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Positronium in quantum electrodynamics of effective particles

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Outline

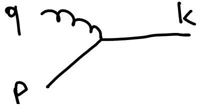
- RGPEP
- BLFQ
- Extrapolations
- Results

- $\mathcal{H} = \mathcal{H}_{\text{QED}} + \frac{1}{2} m_g^2 A_\perp A_\perp$
- Photon energy:

$$p^- = \frac{m_g^2 + p_\perp^2}{p^+}$$

- UV: $p^- \to \infty$, small x: $p^+ \to 0^+$
- Regularization:

$$\exp\left[-t_r \left(p^- + q^- - k^-\right)^2\right]$$



Renormalized Hamiltonian

- $H(m_g, t_r) = \int dx^- dx^\perp \mathcal{H} + \text{counterterms}$

- Renormalization group procedure for effective particles (RGPEP):

$$H(m_g, t_r) \stackrel{\text{RGPEP}}{\longrightarrow} H_t(m_g, t_r)$$

- Renormalized Hamiltonian:

$$H_t(m_g) = \lim_{t_r \to 0^+} H_t(m_g, t_r)$$

- Effective particles

$$b_t^\dagger \;=\; U_t\, b^\dagger\, U_t^\dagger$$

where U_t is a unitary operator, $t \geq 0$.

- $t = \frac{(P^+)^2}{\lambda^4}$

- Rewrite the theory in terms of effective particles

$$\underbrace{H_{t=0}(b, b^\dagger, d, d^\dagger, a, a^\dagger)}_{\text{initial Hamiltonian}} \;=\; \underbrace{H_t(b_t, b_t^\dagger, d_t, d_t^\dagger, a_t, a_t^\dagger)}_{\text{effective Hamiltonian}}$$

- Instead of U_t it is more convenient to write the equation that governs scale evolution of the Hamiltonian (Wegner-like flow equation):

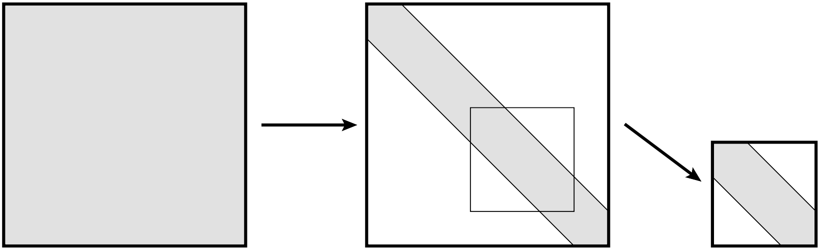
$$\frac{d}{dt}\mathcal{H}_t = [[\mathcal{H}_f, \mathcal{H}_t], \mathcal{H}_t],$$

$$\mathcal{H}_t = H_t(b, b^\dagger, d, d^\dagger, a, a^\dagger),$$

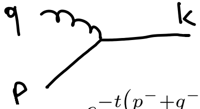
$$U_t = T \exp\left(-\int_0^t d\tau \, [\mathcal{H}_f, \mathcal{H}_\tau]\right).$$

Hamiltonian matrix

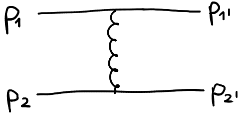
$$H(m_g, t_r) \longrightarrow H_t(m_g, t_r)$$



$$H_t = H_{t\,0} + gH_{t\,1} + g^2 H_{t\,2} + \ldots$$



$$e^{-t\left(p^- + q^- - k^-\right)^2}$$



$$e^{-t\left(p_1^- + p_2^- - p_{1'}^- - p_{2'}^-\right)^2}$$

Effective Hamiltonian

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & H_{t\,0}+g^2 H_{t\,2} & gH_{t\,1} \\ \cdot & \cdot & gH_{t\,1} & H_{t\,0}+g^2 H_{t\,2} \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ |e_t \bar{e}_t\,\gamma_t\rangle \\ |e_t \bar{e}_t\rangle \end{bmatrix} = P^- \begin{bmatrix} \cdot \\ \cdot \\ |e_t \bar{e}_t\,\gamma_t\rangle \\ |e_t \bar{e}_t\rangle \end{bmatrix}$$

$\downarrow$ Up to g^2 $\downarrow$

$$\begin{bmatrix} H_{t\,0} & gH_{t\,1} \\ gH_{t\,1} & H_{t\,0}+g^2 H_{t\,2} \end{bmatrix} \begin{bmatrix} |e_t \bar{e}_t\,\gamma_t\rangle \\ |e_t \bar{e}_t\rangle \end{bmatrix} = P^- \begin{bmatrix} |e_t \bar{e}_t\,\gamma_t\rangle \\ |e_t \bar{e}_t\rangle \end{bmatrix}$$

$\downarrow$ integrate out perturbatively the higher sector $\downarrow$

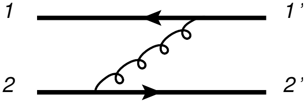
$$H_{\text{eff}\,t}\big|E_t \bar{E}_t\big\rangle = \frac{M^2+P_\perp^2}{P^+}\big|E_t \bar{E}_t\big\rangle$$

RGPEP

$$\left(\frac{m_1^2 + (p_1^\perp)^2}{p_1^+} + \frac{m_2^2 + (p_2^\perp)^2}{p_2^+}\right)\psi$$

$$-g_t^2 \sum_{\sigma_{1'},\sigma_{2'}} \int [1'2']\,\tilde{\delta}_{12.1'2'}\,(U_C + U_X)\,\psi' \quad = \quad \frac{M^2 + (P^\perp)^2}{P^+}\,\psi\ ,$$

where



$$U_C \quad = \quad r\, g_{\mu\nu} j_1^\mu j_2^\nu\, f_t\, \mathcal{F}\ ,$$

$$U_X \quad = \quad r\, f_t\, \frac{j_1^+ j_2^+}{(q^+)^2}\left(1 + \frac{q_1^2 + q_2^2}{2}\,\mathcal{F}\right) - f_t\, X\ ,$$

with

$$\mathcal{F} \quad = \quad \frac{1}{2}\left(\frac{1}{m_g^2 - q_1^2} + \frac{1}{m_g^2 - q_2^2}\right),$$

and

$$f_t \quad = \quad \exp\left[-t\left(\frac{q_2^2 - q_1^2}{p_3^+}\right)^2\right].$$

$$q_1^\mu = p_{1'}^\mu - p_1^\mu\ , \quad q_2^\mu = p_2^\mu - p_{2'}^\mu\ , \quad j_i^\mu = \bar{u}_i \gamma^\mu u_{i'}$$

- N_{max} and b

$$\Psi_n^m(\mathbf{q}) = \frac{1}{b}\sqrt{\frac{4\pi n!}{(n+|m|)!}}\, L_n^{|m|}\left(\frac{q^2}{b^2}\right) e^{-\frac{q^2}{2b^2}} \left|\frac{q}{b}\right|^{|m|} e^{im\varphi}$$

- K and K_n $\qquad p_1^+ = \frac{2\pi}{L}(k_1 + 1/2)$

  Example:

  $k_1 =$ 0, 1, 2, 3, 4, 5, 6, 7, 8 $\qquad K = 9,\ K_n = 9$

  $k_1 =$ 3, 4, 5 $\qquad K = 9,\ K_n = 3$

- Light front helicity M_J

- We take $\alpha = \frac{1}{137}$
- No annihilation channel
- Characteristic momentum: $p_B = \frac{1}{2} m_e \alpha$
- Characteristic energy: $\frac{1}{4} m_e \alpha^2$
- $E = \frac{M - 2m_e}{\frac{1}{4} m_e \alpha^2}$
- $E = -\frac{1}{n^2} + O(\alpha^2)$

Numerically challenging:

$$M_{\text{singlet}} = 2m_e - \frac{1}{2}\frac{m_e}{2}\alpha^2\left(1+\frac{63}{48}\alpha^2\right) \approx \underline{1.9999866\ 79}2326557\ ,$$

$$M_{\text{triplet}} = 2m_e - \frac{1}{2}\frac{m_e}{2}\alpha^2\left(1-\frac{1}{48}\alpha^2\right) \approx \underline{1.9999866\ 80}1788853\ ,$$

$$M_{{}^3P_2} = 2m_e - \frac{1}{2^2}\frac{\frac{1}{2}m_e\alpha^2}{2}\left(1+\frac{43}{960}\alpha^2\right) \approx \underline{1.9999966\ 70}0330782\ .$$

$$E_{\text{singlet}} = -1-\frac{63}{48}\alpha^2 \approx \underline{-1.00\ 00}699291\ ,$$

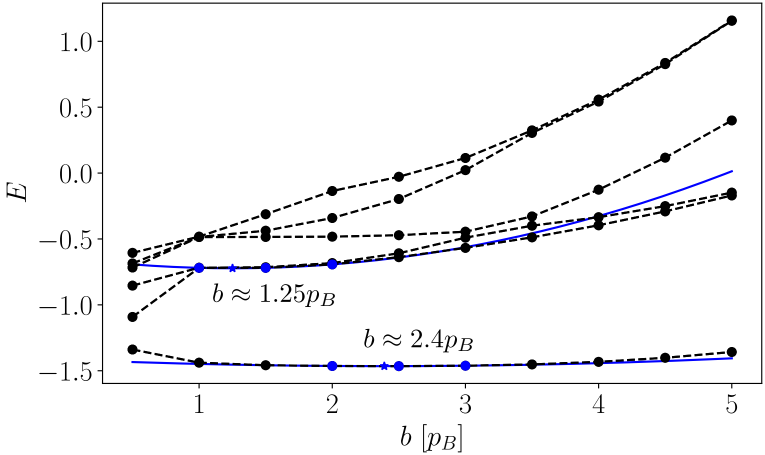
$$E_{\text{triplet}} = -1+\frac{1}{48}\alpha^2 \approx \underline{-0.99\ 99}988900\ ,$$

$$E_{{}^3P_2} = -\frac{1}{4}-\frac{43}{3840}\alpha^2 \approx \underline{-0.25\ 00}005966\ .$$

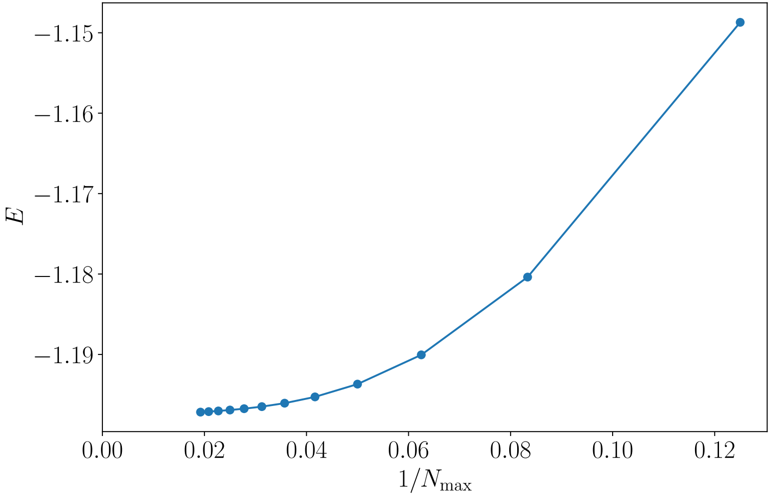
$$E_{\text{triplet}} - E_{\text{singlet}} \approx \underline{7.10}391\cdot 10^{-5}$$

Optimal b

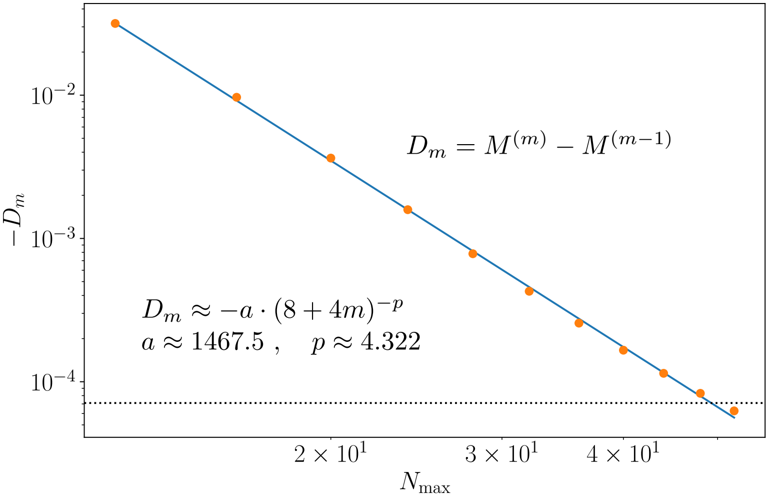
$N_{\max} = 24$, $K = 725$, $K_n = 91$, $m_g = 0.0001$, $t = 10^{-8}$



Convergence with respect to $N_{\max}$

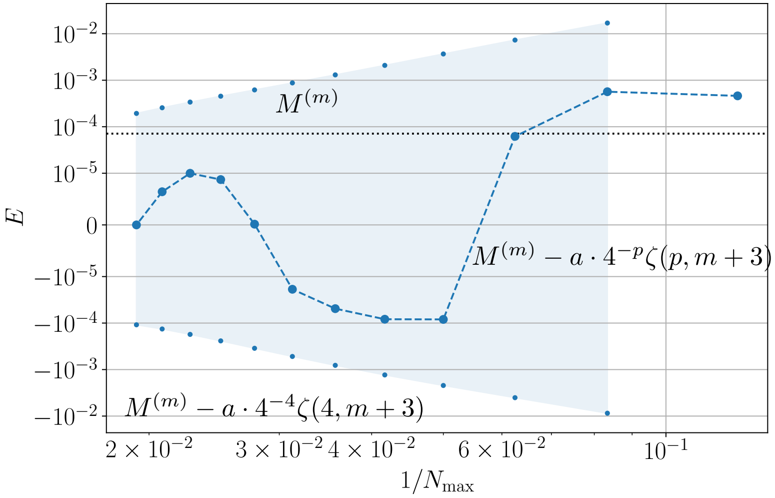


Convergence with respect to $N_{\max}$

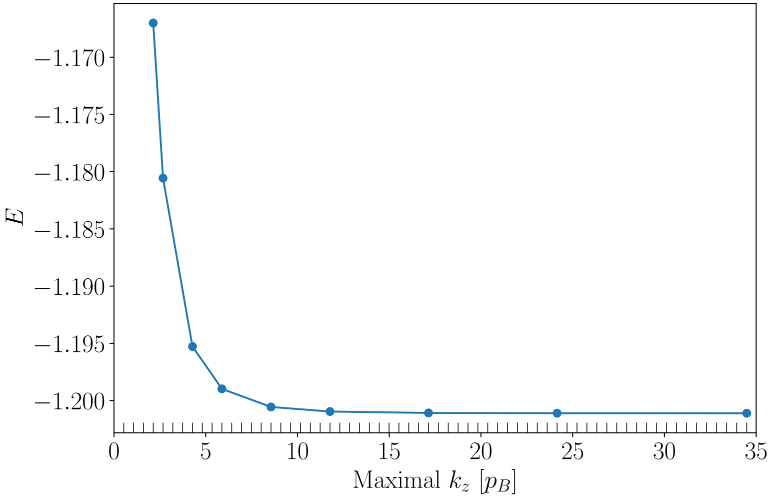


$$M^{(\infty)} = M^{(m)} + \sum_{i=m+1}^{\infty} D_i = M^{(m)} - a \cdot 4^{-p} \zeta(p, m+3)$$

Convergence with respect to $N_{\max}$

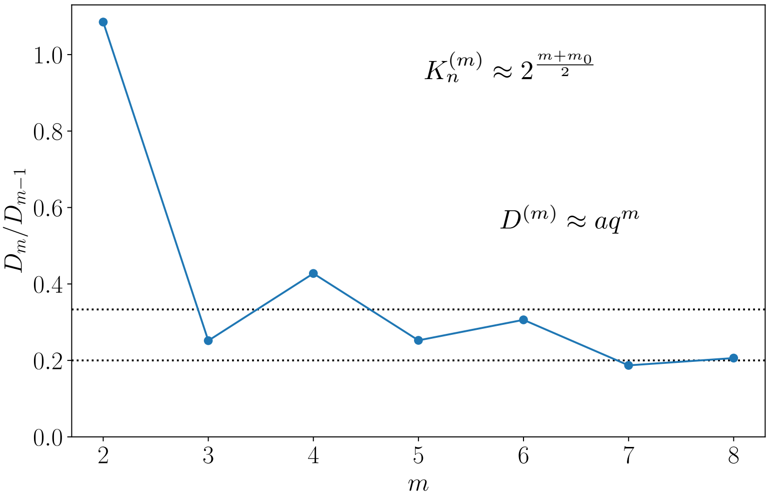


Convergence with respect to K_n



$b = 2.4 p_B$, $K = 1025$, $\Delta k_z \approx 0.53 p_B$

Convergence with respect to K_n



Convergence with respect to K_n

To estimate the uncertainties we assume that

$$\frac{D_{n+1}}{D_n} < Q$$

for sufficiently large n. Hence,

$$\left|M^{(\infty)} - M^{(m)}\right| = \left|\sum_{i=1}^{\infty} D_{m+i}\right| \leq \sum_{i=1}^{\infty} |D_{m+i}| = |D_m| \sum_{i=1}^{\infty} Q^i = \frac{Q}{1-Q} |D_m| .$$

Therefore, for sufficiently large m,

$$\left|M^{(\infty)} - M^{(m)}\right| \leq \frac{Q}{1-Q} |D_m| .$$

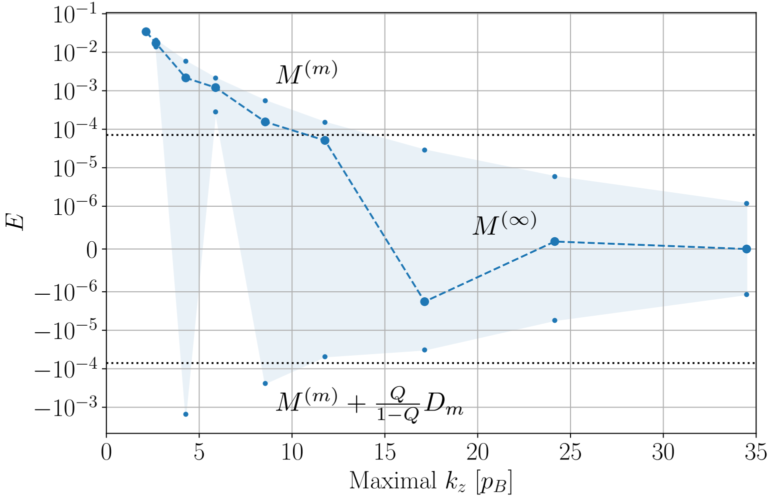
Suppose $D_m < 0$, $D_{m+1} \approx q D_m$, and $|D_{m+1}| < Q|D_m|$. Consistency requries $Q > q > 0$. Then the limit is

$$M^{(\infty)} \approx M^{(m)} + \frac{q}{1-q} D_m ,$$

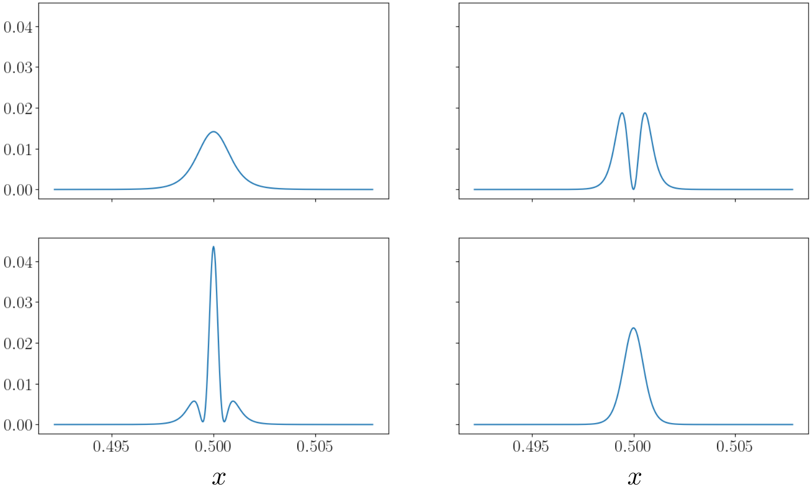
and the bounds are,

$$M^{(m)} + \frac{Q}{1-Q} D_m \leq M^{(\infty)} \leq M^{(m)} .$$

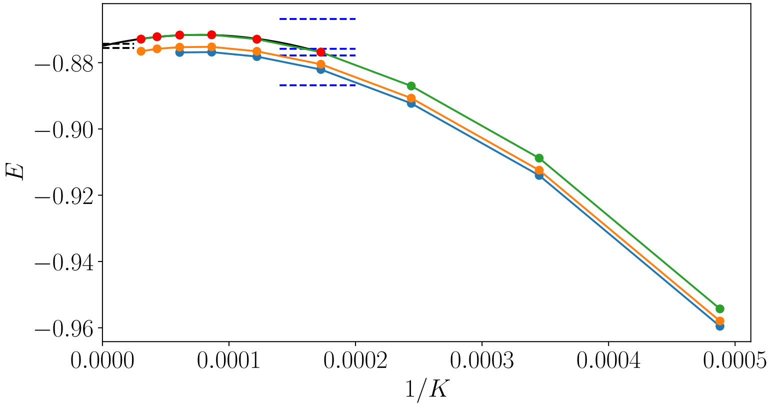
Convergence with respect to K_n



Electron PDFs in different states

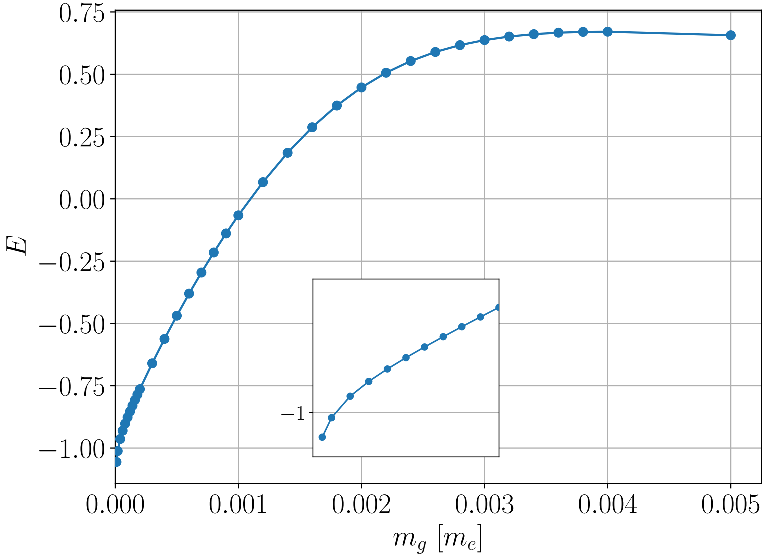


Convergence with respect to K



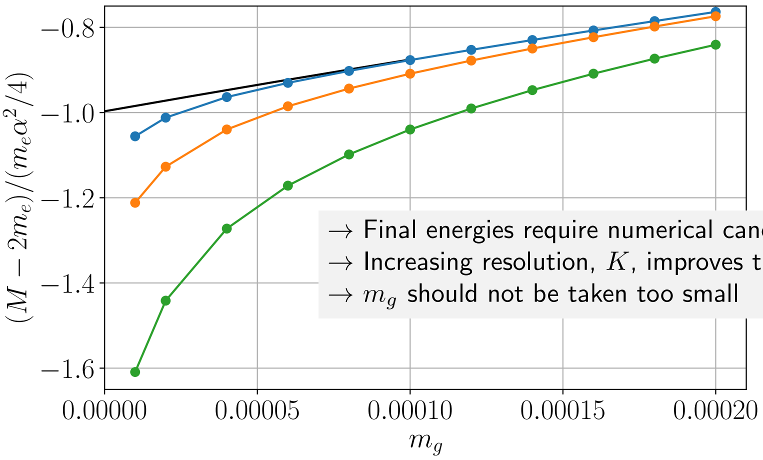
$N_{\max} = 24, 20, 16$. The range of longitudinal momentum is roughly $k_z \in [-4.28 p_B, +4.28 p_B]$ and differs slightly from point to point.

Convergence with respect to m_g



Uncertainties

Convergence with respect to m_g

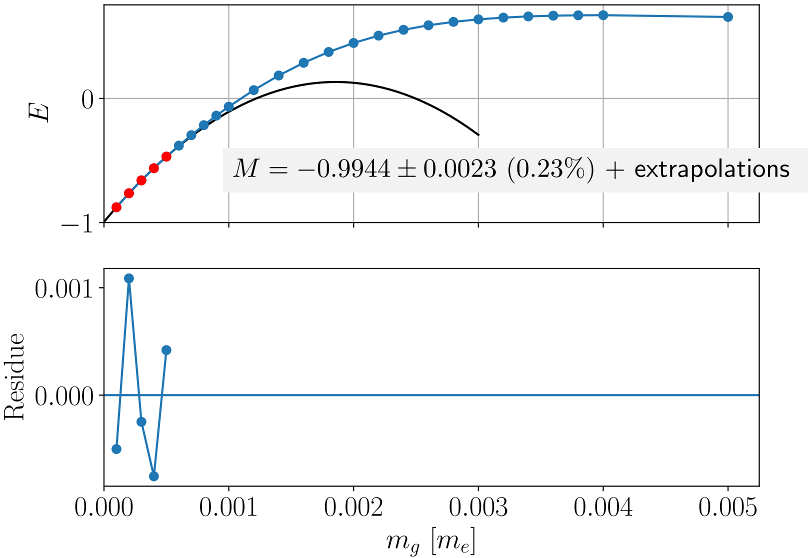


→ Final energies require numerical cancellations

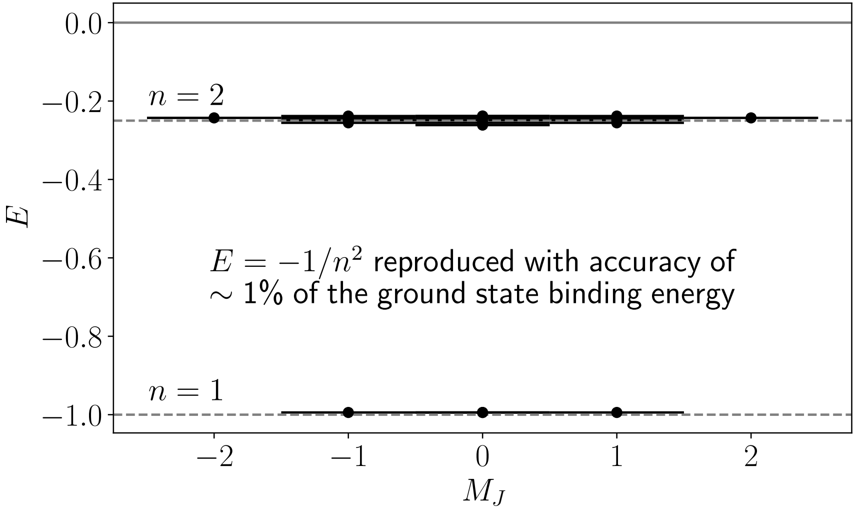
→ Increasing resolution, K, improves the situation

→ m_g should not be taken too small

Convergence with respect to m_g

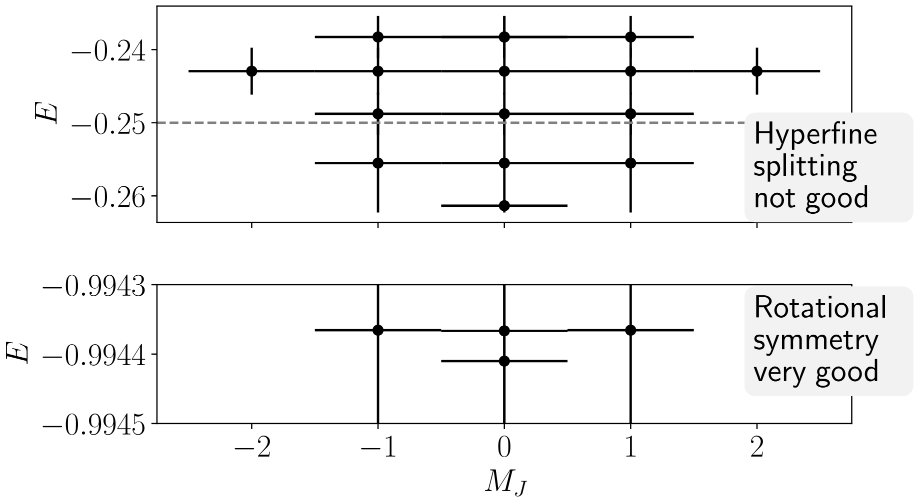


Positronium spectrum



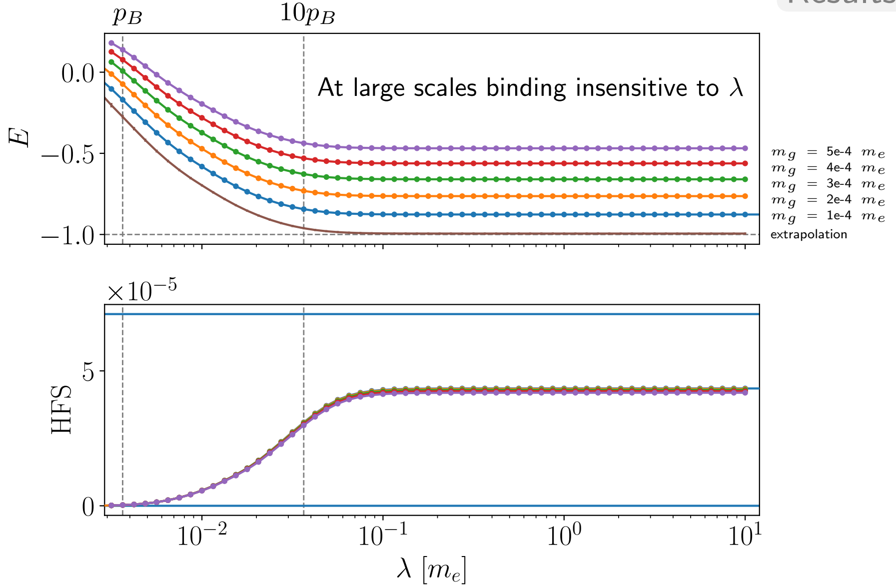
Each point is an $m_g \to 0$ extrapolation obtained for $N_{\max} = 16$, $K = 5793$, and $K_n = 91$.

Positronium spectrum, closer look



Each point is an $m_g \to 0$ extrapolation obtained for $N_{\max} = 16$, $K = 5793$, and $K_n = 91$.

Renormalization scale dependence



- BLFQ framework set up for calculations with RGPEP interactions
- Effective interactions from RGPEP tested in an elementary way
- We control uncertainties due to extrapolations
- Hyperfine structure is challenging for physical positronium ($\alpha = 1/137$)
- We can confidently apply the method for heavy quarkonia in QCD