Resummation in JIMWLK Hamiltonian

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High Energy Scattering

Target (ρ^t) **Projectile (ρ^p)**

$$\langle \mathbf{T} | \quad \rightarrow \qquad\qquad \leftarrow \quad | \mathbf{P} \rangle$$

S-matrix:

$$\mathbf{S(Y)} \;=\; \langle \mathbf{T} \, \langle \, \mathbf{P} | \;\; \hat{\mathbf{S}}(\rho^{\mathrm{t}}, \; \rho^{\mathrm{p}}) \;\; | \mathbf{P} \rangle \, \mathbf{T} \rangle$$

or, more generally, any observable $\hat{\mathcal{O}}(\rho^{\mathrm{t}}, \; \rho^{\mathrm{p}})$

$$\langle \hat{\mathcal{O}} \rangle_{\mathbf{Y}} \;=\; \langle \mathbf{T} \, \langle \, \mathbf{P} | \;\; \hat{\mathcal{O}}(\rho^{\mathrm{t}}, \; \rho^{\mathrm{p}}) \;\; | \mathbf{P} \rangle \, \mathbf{T} \rangle$$

How these averages change with increase in energy of the process?

$$\partial_{\mathbf{Y}} \langle \hat{\mathcal{O}} \rangle_{\mathbf{Y}} \;=\; -\mathcal{H} \, \langle \hat{\mathcal{O}} \rangle_{\mathbf{Y}} \qquad\qquad \mathcal{H} \;\; \rightarrow \;\; \text{the HE effective Hamiltonian}$$

$\mathcal{H}$ **defines the high energy limit of QCD and is universal**

Projectile averaged S-matrix:

$$\mathbf{\Sigma(Y)} \equiv \langle \mathbf{P}| \, \hat{\mathbf{S}}(\rho^{\mathrm{t}}, \, \rho^{\mathrm{p}}) \, |\mathbf{P}\rangle$$

evolves with rapidity as

$$\mathbf{\Sigma(Y + \delta Y)} = \mathbf{e}^{-\delta \mathbf{Y} \, \mathcal{H}} \, \mathbf{\Sigma(Y)}$$

$$\mathcal{H} = \mathcal{H}^{\mathbf{LO}}(\alpha_{\mathrm{s}}) + \mathcal{H}^{\mathbf{NLO}}(\alpha_{\mathrm{s}}^{2}) + \ldots; \qquad \mathcal{H} = \mathcal{H}[\rho^{\mathrm{t}}, \, \delta/\delta\rho^{\mathrm{t}}]$$

JIMWLK Hamiltonian is a limit of $\mathcal{H}$ for dilute partonic system ($\rho^{\mathrm{p}} \to \mathbf{0}$) which scatters on a dense target. It accounts for linear gluon emission + multiple rescatterings. When applied to a dipole, in the large N_c limit, it leads to BK equation.

$\mathcal{H}^{LO}$ **(1997-2002),** $\mathcal{H}^{NLO}$ **with massless quarks (2007-2016),** $\mathcal{H}^{NLO}(m_q)$ **(2022)**

Motivation and Objectives

Precise saturation physics phenomenology at NLO is badly needed.

The JIMWLK Hamiltonian at NLO is known for some years, but there are problems there.

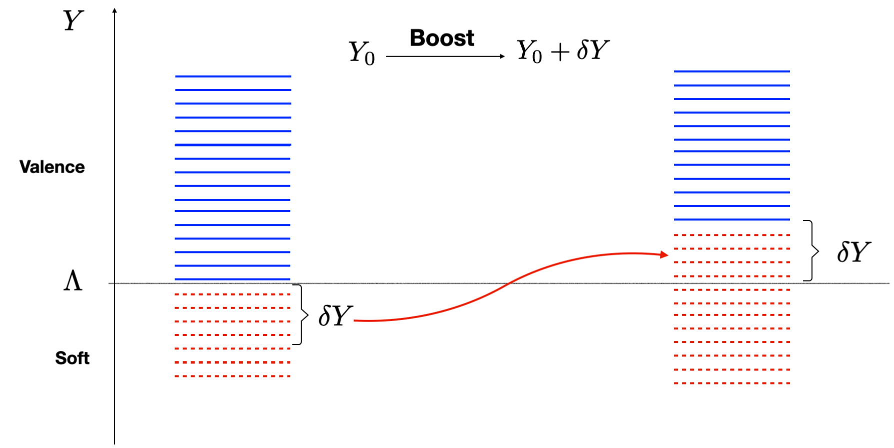
- No known recipe for numerical evaluation
- Large logarithms emerge: $\mathcal{H} \sim \alpha_s(\# + \alpha_s(\# + \mathrm{Log}))$,
  If the Log is large, then $\alpha_s \, \mathbf{Log} \sim \mathbf{1}$ – not a small correction to LO
  There are various types of the large Logs there:
  running coupling effects, (Ioffe) time ordering, DGLAP logs.
  All have to be identified, clearly separated, and independently resummed.

A. Kovner, M. Lublinsky, V. V. Skokov and Z. Zhao,
"DGLAP resummation and the running coupling in NLO JIMWLK," [arXiv:2308.15545 [hep-ph]].

We both resummed the UV divergent Logs into correct form of the running coupling (rcJIMWLK) and derived the RG-improved JIMWLK Hamiltonian. We have not resummed all large Logs, but only the ones related to DGLAP splittings. We also approximately solved the RG in the dilute and dense limits. The result is smearing of the LO kernel.

Light Cone Wave Function in Born-Oppenheimer approximation

$$\mathbf{H}^{\mathrm{LC}}_{\mathrm{QCD}}\,|\Psi\rangle \;=\; \mathbf{E}\,|\Psi\rangle$$



BO: split the modes into hard and soft.
The hard (valence) modes with $k^+ \;>\; \Lambda$
They act as an external background current
$j_a^+ \;=\; \delta(x^-)\,\rho^a$ **for the soft modes.**

$$\mathbf{H}^{\mathrm{LC}}_{\mathrm{QCD}} \;=\; \mathbf{H}[\rho,\,\mathbf{a},\,\mathbf{a}^\dagger] \;=\; \mathbf{H}_{\mathrm{V}}[\rho] \;+\; \mathbf{H}_{\mathrm{free}}[\mathbf{a},\,\mathbf{a}^\dagger] \;+\; \mathbf{H}_{\mathrm{int}}[\rho,\,\mathbf{a},\,\mathbf{a}^\dagger]$$

LCWF with no soft modes

$$\mathbf{H}_{\mathrm{V}}\,|\mathrm{v},\,0_{\mathrm{a}}\rangle \;=\; \mathbf{E}_0\,|\mathrm{v},\,0_{\mathrm{a}}\rangle\,; \qquad \mathbf{a}\,|\mathrm{v},\,0_{\mathrm{a}}\rangle \;=\; 0\,; \qquad \mathbf{E}_0 \;=\; 0$$

LCWF with soft gluon/quark dressing

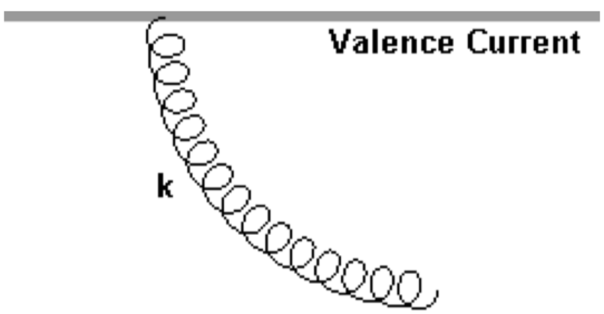
$$|\Psi\rangle \;=\; \mathbf{\Omega}(\rho,\,\mathbf{a},\,\mathbf{a}^\dagger)\,|\mathrm{v},\,0_{\mathrm{a}}\rangle\,; \qquad \mathbf{\Omega}^\dagger\,(\mathbf{H}_{\mathrm{free}}+\mathbf{H}_{\mathrm{int}})\,\mathbf{\Omega} \;=\; \mathbf{H}_{\mathrm{diagonal}}$$

Find Ω in perturbation theory

LCWF at LO

Eikonal coupling between valence and soft gluons due to separation of scales

$$\mathbf{H_{int}} = -\int \frac{dk^+}{2\pi}\frac{d^2k_\perp}{(2\pi)^2}\frac{g\,k_i}{\sqrt{2}\,|k^+|^{3/2}}\left[a_i^{\dagger a}(k^+,\,k_\perp)\,\rho^a(-k_\perp) + a_i^a(k^+,\,-k_\perp)\,\rho^a(k_\perp)\right]$$



A cloud of classical Weizsaker-Williams gluons dressing the valence ones

$$b_i^a(z) = \frac{g}{2\pi}\int d^2x\frac{(z-x)_i}{(z-x)^2}\;\rho^a(x)$$

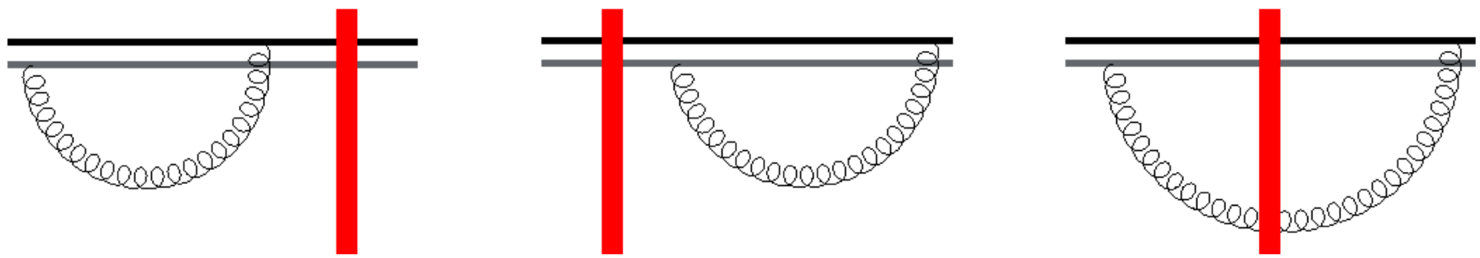
$$\Omega_Y(\rho\to 0) \equiv C_Y = \mathrm{Exp}\left\{i\int d^2z\,b_i^a(z)\int_{e^{Y_0}\Lambda}^{e^Y\Lambda}\frac{dk^+}{\pi^{1/2}|k^+|^{1/2}}\left[a_i^a(k^+,z) + a_i^{\dagger a}(k^+,z))\right]\right\}$$

LO JIMWLK Hamiltonian

Jalilian Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (1997-2002)

$$\mathcal{H}^{JIMWLK}_{LO} = \int_{x,y,z} K_{LO} \left\{ J^a_L(x) J^a_L(y) + J^a_R(x) J^a_R(y) - 2 J^a_L(x) S^{ab}_A(z) J^b_R(y) \right\}$$

$$K_{LO}(x, y, z) = \frac{\alpha_s}{2\,\pi^2} \frac{(x - z)_i (y - z)_i}{(x - z)^2 (y - z)^2}$$



$$S_A{}^{cd}(z) = \mathcal{P} \exp\left\{ i \int dx^+ \, T^a \, \alpha^a_t(z, x^+) \right\}^{cd} . \qquad \Delta" \alpha_t = \rho_t \quad (YM)$$

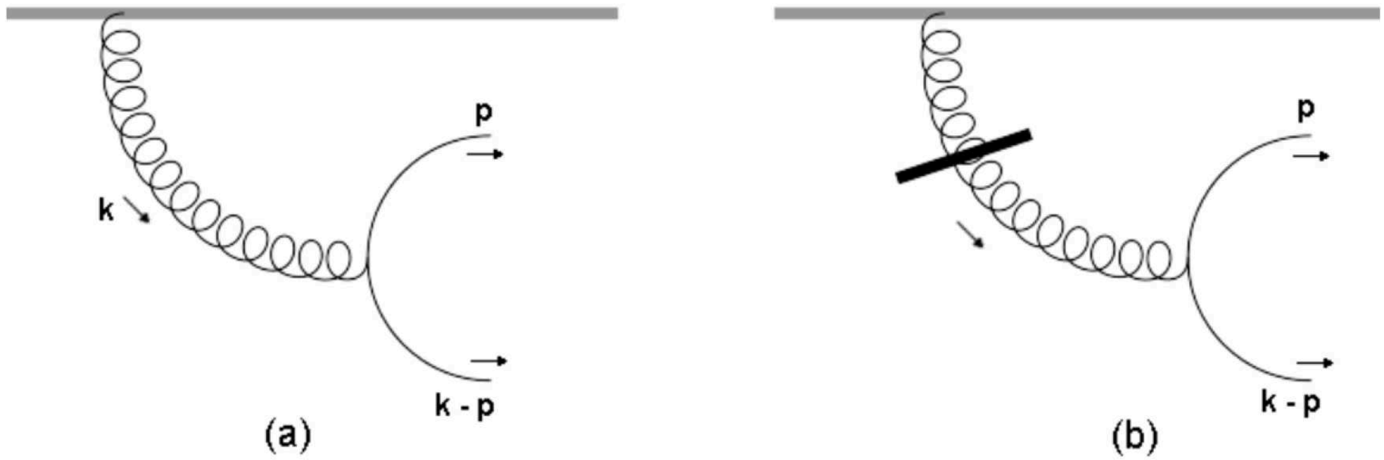
Here $\rho^p \to J_L$ and $\hat{S}\rho^p \to J_R$ are left and right SU(N) generators:

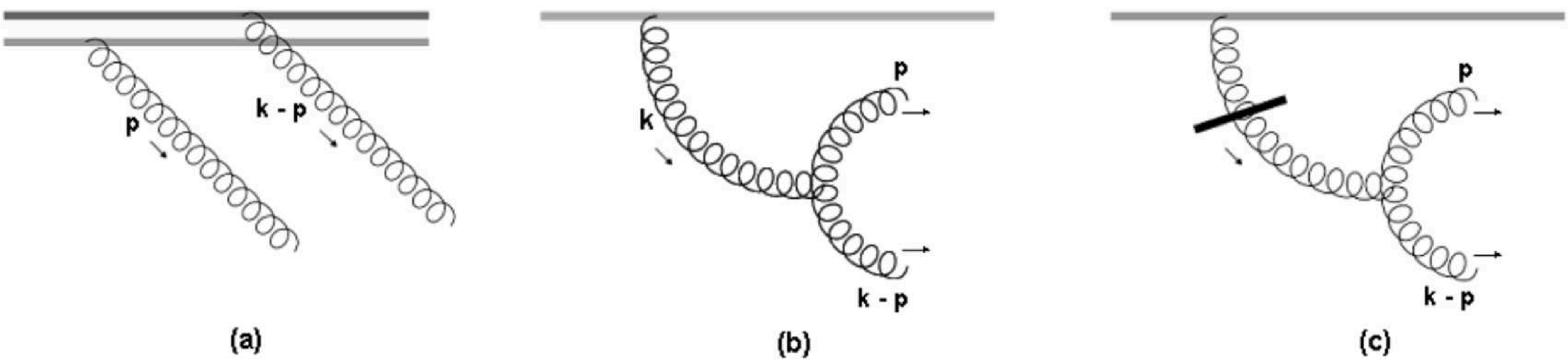
$$J^a_L(x) S^{ij}_A(z) = (T^a S_A(z))^{ij} \, \delta^2(x - z) \qquad J^a_R(x) S^{ij}_A(z) = (S_A(z) T^a)^{ij} \, \delta^2(x - z)$$

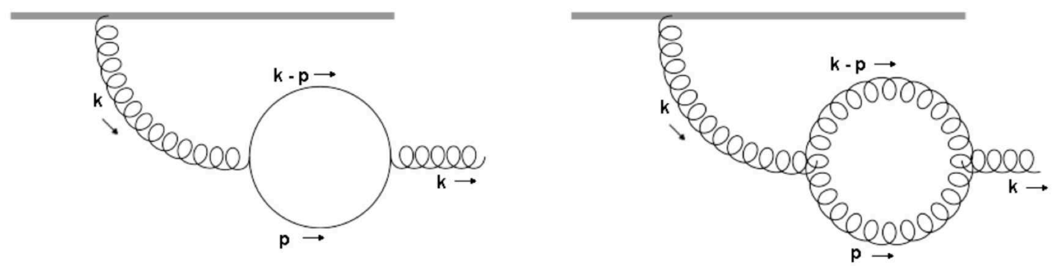
$\mathcal{H}^{JIMWLK}$ contains all the LO BFKL / BKP / TPV physics

LCWF at NLO

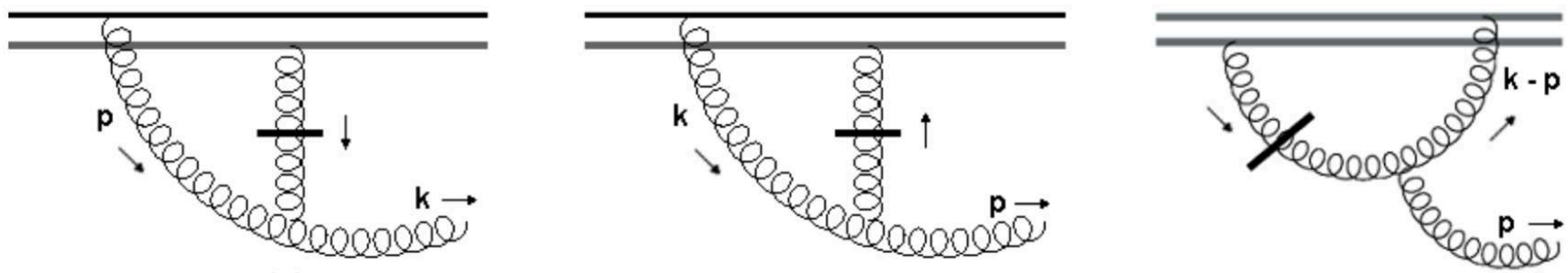
ML and Yair Mulian, arXiv:1610.03453



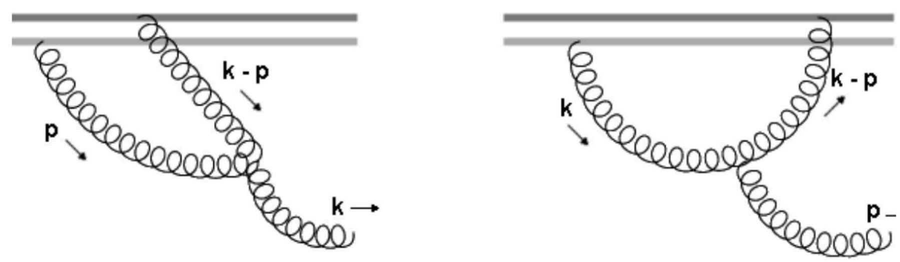




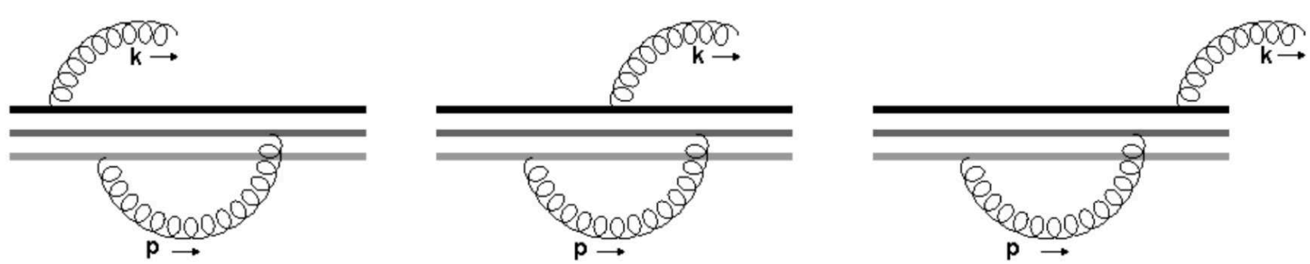
(a) (b)



(a) (b) (c)



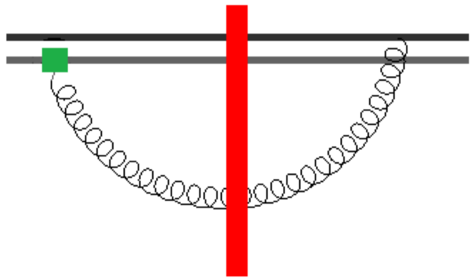
(a) (b)



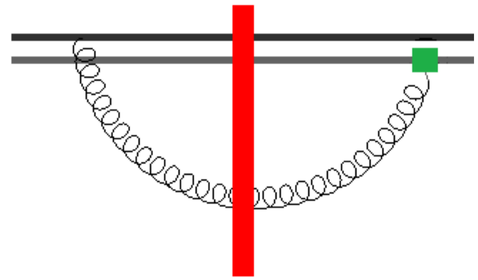
(a) (b) (c)

JIMWLK Hamiltonian @ NLO

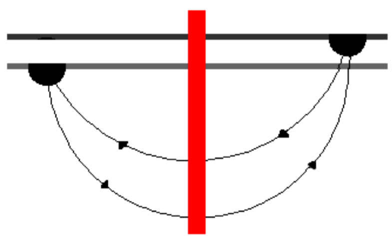
Σ_{JSJ}



Σ_{JSJ}

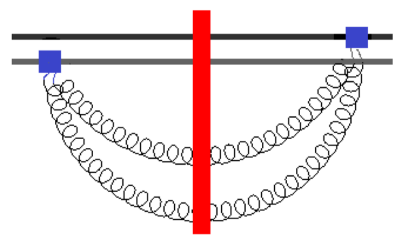


Σ_{qq}



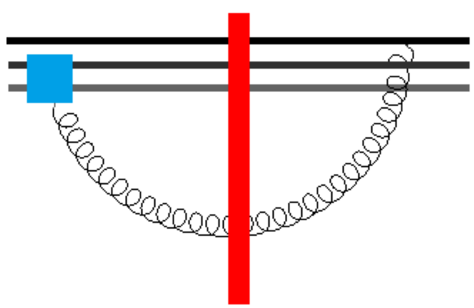
(a)

Σ_{JSSJ}

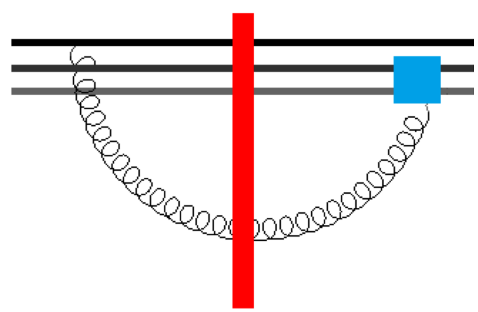


(b)

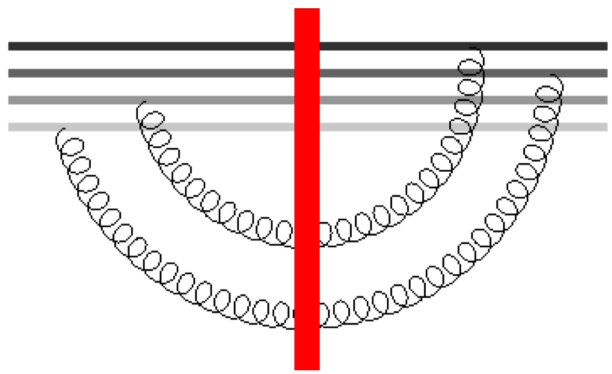
Σ_{JJSJ}



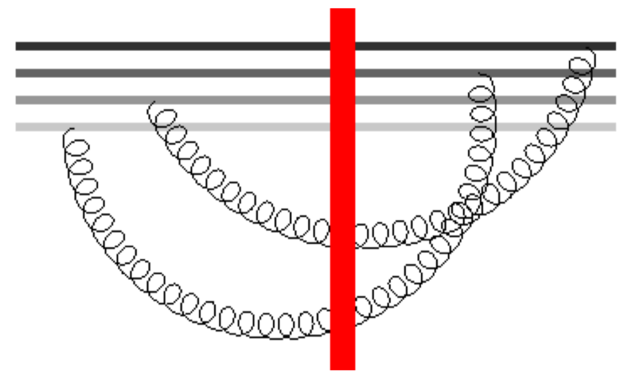
Σ_{JJSJ}



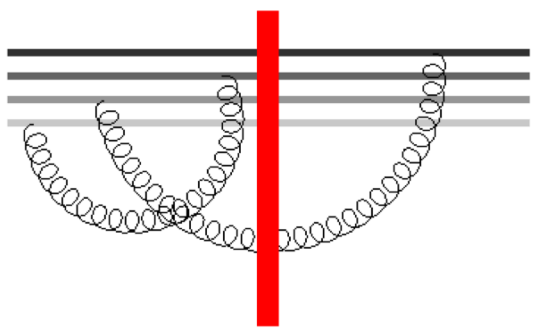
$\Sigma_{\mathbf{JJSSJJ}}$

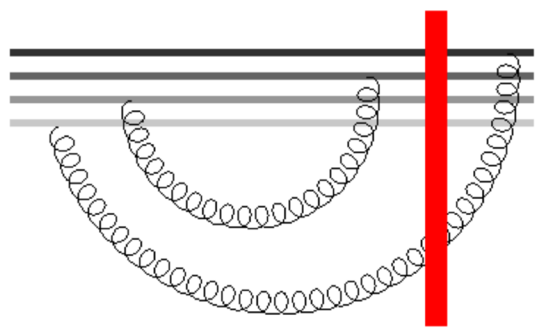


$\Sigma_{\mathbf{JJSSJJ}}$

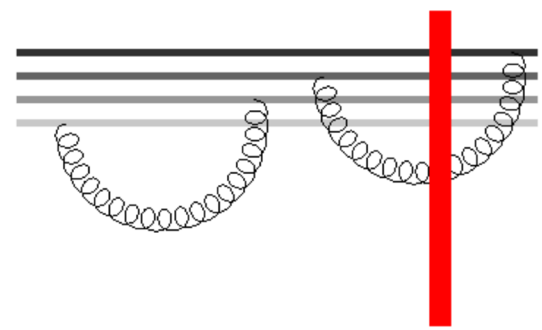


$\Sigma_{\mathbf{JJJSJ}}$





$\Sigma_{\mathbf{JJJSJ}}$



JIMWLK Hamiltonian @ NLO

Kovner, ML & Mulian (2013) based on Balitsky & Chirilli (2007), Grabovsky (2013); ML & Mulian (2016)

$$\mathcal{H}^{NLO\ JIMWLK} = \int_{x,y,z} K_{JSJ}(x,y;z)\left[J_L^a(x)J_L^a(y) + J_R^a(x)J_R^a(y) - 2J_L^a(x)S_A^{ab}(z)J_R^b(y)\right]$$

$$+ \int_{x\,y\,z\,z'} K_{JSSJ}(x,y;z,z')\left[f^{abc}f^{def}J_L^a(x)S_A^{be}(z)S_A^{cf}(z')J_R^d(y) - N_c J_L^a(x)S_A^{ab}(z)J_R^b(y)\right]$$

$$+ \int_{x,y,z,z'} K_{q\bar{q}}(x,y;z,z')\left[2\,J_L^a(x)\,tr[S_F^\dagger(z)\,t^a\,S_F(z')t^b]\,J_R^b(y)\ -\ J_L^a(x)\,S_A^{ab}(z)\,J_R^b(y)\right]$$

$$+ \int_{w,x,y,z,z'} K_{JJSSJ}(w;x,y;z,z')f^{acb}\left[J_L^d(x)\,J_L^e(y)\,S_A^{dc}(z)\,S_A^{eb}(z')\,J_R^a(w)\right.$$
$$\left. - J_L^a(w)\,S_A^{cd}(z)\,S_A^{be}(z')\,J_R^d(x)\,J_R^e(y)\right]$$

$$+ \int_{w,x,y,z} K_{JJSJ}(w;x,y;z)\,f^{bde}\left[J_L^d(x)\,J_L^e(y)\,S_A^{ba}(z)\,J_R^a(w) - J_L^a(w)\,S_A^{ab}(z)\,J_R^d(x)\,J_R^e(y)\right]$$

$$+ \int_{w,x,y} K_{JJJ}(w;x,y)f^{deb}\left[J_L^d(x)\,J_L^e(y)\,J_L^b(w)\ -\ J_R^d(x)\,J_R^e(y)\,J_R^b(w)\right].$$

NLO Kernels

$$\mathbf{X = x - z, \quad X' = x - z', \quad Y = y - z, \quad Y' = y - z', \quad W = w - z}$$

$$K_{JSSJ}(x,y;z,z') = \frac{\alpha_s^2}{16\,\pi^4}\Bigg[-\frac{4}{(z-z')^4} + \Big\{2\frac{X^2Y'^2 + X'^2Y^2 - 4(x-y)^2(z-z')^2}{(z-z')^4[X^2Y'^2 - X'^2Y^2]}$$

$$+ \frac{(x-y)^4}{X^2Y'^2 - X'^2Y^2}\Big[\frac{1}{X^2Y'^2}+\frac{1}{Y^2X'^2}\Big]+\frac{(x-y)^2}{(z-z')^2}\Big[\frac{1}{X^2Y'^2}-\frac{1}{X'^2Y^2}\Big]\Big\}\ln\frac{X^2Y'^2}{X'^2Y^2}\Bigg] + \tilde{K}(x,y,z,z').$$

$$K_{JSJ}(x,y;z) = -\frac{\alpha_s^2}{16\pi^3}\frac{(x-y)^2}{X^2Y^2}\Big[b\ln(x-y)^2\mu^2 - b\frac{X^2-Y^2}{(x-y)^2}\ln\frac{X^2}{Y^2} + (\frac{67}{9}-\frac{\pi^2}{3})N_c - \frac{10}{9}n_f\Big]$$

$$-\frac{N_c}{2}\int_{z'}\tilde{K}(x,y,z,z').$$

Here μ is the normalization point, $\mathbf{b} = \frac{11}{3}\mathbf{N_c} - \frac{2}{3}\mathbf{n_f}$

Resummation of large Logs

$$\mathcal{H} = \int_{x,y,z} \mathbf{K(x,y;z)} \left[\mathbf{J^a_L(x)\,J^a_L(y) \,+\, J^a_R(x) J^a_R(y) \,-\, 2 J^a_L(x)\, S^{ab}_A(z)\, J^b_R(y)} \right]$$

Resum large Logs into an effective kernel $\mathbf{K \,=\, K_{LO} \,+\, K_{JSJ} \,+\, \ldots.}$

Action on a dipole $\mathbf{S(u,v)}$

$$\mathcal{H}\,\mathbf{S(u,v) \,=\, N_c \int_z K_{dipole}(u,v,z)\, [S(u,z)\,S(z,v) \,-\, S(u,v)]}$$

$$\mathbf{K_{dipole}(u,v,z) \,=\, K(u,u,z) \,+\, K(v,v,z) \,-\, K(u,v,z) \,-\, K(v,u,z)}$$

Positive semidefinite kernels:

$$\int \mathbf{d^2X\, d^2Y\, f(X)\, K(X,Y)\, f(Y) \,\geq\, 0} \qquad \text{for } \mathbf{f}$$

$$\mathbf{K(X,X) \,+\,} \mathcal{K}\mathbf{(Y,Y) \,-\, 2K(X,Y) \,\geq\, 0} \qquad \rightarrow \qquad \mathbf{K_{dip}(X,Y) \,\geq\, 0}$$

$$\mathbf{K(X,X) \,+\, K(Y,Y) \,-\, K^2(X,Y) \,\geq\, 0}$$

Running coupling

Large UV Logs:

$$\mathbf{K_{JSJ}}(\mathrm{b\,terms}) = \frac{\alpha_s^2}{16\pi^3}\left\{-\mathbf{b}\frac{(\mathbf{x}-\mathbf{y})^2}{\mathbf{X}^2\mathbf{Y}^2}\ln(\mathbf{x}-\mathbf{y})^2\mu^2 + \frac{\mathbf{b}}{\mathbf{X}^2}\ln\mathbf{Y}^2\mu^2 + \frac{\mathbf{b}}{\mathbf{Y}^2}\ln\mathbf{X}^2\mu^2\right\}$$

$$\mathbf{K}^{\mathrm{Bal}} = \frac{\alpha_s(\mathbf{X}-\mathbf{Y})}{2\pi^2}\frac{\mathbf{X}\cdot\mathbf{Y}}{\mathbf{X}^2\mathbf{Y}^2} + \frac{\alpha_s(\mathbf{X})}{4\pi^2}\frac{\mathbf{1}}{\mathbf{X}^2}\left(\mathbf{1} - \frac{\alpha_s(\mathbf{X}-\mathbf{Y})}{\alpha_s(\mathbf{Y})}\right) + \frac{\alpha_s(\mathbf{Y})}{4\pi^2}\frac{\mathbf{1}}{\mathbf{Y}^2}\left(\mathbf{1} - \frac{\alpha_s(\mathbf{X}-\mathbf{Y})}{\alpha_s(\mathbf{X})}\right)$$

$$\mathbf{K}^{\mathrm{KW}} = \frac{\alpha_s(\mathbf{X})\alpha_s(\mathbf{Y})}{\alpha_s(\mathbf{R}(\mathbf{X},\mathbf{Y}))}\frac{\mathbf{1}}{2\pi^2}\frac{\mathbf{X}\cdot\mathbf{Y}}{\mathbf{X}^2\mathbf{Y}^2}$$

Finite terms are different too due to different UV subtraction/reshuffling schemes

None of the kernels is positive semidefinite T. Altinoluk, G. Beuf, ML, and V. Skokov, to appear

These results are problematic: what α_s is doing in the denominators? why is the charge renormalization of the emitter at position x sensitive to position of another emitter at y?

A different approach

1. Proper definition of the considered observable:
   Finite resolution Q for gluon splitting into two must be introduced (this could be identified with non-linear wavefunction renormalization of the soft gluon field).
   Bare gluons $\rightarrow$ dressed gluons: bare Wilson lines $\rightarrow$ dressed Wilson lines, $\mathbf{S} \rightarrow \mathbf{S_Q}$

2. Identification of running coupling correction among higher order terms
   Not all Logs proportional to b are due to charge renormalization and should be absorbed into some form of α_s running. Zoom on UV divergent logs.

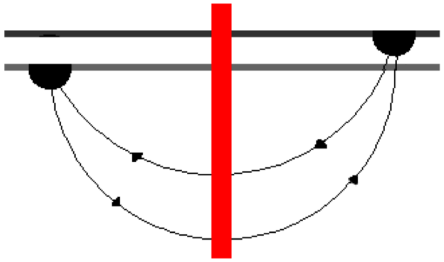
3. Resummation of the identified running coupling effects.
   Charge in the amplitude and the conjugate amplitude must be renormalized independently. We will get the most intuitive result:

$$\mathbf{K_{LO}} = \frac{\alpha_s}{2\pi^2}\frac{\mathbf{XY}}{\mathbf{X^2Y^2}} \rightarrow \frac{\sqrt{\alpha_s(\mathbf{X})\alpha_s(\mathbf{Y})}}{2\pi^2}\frac{\mathbf{XY}}{\mathbf{X^2Y^2}}$$

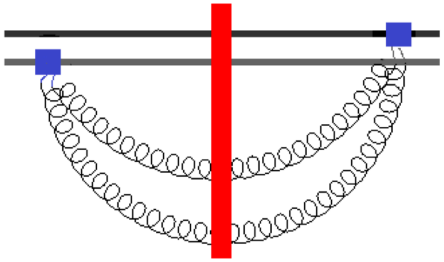
4. Resummation of additional large logs.
   Large Logs associated with the resolution Q are resummed a la DGLAP.

$$\frac{\sqrt{\alpha_s(\mathbf{X})\alpha_s(\mathbf{Y})}}{2\pi^2}\frac{\mathbf{XY}}{\mathbf{X^2Y^2}} \rightarrow \frac{\sqrt{\alpha_s(\mathbf{X})\alpha_s(\mathbf{Y})}}{2\pi^2}\frac{\mathbf{XY}}{\mathbf{X^2Y^2}} \times \delta\mathbf{K}^{\mathbf{DGLAP}}$$

UV divergence in two-gluon (and quark) sector



(a)



(b)

$$\int_{x\,y\,z,z'} K_{JSSJ}(x, y; z, z')\, J_L^a(x) J_R^b(y) \left[D^{ab}(z, z') \right] \quad \sim \; b \times (\text{UV divergent Log})$$

$$D^{ab}(z_1, z_2) \equiv Tr[T^a S_A(z) T^b S_A^+(z')]$$

The UV divergence in $JSSJ$ is trivial: when the two gluons are too close to each other, they cannot be resolved by the target and hence should be counted as a single gluon scattering. We are thus prompted to introduce a "resolution scale" Q pretty much the same way as it is done in the DGLAP.

Dressed Wilson line

$$\mathbf{S}^{ab}_{Q}(\mathbf{z}) = \mathbf{S}^{ab}_{A}(\mathbf{z}) + \frac{\alpha_s}{2\pi^2}\int_0^1 d\xi\,\sigma(\xi)\int^{Q^{-1}}\frac{d^2Z}{Z^2}\left(\mathbf{D}^{ab}(\mathbf{z}+(1-\xi)\mathbf{Z},\mathbf{z}-\xi\mathbf{Z}) \;-\; N_c\,\mathbf{S}^{ab}_{A}(\mathbf{z})\right)$$

ξ is the fraction of longitudinal momentum carried by one of the gluons.

$$\sigma(\xi) = \left[\frac{1}{\xi(1-\xi)}\left(\xi^2+(1-\xi)^2+\xi^2(1-\xi)^2\right)\right]_+ \;;\quad 2N_c\int_0^1 d\xi\,\sigma(\xi) \;=\; -\frac{11N_c}{3} \to -b$$

This is a P_{gg} splitting function except that we introduce the "+" prescription both for $\xi = 1$ and $\xi = 0$ poles (they enter fully symmetrically). In the usual DGLAP, $\xi > x_{bj}$. In the eikonal approximation we have lost a track of the longitudinal momenta, equivalent to $x_{bj} \to 0$. Further extension is to go beyond the eikonal $S \to S_\xi$.
The "+" prescription emerges from the $1/\xi$ subtraction absorbed into (LO)^2 part of the evolution.

The sign is negative – correcting for the over-subtraction in the LO.

Apart a somewhat modified splitting function, we go beyond the usual DGLAP: we allow simultaneous scattering of all gluons.

Resolution scale

Express S in terms of $\mathrm{S_Q}$ and substitute it into the JIMWLK Hamiltonian. Consequently, the Hamiltonian will feature $\ln \mathbf{Q}^2$ terms, such as $\ln(\mathbf{Q}^2\mathbf{X}^2)$.

What value of Q should we take?

- If we take $\mathbf{Q} = \mathbf{Q_s^T}$ then $\mathrm{S_Q} = \mathrm{S_A}$ - the target does not resolve gluon splitting at distances smaller than $1/\mathbf{Q_s^T}$. This is also the scale of the real-virtual cancelation inside the JIMWLK Hamiltonian. The $\ln \mathbf{Q}^2$ terms in the Hamiltonian will be large and have to be resummed.
- We assume existence of a typical scale $\mathbf{Q_s^P} \ll \mathbf{Q_s^T}$ associated with the projectile, such that $\ln(\mathbf{Q_s^P}\mathbf{X}^2)$ are small. If we set $\mathbf{Q} = \mathbf{Q_s^P}$, then there will be no large Logs in the Hamiltonian, but $\mathrm{S_Q}$ is very different from $\mathrm{S_A}$, roughly $\mathrm{S_Q} \sim \mathrm{S_A}\,[1 + \alpha_s\,\#\,\mathrm{Log}(\mathbf{Q}^2/\mathbf{Q_s^T})]$. This large Log has to be resummed into $\mathrm{S_Q}$ via inclusion of multiple consecutive DGLAP splittings:

$$\frac{\partial \mathbf{S_Q(z)}}{\partial \ln \mathbf{Q}} = -\frac{\alpha_s}{2\pi^2}\int_\xi \sigma(\xi)\int_{\phi_\mathbf{Q}}[\mathbf{D_Q(z)} - \mathbf{N_c}\,\mathbf{S_Q(z)}]\,;\quad \mathbf{D_Q(z_1,z_2)} \equiv \mathbf{Tr}[\mathbf{T^a S_Q(z_1) T^b S_Q^+(z_2)}]$$

RG

- **The resummed Hamiltonian should be Q-independent:**

$$\frac{\mathbf{d}\mathcal{H}}{\mathbf{d}\ln\mathbf{Q}} = \frac{\partial\mathcal{H}}{\partial\ln\mathbf{Q}} + \int_{\mathbf{u}}\left[\frac{\delta\mathbf{H}}{\delta\mathbf{S_Q(u)}}\frac{\partial\mathbf{S_Q(u)}}{\partial\ln\mathbf{Q}}\right] = \mathbf{0}$$

Recall that $\mathcal{H}$ (or rather Σ) is the diagonal element of the S-matrix.

$$\mathcal{H} = \int_{\mathbf{u}}\left[\frac{\delta\mathcal{H}}{\delta\mathbf{S_Q(u)}}\mathbf{S_Q(u)}\right] \text{ ''}={}\text{'' probability to emit gluon} \otimes \text{hard scattering}$$

DGLAP-like evolution for the Hamiltonian (evolution in the space of Hamiltonians):

$$\frac{\partial\mathcal{H}}{\partial\ln\mathbf{Q}} = -\int_{\mathbf{u}}\left[\frac{\delta\mathcal{H}}{\delta\mathbf{S_Q(u)}}\frac{\partial\mathbf{S_Q(u)}}{\partial\ln\mathbf{Q}}\right] = \frac{\alpha_s}{\mathbf{2}\pi^{\mathbf{2}}}\int_{\mathbf{u}}\left[\frac{\delta\mathcal{H}}{\delta\mathbf{S_Q(u)}}\int_{\xi}\sigma(\xi)\int_{\phi_{\mathbf{Q}}}[\mathbf{D_Q(u)} - \mathbf{N_c}\,\mathbf{S_Q(u)}]\right]$$

Initial conditions and the running coupling

Initial conditions: at $\mathbf{Q} = \mathbf{Q_s^P}$

$$\mathcal{H}_{\rm in} = \int \mathbf{K_{in}} \left[\mathbf{S_Q(z)S_Q^\dagger(z) + S_Q(x)S_Q^\dagger(y) - S_Q(x)S_Q^\dagger(z) - S_Q(z)S_Q^\dagger(y)}\right]^{\mathbf{ab}} \mathbf{J_L^a(x)\, J_L^b(y)}$$

$$\mathbf{K_{in}} = \mathbf{K_{LO}} \left(\mathbf{1} + \frac{\alpha_s}{\mathbf{4}\pi}\, \mathbf{b}\, (\ln \mathbf{X}^2\mu^2 + \ln \mathbf{Y}^2\mu^2 - \ln \mathbf{Q}^{-2}\mu^2)\right) + \text{other } \mathrm{O}(\alpha_s^2) \text{ stuff}$$

The remaining part of the Hamiltonian proportional to $\mathbf{b}$ is UV finite and does not have any large Logs. So, we ignore it for now. (To be precise, there are large Logs hidden in JJSJ/JJSSJ kernels)

$$\mathbf{K_{in}} = \frac{\sqrt{\alpha_s(\mathbf{X})\, \alpha_s(\mathbf{Y})}}{\mathbf{2}\pi^{\mathbf{2}}} \frac{\mathbf{XY}}{\mathbf{X^2Y^2}} \left[\mathbf{1} + \frac{\alpha_s}{\mathbf{8}\pi}\mathbf{b}\, (\ln \mathbf{X}^2\mathbf{Q_s^{P\,2}} + \ln \mathbf{Y}^2\mathbf{Q_s^{P\,2}})\right]$$

The Log terms in $\mathrm{K_{in}}$ are small NLO corrections.

Evolve up to $\mathbf{Q} = \mathbf{Q_s^T}$.

Formal Solution

$$\mathcal{H} \;=\; \mathbf{Exp}\left[\int_{Q_s^P}^{Q_s^T} \frac{dQ}{Q} H_{DGLAP}\right] \mathcal{H}_{in}$$

$$H_{DGLAP} \;=\; \frac{\alpha_s(Q^2)}{2\pi^2} \int_u \int_\xi \sigma(\xi) \int_{\phi_Q} \mathbf{Tr}\left([D_Q(u) \;-\; N_c\, S_Q(u)]\, \frac{\delta}{\delta S_Q(u)} \right)$$

where we have promoted $\alpha_s = \alpha_s(Q^2)$ into running coupling, formally exceeding the accuracy of the derivation.

Weak target field approximation – linearization

$$\mathbf{S^{ab}_Q} = \delta^{\mathbf{ab}} + \mathbf{f^{abc}}\alpha^{\mathbf{c}}_{\mathbf{Q}}\,; \qquad \mathbf{D^{ab}_Q(z_1, z_2)} = \mathbf{N_c}\left(\delta^{\mathbf{ab}} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{f^{abc}}\left[\alpha^{\mathbf{c}}_{\mathbf{Q}}(\mathbf{z_1}) + (\alpha^{\mathbf{c}}_{\mathbf{Q}}(\mathbf{z_2}))^*\right]\right)$$

Expand the Hamiltonian (BFKL-like)

$$\mathbf{H_{DGLAP}} \sim \alpha_{\mathbf{Q}}\frac{\delta}{\delta\alpha_{\mathbf{Q}}}$$

$\mathbf{H_{DGLAP}}$ **is homogeneous and hence solvable**

Saturation region

$$\mathbf{H_{DGLAP}} = \frac{\alpha_{\mathbf{s}}}{\mathbf{2}\pi^{\mathbf{2}}}\int_{\mathbf{u}}\int_{\xi}\sigma(\xi)\int_{\phi_{\mathbf{Q}}}\mathbf{Tr}\left([\mathbf{Tr}[\mathbf{T^a S_Q(z_1) T^b S^+_Q(z_2)}]_{\mathbf{u}} - \mathbf{N_c}\,\mathbf{S_Q(u)}]\frac{\delta}{\delta\mathbf{S_Q(u)}}\right)$$

Since $|\mathbf{z_1} - \mathbf{z_2}| = \mathbf{1/Q} > \mathbf{1/Q^T_s}$, the two gluons are well separated and outside the correlation region in the target (in a sense of averaging over the target). Neglect the first term. H_{DGLAP} is again homogeneous

Summary/Outlook

- DGLAP-like resummation inside the JIMWLK Hamiltonian has been performed. These DGLAP corrections are large whenever there is a large disparity between the correlation lengths (or saturation momenta) in the projectile and the target. This is precisely JIMWLK's regime of validity.
  The result is a smearing of the WW fields within the $1/Q_s^T$ distance

- rcJIMWLK emerges with the scale choice for the running coupling:
  $K \sim \sqrt{\alpha_s(X)\alpha_s(Y)}$

- Numerical implementations are to follow