THE CHIRAL PION DECAY CONSTANT
with massive gluons

Marcela Peláez, N. Barrios, U. Reinosa, J. Serreau,
M. Tissier and N. Wschebor

Light Cone

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UNIVERSIDAD DE LA REPÚBLICA URUGUAY
PEDECIBA MEC-UDELAR
FACULTAD DE CIENCIAS UDELAR fcien.edu.uy

Outline

The pion decay constant: F_π

- It contributes in the leptonic pion decay $\pi^- \to \ell^- \bar{\nu}_\ell$ for $\ell = e, \mu$
- The contribution due to the strong interaction comes from the coupling of the pion to the axial current

$A^{\mu,i} = \bar{\psi}(x) i\gamma_5 \gamma^\mu \sigma^i \psi(x)$

$$\langle 0 | A^i_{\mu,R}(x) \left| \Pi^j(p) \right\rangle = i p_\mu e^{ipx} \delta^{ij} F_\pi$$

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Which diagrams we must include here?

Small parameters in ghost-gluon sector

- Small parameters even in the infrared.
- **Advantage of small parameters:** allow the contributions of the different diagrams to be arranged in a controlled way.

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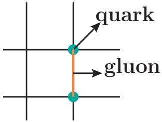
Infrared regime

Perturbation theory not valid at low momentum for the standard Faddeev-Popov Lagrangian.

- Alternatives: e.g. Dyson-Schwinger equations, fRG, Hamiltonian approach...
- Lattice simulations



$$\langle \mathcal{O} \rangle = \underbrace{\frac{1}{\text{conf}} \sum_{\text{conf}} \mathcal{O}[\text{conf}]}_{\text{conf probability } e^{-S}}$$

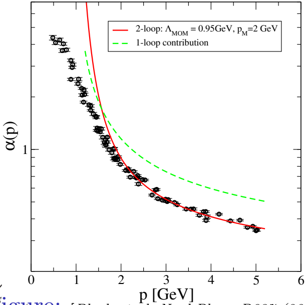


Figure: [*Bloch et al, Nucl.Phys. B687 (2004)*]

Infrared regime

- **Observation: Finite coupling constant.**

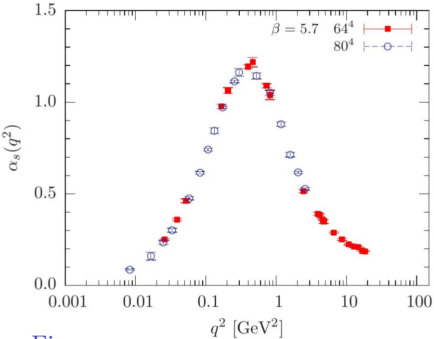


Figure: [*Bogolubsky et al. Phys. Lett. B676 (2009)*]

So, why does perturbation theory not work at all?

The Faddeev-Popov procedure is not justified at low momentum.

Moreover

- Lattice simulations show a mass generation for the gluon propagator

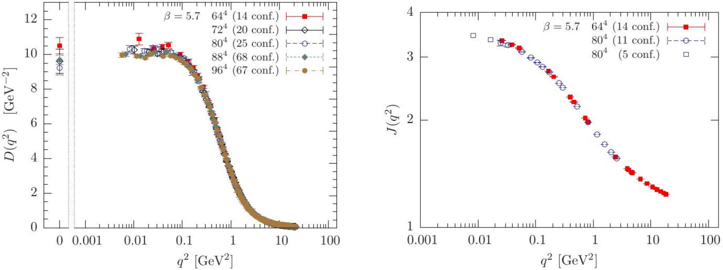


Figure: *[I. L. Bogolubsky et al. Phys. Lett. B676 (2009)]*

The model: Massive gluons (Curci-Ferrari)

- **Curci-Ferrari Lagrangian in Landau gauge:**

$$\mathcal{L} = \mathcal{L}_{\text{inv}} + ih^a\partial_\mu A_\mu^a + \partial_\mu \bar{c}^a (D_\mu c)^a + \frac{\mathbf{m^2}}{\mathbf{2}} \mathbf{A_\mu^a A_\mu^a}$$

[*Curci-Ferrari (1975)*]

- It still has a modified-BRST symmetry which allows to prove renormalizability.
- It is posible to use a Infrared safe renormalization scheme.

We would like to check

... if the perturbative analysis reproduces the lattice data

Gluon propagator & Ghost dressing function

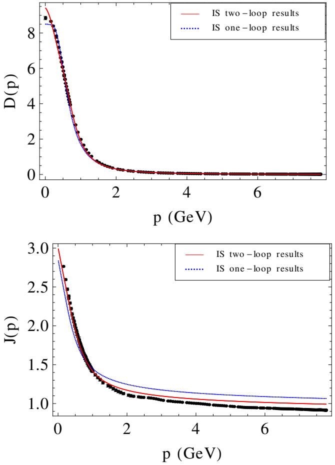
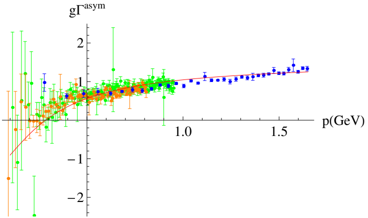


Figure: [J.A. Gracey et al. Phys.Rev.D 100 (2019) 3, 034023]

Vertices

Three-gluon vertex & Ghost-gluon vertex



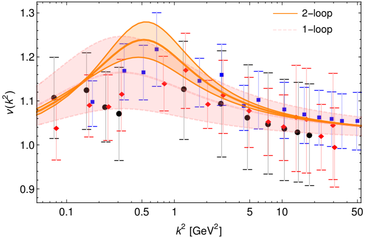


Figure: Lattice data from [A.C. Aguilar et al. Phys.Lett.B 818 (2021) 136352] & [E. Ilgenfritz et al. Braz. J. Phys. 37 (2007)]

Two loop results: Coupling constant

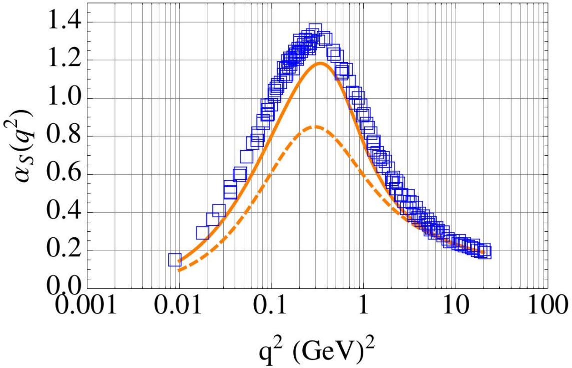


Figure: [J.A. Gracey, MP et al Phys.Rev.D 100 (2019) 3, 034023]

Ghost-Gluon sector

Correlation functions in the ghost-gluon sector

$\Downarrow$

Perturbative despription in Curci-Ferrari model.

Quark-Gluon coupling VS Ghost-Gluon coupling

- Quark-gluon coupling constant not too small.

$$g_q(\mu) = g_g(\mu)\lambda_1(\mu)$$

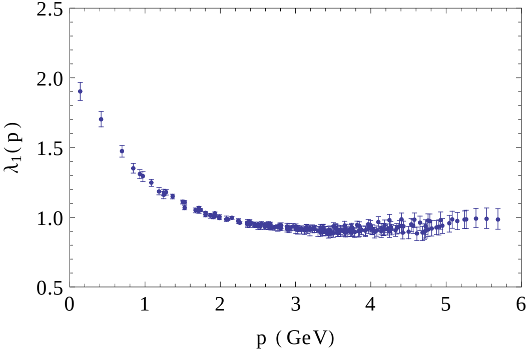
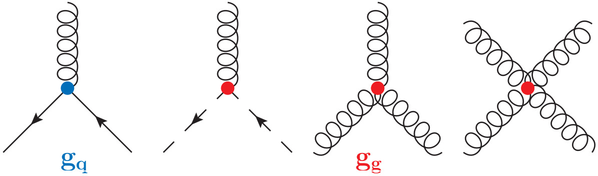


Figure: Data from [*Skullerud et al. JHEP* **0304**, *047 (2003)*]

Quark-Gluon coupling VS Ghost-Gluon coupling

- As the quark-gluon g_q and YM g_g running coupling constants are different in the infrared, we treat them separetly,



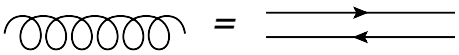
- g_g is considered as small parameter. Yang-Mills sector can be studied perturbately in the infrared.

Large N_c limit

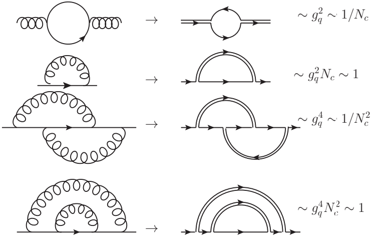
- Large N_c limit shows the same general features of QCD.

[*G. 't Hooft, Nucl. Phys. B* **75**, *461 (1974). Witten, Nucl. Phys. B* **160**, *57 (1979)*]

In the large N_c limit, gluon propagators can be replaced by double color lines and



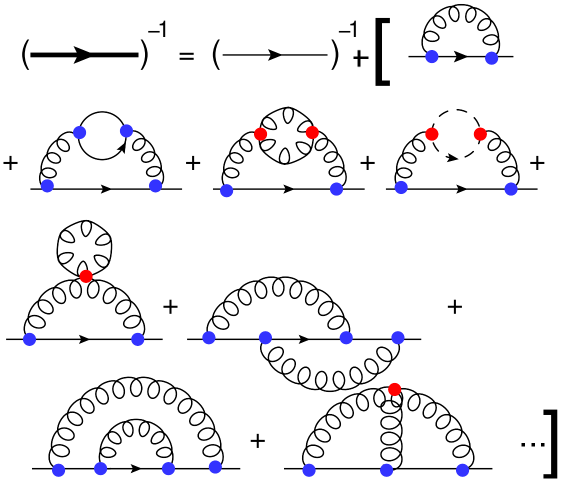
$$g_q \sim 1/\sqrt{N_c}$$



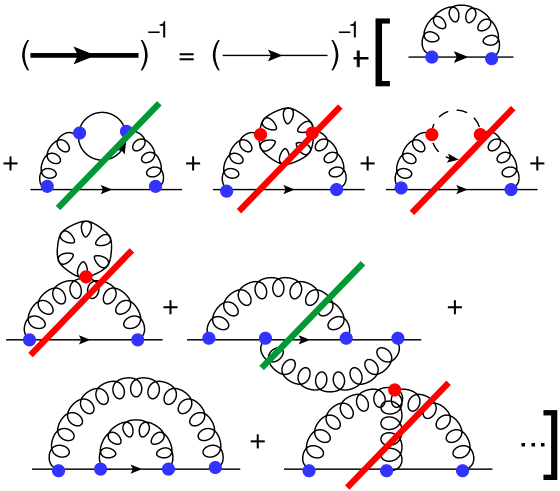
Organizing the systematic expansion:

- How to implement the systematic expansion, ℓ-order improved expansion:
  - We write all diagrams until ℓ-loops
  - We count the powers of g_g and $1/N_c$
  - We also add higher loop order diagrams with the same powers of g_g and $1/N_c$.

Quark propagator

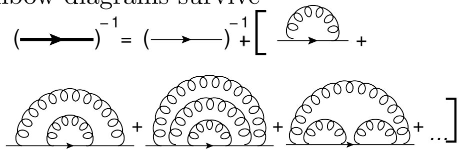


Quark propagator

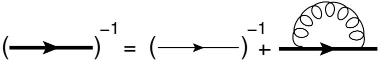


Rainbow equation

- Only Rainbow diagrams survive



- They can be resumed in:



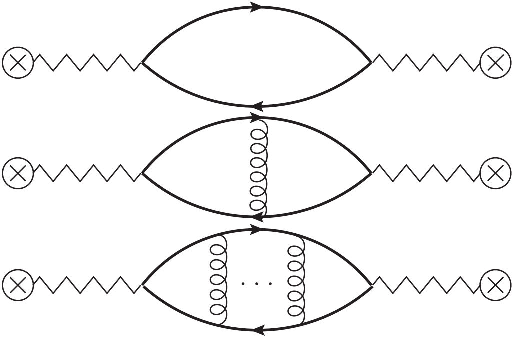
which is the well-known Rainbow approximation for the quark propagator.

see e.g. [*Johnson et al, PRB (1964). Maskawa, PTP (1975). Atkinson et al, PRD (1988). Miransky et al, PRC (2004).*]

[Maris et al, IJMP (2003). Roberts et al,EPJST (2007). Eichmann et al, PRC (2008).]

Meson propagator

We would like to use the one-loop NLO-RI approximation to study meson properties.



- We can answered:

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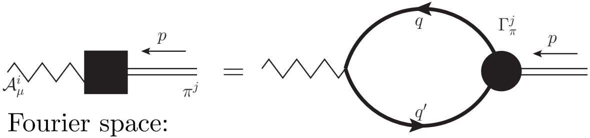
Which diagrams we must include here?

Pion decay constant

- The diagrams we must include correspond to



In the Fourier space:

$$G^{ij}_{\mathcal{A}_\mu \pi}(p) = -\int_q \mathrm{tr}\left[i\gamma_\mu \gamma_5 \sigma^i S(q) \Gamma^j_\pi(q,q') S(q')\right]$$

Rainbow-ladder Bethe-Salpeter equation

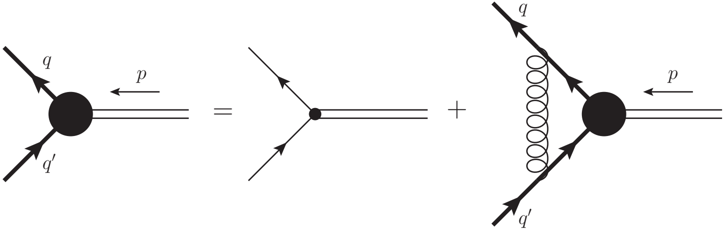


Figure: Meson Bethe-Salpeter equation at RI-one-loop order

The ladder aproximation for Bethe-Salpeter equation is:

$$\Gamma^i_\pi(q,q') = i\gamma_5\sigma^i - \lambda_\Lambda \int_k G_{\mu\nu}(k)\gamma_\mu S(\ell)\Gamma^i_\pi(\ell,\ell')S(\ell')\gamma_\nu$$

where $\lambda_\Lambda = C_F g^2_\Lambda$

- Writing

$$\Gamma^i_\pi(q,q') = i\gamma_5\sigma^i \frac{m_\pi^2}{\mathcal{M}_\Lambda}\frac{\gamma_\pi(q,q')}{p^2+m_\pi^2},$$

where $\gamma_\pi(q,q')$ is regular when $p^2 \to -m_\pi^2$.

- The **symmetries** of the problem give:

$$\gamma_\pi(q,q') = \boldsymbol{\gamma_P}(q,q') + i\sigma_{\mu\nu}q_\mu q'_\nu \boldsymbol{\gamma_T}(q,q') + i\gamma_\mu\left[q_\mu \boldsymbol{\gamma_A}(q,q') - q'_\mu \boldsymbol{\gamma_A}(q',q)\right]$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu]$ and where $\gamma_{P,T,A}$ are real scalar functions.

The chiral limit

- Introducing the quark-antiquark relative momentum $r = (q + q')/2$, **expanding Bethe-Salpeter around** $p^2 = 0$ **up to linear order**

$$\gamma_{P,T}(q, q') = \gamma_{P,T}(r^2) + \mathcal{O}(p^2)$$
$$\gamma_A(q, q') = \gamma_A(r^2) + \frac{p \cdot r}{r^2}\left[\gamma_A(r^2) - \gamma_B(r^2)\right] + \mathcal{O}(p^2),$$

- This reduces to a set of one-dimensional integral equations for the functions $\gamma_{P,T,A,B}(r^2)$.

  - The equation for $\gamma_P(r^2)$ actually decouples.
  - $\gamma_P(r^2) = M(r^2)/Z(r^2)$, as expected from the **Ward identity**.

Renormalization

- Afeter renormalization:

$$\gamma_R(x;\mu_0^2) = g_q^2(\mu_0^2)\int_0^\infty dy\,\mathcal{K}_R(x,y;\mu_0^2)\gamma_R(y;\mu_0^2)$$

- Using renormalization group methods: $\mathcal{K}_R$ and γ_R at different scales:

$$\gamma_R(y;x) = \frac{Z_\psi^{-1}(y)}{Z_\psi^{-1}(x)}\gamma_R(y;y), \quad \mathcal{K}_R(x,y;x) = \frac{Z_\psi^2(y)}{Z_\psi^2(x)}\mathcal{K}_R(x,y;y)$$

We also note that, at this order, $Z_\psi(x)$ is finite.

- Defining $\hat{\gamma}(x) = Z_\psi(x)\gamma_R(x;x)$, we obtain the RG-improved equation

$$\hat{\gamma}(x) = g_q^2(x)\int_0^\infty dy\mathcal{K}_R(x,y;y)\hat{\gamma}(y)$$

- The system of $\hat{\gamma}_{T,A,B}(x)$ is coupled with the running of $M(x), Z_\psi(x), g_q(x)$ and the running of the gluon mass $m(x)$.

$$\hat{\gamma}_T(x) = \frac{\lambda(x)}{16\pi^3}\int_0^\infty dy \left(\frac{x+y}{2x} f_{m^2}(x,y) + \frac{(x-y)^2}{2x}\Delta f_{m^2}(x,y) + \Delta I_{m^2}(x,y)\right)\hat{N}(y)$$

$$\hat{\gamma}_A(x) = \frac{\lambda(x)}{16\pi^3}\int_0^\infty dy \Big\{ \left[f_{m^2}(x,y) - \Delta I_{m^2}(x,y)\right]\hat{H}(y) - \left[2I_{m^2}(x,y) + (x-y)\Delta I_{m^2}(x,y)\right]\hat{L}(y)\Big\}$$

$$\hat{\gamma}_B(x) = \frac{3\lambda(x)}{16\pi^3}\int_0^\infty dy \Big\{ -\Delta I_{m^2}(x,y)\hat{H}(y) + \left[y f_{m^2}(x,y) - 2I_{m^2}(x,y) + y\Delta I_{m^2}(x,y)\right]\hat{L}(y)\Big\}$$

where the gluon mass is the running one at the scale x, $m^2 \equiv m^2(x)$

$$\hat{N}(x) = \hat{N}^{\mathrm{source}}(x) + \frac{[x - M^2(x)]\hat{\gamma}_T(x) - 2M(x)\hat{\gamma}_A(x)}{\left[x + M^2(x)\right]^2}$$

$$\hat{H}(x) = \hat{H}^{\mathrm{source}}(x) + \frac{2xM(x)\hat{\gamma}_T(x) + \left[x - M^2(x)\right]\hat{\gamma}_A(x)}{\left[x + M^2(x)\right]^2}$$

$$\hat{L}(x) = \hat{L}^{\mathrm{source}}(x) + \frac{xM(x)\hat{\gamma}_T(x) + \left[2x + M^2(x)\right]\hat{\gamma}_A(x) - \left[x + M^2(x)\right]\hat{\gamma}_B(x)}{x\left[x + M^2(x)\right]^2}$$

Chiral Pion decay constant

In the Fourier space:

$$G^{ij}_{\mathcal{A}_\mu \pi}(p) = -\int_q \mathrm{tr}\left[i\gamma_\mu\gamma_5\sigma^i S(q)\Gamma^j_\pi(q,q')S(q')\right]$$

And then in the chiral limit:

$$-ip_\mu f_\pi^2 = N_c \int_q \mathrm{tr}\left[\gamma_\mu S(-q)\gamma_\pi(q,q')S(q')\right]_{p^2\to 0}$$

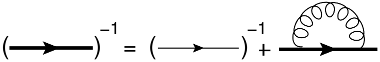
$$f_\pi^2 = \frac{N_c}{4\pi^2}\int_0^\infty dx \frac{x Z_\psi(x)}{\left[x+M^2(x)\right]^2}\Big\{M(x)\Big[M(x)-\frac{x}{2}M'(x)\Big] + \frac{3}{2}M(x)[x\hat{\gamma}_T(x)-M(x)\hat{\gamma}_A(x)] + \frac{x+M^2(x)}{2}\hat{\gamma}_B(x)\Big\}.$$

- This equation is exact in the chiral limit.
- Retaining only the first line corresponds to the Pagel-Stokar approximation [1], which, thanks to Ward identity ($\gamma_P(x) = \frac{M(x)}{Z(x)}$), provides an expression involving only the quark propagator.

[Peláez et al. Phys.Rev.D 107 (2023) 5, 054025]

Numerical solution.

- First we need to compute the quark propagator.



Including the running of the gluon mass and the couplings.

- The β-function for g_q takes the form:

$$\beta_{g_q} = \mu \frac{dg_q}{d\mu}|_{g_\Lambda} = g_q(\gamma_\psi + \frac{1}{2}\gamma_A) + g_q \mu \frac{d\lambda_1^\Lambda(ren)}{d\mu}$$

$$\beta_{g_g} = g_g(\gamma_C + \frac{1}{2}\gamma_A)$$

$$\beta_{m^2} = m^2(\gamma_C + \gamma_A)$$

- The relation between both coupling constants at $10GeV$ (the starting point of the flow) can be done perturbately.
- In this case the flow of the coupling constant is determined only by one parameter, $g_g(10GeV)$.
- γ_C and γ_A are computed in its one loop form (with full quark propagators).
- λ_1, γ_A and γ_ψ are also coupled with $M(p)$ and $z_\psi(p)$.

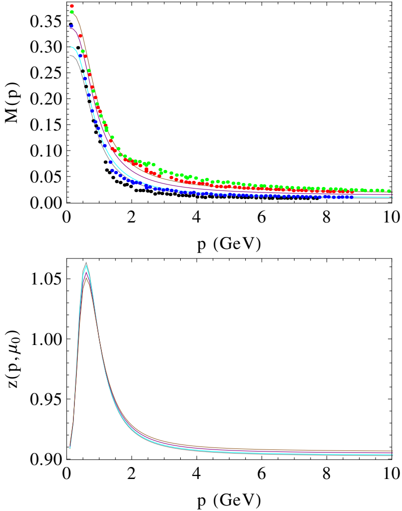


Figure: M(p), with initial condition M(10GeV)=0.008, 0.01, 0.015, 0.02 GeV, g_g(10GeV)=1.57 and m(10 GeV)=0.21 GeV
[M. Peláez et al Phys.Rev.D 103 (2021) 9, 094035]

- We reproduce spontaneus chiral symmetry breaking.
- Same UV behaviour as in [Aguilar et al PRD **83** (2011).]

Bethe-Salpeter

- Once $M(x), Z(x), g_q(x), m(x)$ are computed. We solve RG-improved Bethe-Salpeter equation:

$$\hat{\gamma}(x) = g_q^2(x) \int_0^\infty dy \mathcal{K}_R(x, y; y) \hat{\gamma}(y)$$

with initial condition: $\hat{\gamma}_{T,A,B}(x) = 0$.

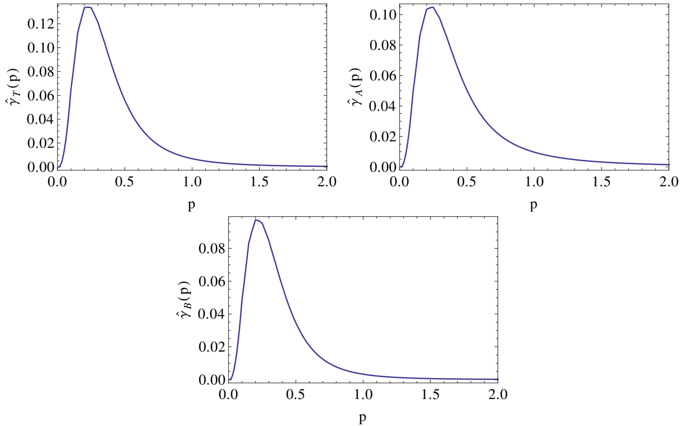


Figure: Scalar functions $\hat{\gamma}_{T,A,B}$ as functions of the momentum p for $g_0 = 1.93$ and $m_0 = 0.11$ GeV. All units are in GeV. [M. Peláez, U. Reinosa, J. Serrau,N. Wschebor, Phys.Rev.D 107 (2023) 5, 054025]

Solving by iterations.

$$f_\pi^2 = \frac{N_c}{4\pi^2}\int_0^\infty dx \frac{x Z_\psi(x)}{\left[x + M^2(x)\right]^2}\Big\{M(x)\Big[M(x) - \frac{x}{2}M'(x)\Big]$$
$$+\frac{3}{2}M(x)[x\hat{\gamma}_T(x) - M(x)\hat{\gamma}_A(x)] + \frac{x+M^2(x)}{2}\hat{\gamma}_B(x)\Big\}.$$

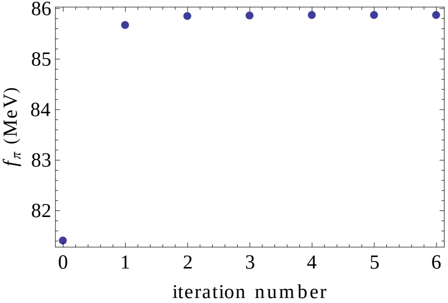


Figure: Evolution of f_π with the number of iterations for $g_0 = 1.93$

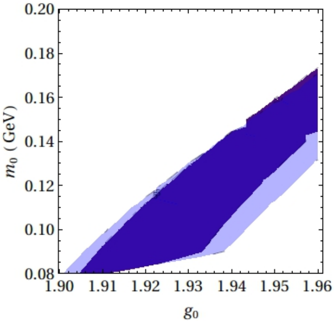


Figure: The region in parameter space where $|f_\pi - f_\pi^*|/f_\pi^*$ is less than 5% (MeV (dark blue) compared to that where the overall error for the quark mass function less than 15% (light blue).

Conclusions

To sumarize:

- **Perturbative CF in ghost-gluon sector✓**.
- Of course pCF does not reproduce the hole picture for quarks.
- Based on the fact that, at low energies, **the coupling g_g differs significantly from the coupling g_q in the matter sector** we treat both constant on different footing.

Conclusions

To sumarize:

- We propose a **systematic expansion** scheme for QCD at low energy based on a **double expansion** in powers of the coupling strength $\mathbf{g_g}$ in the Yang-Mills sector of the theory and in powers of $\mathbf{1/N_c}$.
- At leading order, this scheme reproduces the well-known **rainbow approximation**.

Conclusions

Conclusions

- The same scheme takes us to the **Rainbow-ladder Bethe Salpeter equation**
- We can give an **expression for** f_π within CF model in the chiral limit.
- We obtain the expected f_π factor for the parameters that well reproduce the quark propagator.

Future work

- Computing the masses of the pion and kaon.

Thanks