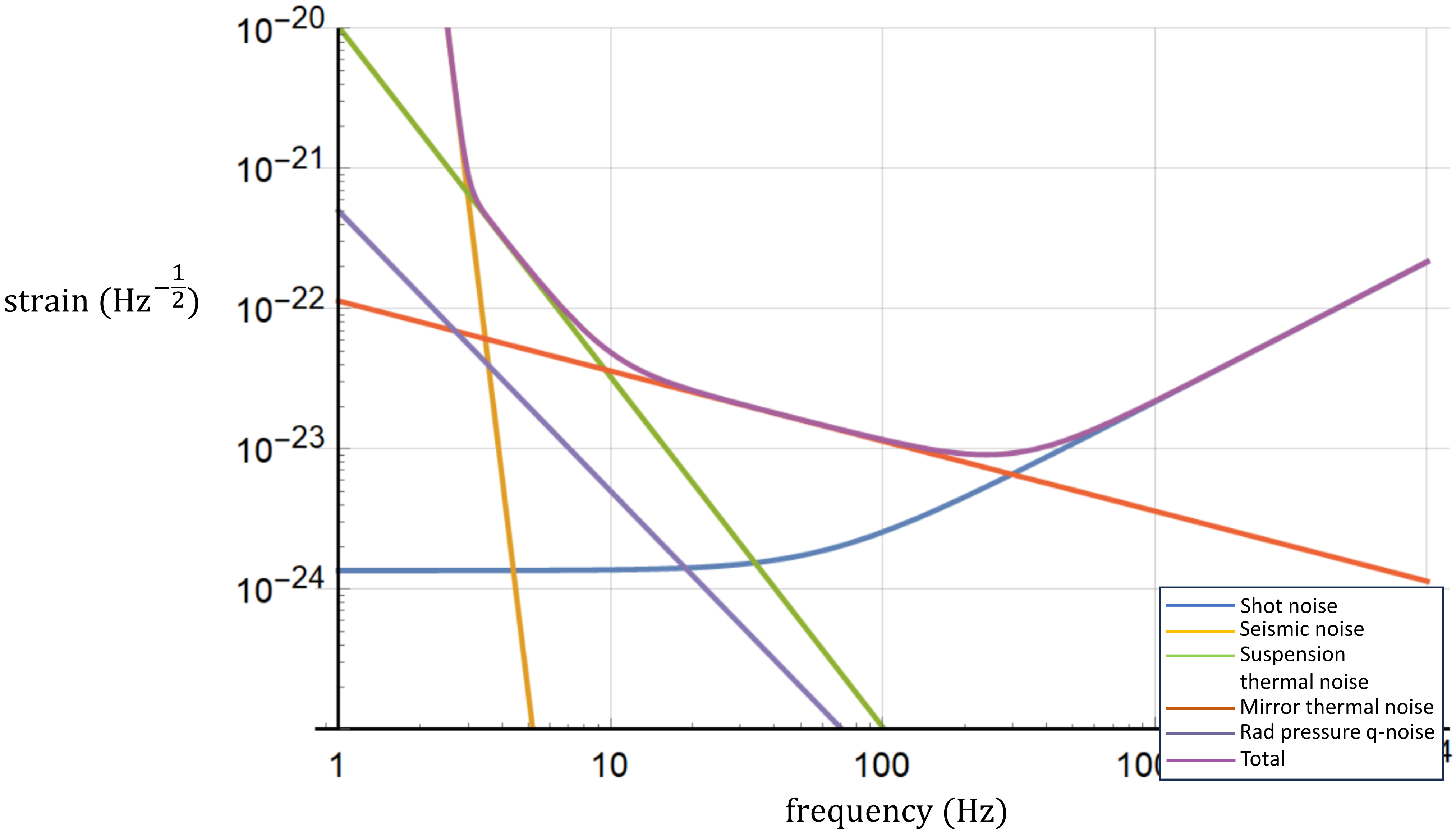
Quantum noise : sensitivity curve

$$S_{rad,p}(f) \simeq \frac{1}{L^2} \times 2 \times \frac{h_P \nu_0}{m^2 c^2 \pi^5 f^4} GF \frac{P_0}{2}$$





Sensitivity curve



Outline

Some math : The noise

Detection principle

Interferometer output and tuning :

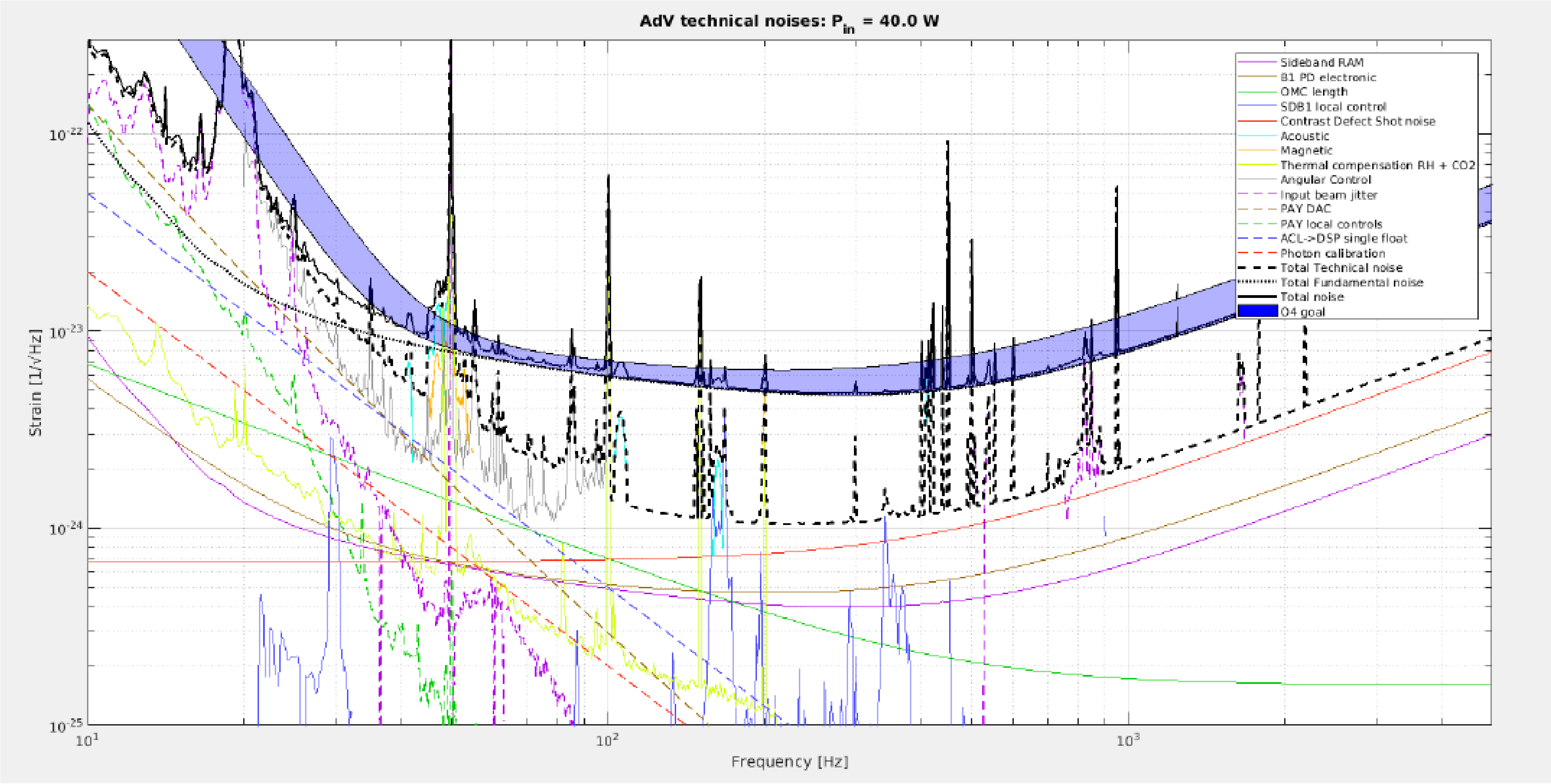
- **The Michelson interferometer**
- **Sensitivity enhancement : The Fabry Perot cavity**
- **Shot noise limited detector**

Noise contribution:

- **Harmonic oscillator model**
- **Seismic noise**
- **Thermal noise**
- **Quantum noise**

Conclusion

A lot of other contributions



…and a lot of other aspects

Thermal compensation system
Controls
High power laser
Electronics
Vacuum
Parametric instabilities
Newtonian noise
Squeezing, dependent and independent
Calibration
Coatings
Simulations
….and R&D