Hot perturbative QCD in a very strong magnetic background

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Strong motivation: in-medium strong interactions under extreme magnetic fields are:

- of experimental relevance
  - HICs, early universe, magnetars
- rich in new phenomenology
  - Chiral magnetic effect, new QCD phase diagram, vacuum SUC
- amenable to lattice simulations: new open channel for comparison!
  - model constraining, tests for pQCD and nonpert. methods, ...

However…

Strong motivation: in-medium strong interactions under extreme magnetic fields are:

- of experimental relevance
  - ✧ HICs, early universe, magnetars

Real deal, BUT very hard for theory: difficult to compute with reasonable control over approximations & hypotheses!

- rich in new phenomenology
  - ✧ Chiral magnetic effect, new QCD phase diagram, vacuum SUC

- amenable to lattice simulations: new open channel for comparison!
  - ✧ model constraining, tests for pQCD and nonpert. methods, …

Here we can play!

From my perspective:

failure of effective chiral models (long story...)

Let's try something else

Bag model

[ESF & Palhares (2012)]

surprisingly good qualitative results - puzzling, hard to justify by itself.

Large N

[ESF, Noronha & Palhares (2013)]

good qualitative results - QCD in that limit! But hard to get more quantitative or improve...

Magnetic pQCD

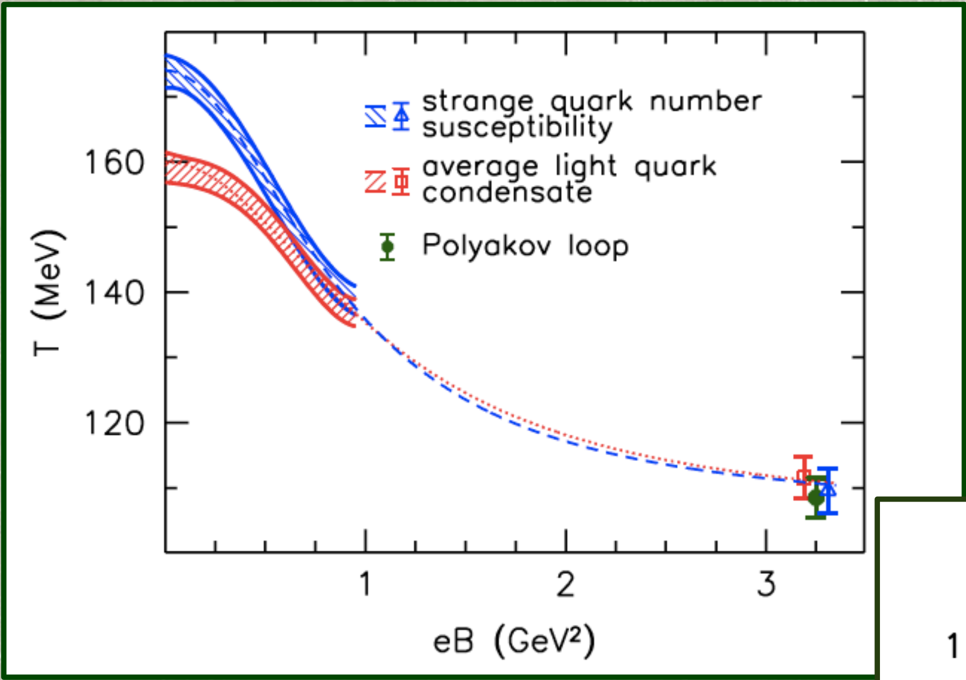
[Palhares (2012); Blaizot, ESF & Palhares (2013)]

QCD in that limit! Systematic & controllable approximations, but...

...needs really high B fields! Actually, needs

$$m_s \ll T \ll \sqrt{eB}$$

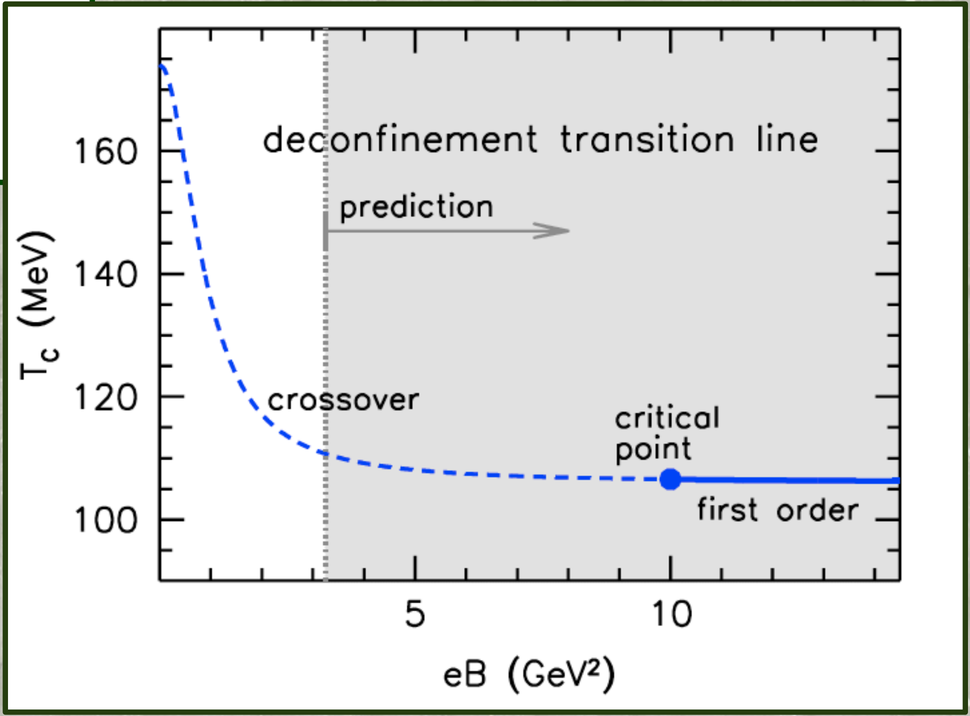
At extremely large values of B, interesting things might happen…



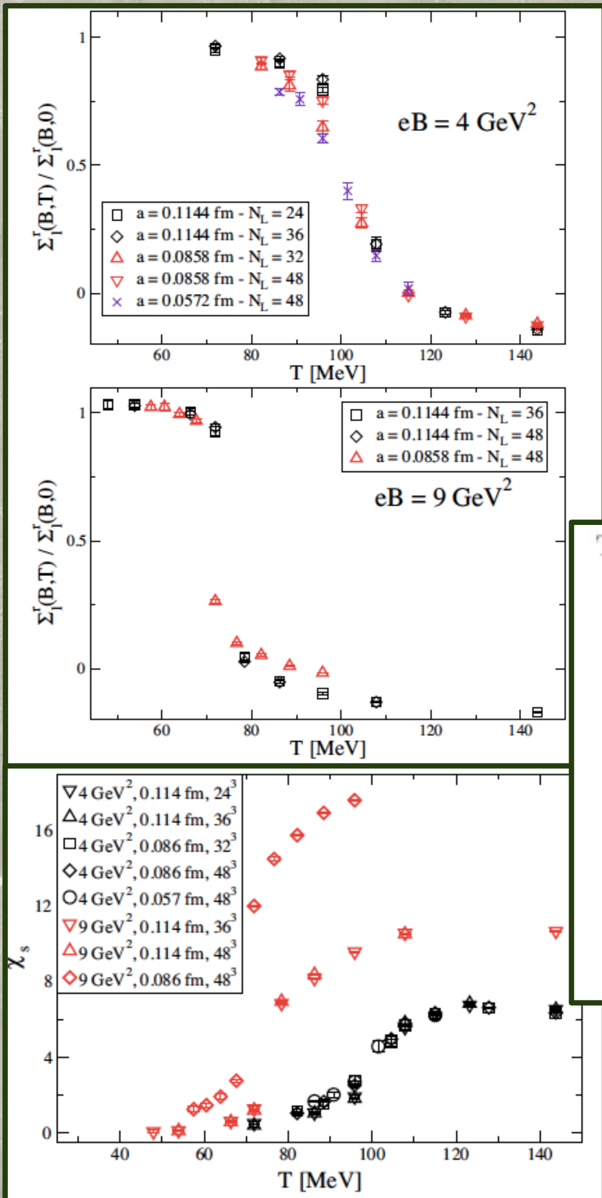
Very exciting lattice results at larger B and predictions for extremely large B !

[Endrodi (2015)]

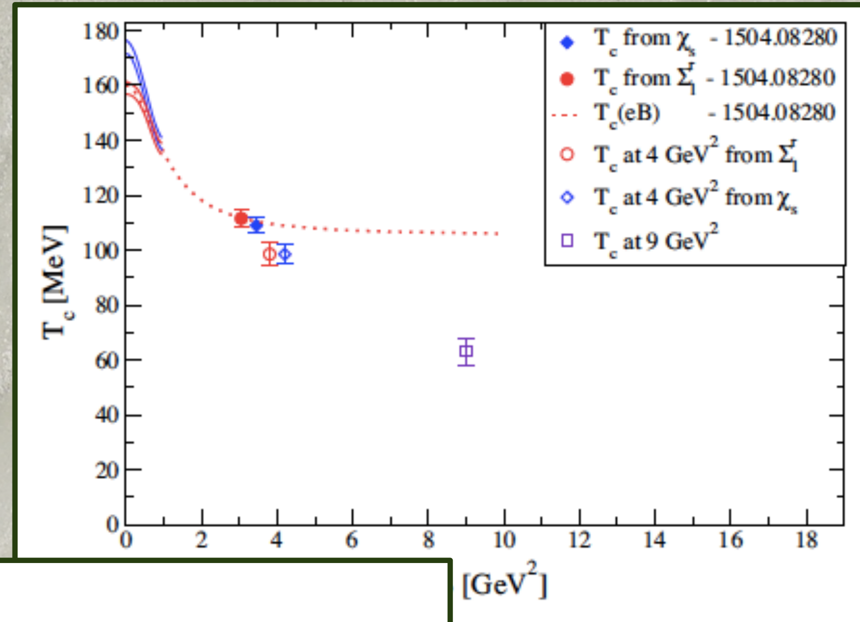
… as was discussed ~10 years ago in Endrodi's pioneering investigation.

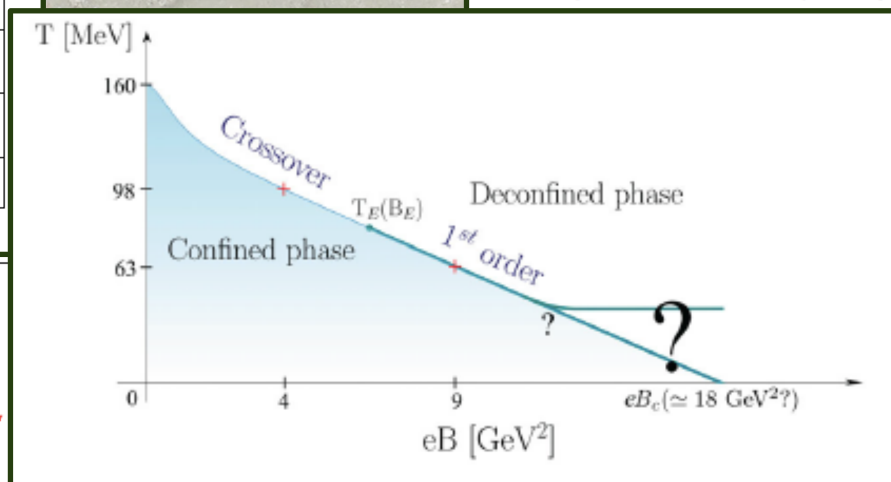


Almost ten years later, D'Elia et al revisited this problem... for higher fields!



[D'Elia et al (2022)]





$$m_s \ll T \ll \sqrt{eB}$$

Not quite yet, but we can play!

Besides Lattice QCD, which other tool can provide predictions from the actual fundamental gauge theory?

Perturbation theory!

pQCD to $O(g^2)$ in a nonperturbative magnetic background

[Blaizot, ESF & Palhares (2013)]

Framework: perturbative QCD in a nonperturbative magnetic background.

Quark-gluon interaction

up to $O(g^2)$

Exact quark propagator in a constant and uniform magnetic field:

$$S_0 = \left[i\partial\!\!\!/ - q_f A\!\!\!/_{\rm cl}(x) - m_f \right]^{-1}$$

$$A_{\rm cl}(x) = (0, \vec{A}(x)) \;\; | \;\; \nabla \times \vec{A} = B\hat{z}$$

Basic ingredients

- Exact fermion propagator [Schwinger (1951); Chodos et al (1990)]

$$S_0(x,y) \;=\; \Phi(x,y)\,\overline{S}_0(x-y)$$

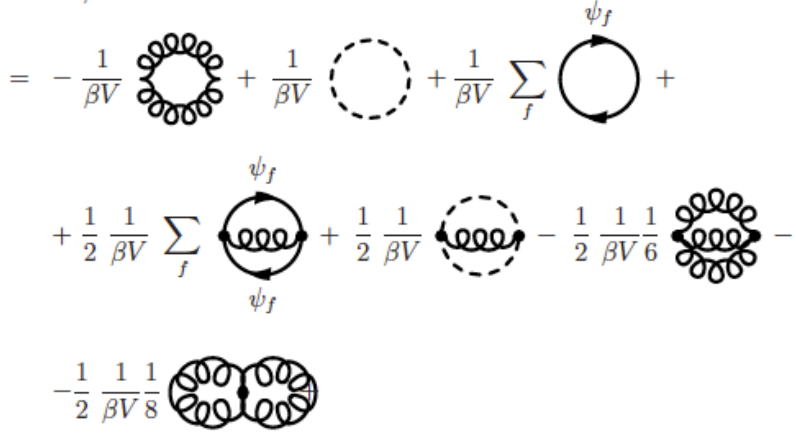
$$\Phi(x,y) \;\equiv\; \exp\left[iq\int_x^y dx'_\mu A^\mu_{\rm cl}(x')\right]$$

$$\overline{S}_0(P) \;=\; i\exp\left[-\frac{\mathbf{p}_T^2}{qB}\right]\sum_{n=0}^{\infty}(-1)^n\frac{D_n(qB,P)}{\mathbf{p}_L^2-m_f^2-2nqB} \quad =$$

- Thermodynamic potential:

(gluonic part from usual hot pQCD + magnetically-dressed quarks)

$$\Omega_{QCD} \;\equiv\; -\frac{1}{\beta V}\ln Z_{QCD}$$



$$+\ [\textit{diagrams with counterterms}]\ +\ O(3\ \textit{loops}),$$

Exchange diagram in a magnetic background (in the LLL approx.):

[Palhares (2012); Blaizot, ESF & Palhares (2013)]

 $^{\mathrm{LLL}} = \left(\frac{q_f B}{2\pi}\right) \int \frac{dk_1 dk_2}{(2\pi)^2}\, \mathrm{e}^{-\frac{k_1^2+k_2^2}{2q_f B}}$  $^{\bar{d}=2;\; m_k^2 = k_1^2+k_2^2}$.

Average over gluon "transverse mass"

exchange contribution in
dim. 2 hot QCD with a "massive gluon"

Results:

- Clear dimensional reduction in the quark dynamics.
- There are no UV divergences.
- In D=1+1, the Dirac trace is proportional to the quark mass: trivial chiral limit!

$$
\begin{aligned}
\frac{P_{\mathrm{exch}}}{N_c} = & -\frac{1}{2}\, g^2 \left(\frac{N_c^2-1}{2}\right) m_f^2 \times \\
& \left(\frac{q_f B}{2\pi}\right) \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2}\, \mathrm{e}^{-\frac{\mathbf{k}_\perp^2}{2q_f B}} \int \frac{dp_3 dq_3 dk_3}{(2\pi)^3}\, (2\pi)\delta(k_3 - p_3 + q_3) \times \\
& \frac{1}{\omega E_{\mathbf{p}} E_{\mathbf{q}}} \left\{ \frac{\omega\, \Sigma_+}{E_-^2 - \omega^2} + \frac{\omega\, \Sigma_-}{E_+^2 - \omega^2} + 2\left[\frac{E_+}{E_+^2 - \omega^2} - \frac{E_-}{E_-^2 - \omega^2}\right] n_B(\omega)\, N_F(1) \right. \\
& \left. - \left[\frac{2(E_{\mathbf{q}} + \omega)}{(E_- - \omega)(E_+ + \omega)}\right] N_F(1) - 2\, \frac{E_+}{E_+^2 - \omega^2}\, n_B(\omega) - \frac{1}{E_+ + \omega} \right\},
\end{aligned}
$$

Very evil (IR-divergent) sum-integrals to deal with, BUT:

- NLO always 1-2 orders of magnitude below the leading contribution.
  - ➡ Improved convergence of the perturbative series at high T and extremely large B?
- No IR divergence -> trivial chiral limit! [strong suppression for small masses]

At the time (2013):

[Blaizot, ESF & Palhares (2013)]

- No running coupling, no running mass, no bands.
- No other observables.
- No test for ultra-high B fields.

~10 years later, we revisited magnetic pQCD. Now with:

[ESF, Palhares & Restrepo (2023)]

- Running coupling, running mass & bands.
- Dependence on the choice of the renormalization scale.
- Pressure, quark condensate, strange quark number susceptibility.
- Tests for ultra-high B fields ("reached" on the lattice!).

$$m_s \ll T \ll \sqrt{eB}$$

Differently from what was done in 2013, it is more convenient to perform momentum integrals first, and leave one integral over the transverse mass and the Matsubara sums to the end (valid for μ = 0). In the LLL limit:

[ESF, Palhares & Restrepo (2023)]

NLO (exchange) contribution to the pressure:

$$\frac{P_{\rm exch}^{\rm LLL}}{N_c} = \frac{1}{2}g^2\left(\frac{N_c^2-1}{2N_c}\right)T^2\sum_f m_f^2\left(\frac{q_fB}{2\pi}\right)\sum_{\ell,n_2}\int\frac{dm_k}{2\pi}m_k e^{-\frac{m_k^2}{2q_fB}}\frac{\mathcal{E}_\ell-\mathcal{E}_{n_2}}{\mathcal{E}_\ell\mathcal{E}_{n_1}\mathcal{E}_{n_2}|\mathcal{E}_\ell-\mathcal{E}_{n_2}|(|\mathcal{E}_\ell-\mathcal{E}_{n_2}|+\mathcal{E}_{n_1})}$$

$$\mathcal{E}_\ell = \sqrt{\omega_\ell^2+m_k^2},\ \mathcal{E}_{n_1} = \sqrt{(\omega_{n_2}+\omega_\ell)^2+m_f^2},\ \text{and}\ \mathcal{E}_{n_2} = \sqrt{\omega_{n_2}^2+m_f^2}$$

$$\omega_\ell = 2\pi\ell T\ \text{and}\ \omega_{n_2} = (2n_2+1)\pi T$$

We also need (LO, quarks):

$$\frac{P_{\rm free}^{\rm LLL}}{N_c} = -\sum_f\frac{(q_fB)^2}{2\pi^2}[x_f\ln\sqrt{x_f}] + T\sum_f\frac{q_fB}{2\pi}\int\frac{dp_z}{2\pi}\left\{\ln\left(1+e^{-\beta[E(0,p_z)-\mu_f]}\right) + \ln\left(1+e^{-\beta[E(0,p_z)+\mu_f]}\right)\right\}.$$

$$P_{\rm free}^G = 2(N_c^2-1)\frac{\pi^2T^4}{90}$$

(gluons)

$$P_2^G = -N_c(N_c^2-1)\frac{g^2T^4}{144}$$

Chiral condensate

$$\langle\bar{\psi}_f\psi_f\rangle = -\frac{\partial P_f}{\partial m_f} = -\frac{\partial P_{\rm free}^{\rm LLL}}{\partial m_f} - \frac{\partial P_{\rm exch}^{\rm LLL}}{\partial m_f}$$

- Massless quarks: true order parameter for the chiral transition.
- Light quark masses: pseudo order parameter.
- Perturbative analysis reliable only for very large T & even larger B - cannot bring information near the phase transition or crossover.
- Nevertheless: there are lattice results for high T & B —> comparison of two first-principle calculations in this region is certainly relevant!

From magnetic pQCD

$$\frac{\partial P_{\rm free}^{\rm LLL}}{\partial m_f} = -N_c m_f \frac{q_f B}{(2\pi)^2}\left[1+\ln x_f + \int dp_z \frac{2n_F(E_p)}{E_p}\right]$$

$$\frac{\partial P_{\rm exch}^{\rm LLL}}{\partial m_f} = -\frac{1}{2}g^2\left(\frac{N_c^2-1}{2}\right)T^2\left(\frac{q_f B}{2\pi}\right)\sum_{l,n_2}\int\frac{dm_k}{2\pi}m_k e^{-\frac{m_k^2}{2q_f B}}$$
$$\times\left\{\frac{m_f^3[\mathcal{E}_{n_2}|\mathcal{E}_l-\mathcal{E}_{n_2}|-\mathcal{E}_{n_1}(\mathcal{E}_l-\mathcal{E}_{n2})]}{\mathcal{E}_l\mathcal{E}_{n_1}^2\mathcal{E}_{n_2}^2(\mathcal{E}_l-\mathcal{E}_{n_2})(|\mathcal{E}_l-\mathcal{E}_{n_2}|+\mathcal{E}_{n_1})^2} - \frac{m_f(\mathcal{E}_l-\mathcal{E}_{n_2})[2(\omega_{n_2}+\omega_l)^2\omega_{n_2}^2+m_f^2((\omega_{n_2}+\omega_l)^2+\omega_{n_2}^2)]}{\mathcal{E}_l\mathcal{E}_{n_1}^3\mathcal{E}_{n_2}^3|\mathcal{E}_l-\mathcal{E}_{n_2}|(|\mathcal{E}_l-\mathcal{E}_{n_2}|+\mathcal{E}_{n_1})}\right\}$$

• On the lattice, one computes the f-flavor renormalized condensate (to eliminate additive and multiplicative divergences):

$$\Sigma_f^r(B,T) = \frac{m_f}{m_\pi^2 f_\pi^2}\left[\langle\bar{\psi}_f\psi_f\rangle_{B,T} - \langle\bar{\psi}_f\psi_f\rangle_{0,0}\right]$$

• NB: To obtain the vacuum condensate, one cannot simply take the zero-field limit since we assumed very large fields from the outset.

Strange quark number susceptibility

$$\chi^s = \frac{1}{T^2}\frac{\partial^2 P}{\partial\mu_s^2},$$

• Given the presence of a derivative with respect to the chemical potential, pure vacuum terms are excluded. —> advantage when comparing lattice results to pQCD, even if the T range in the simulations is still far from optimal for this purpose.

Running

$$\alpha_s(\bar{\Lambda}) = \frac{4\pi}{\beta_0 L}\left(1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln L}{L}\right)$$

$$m_s(\bar{\Lambda}) = \hat{m}_s\left(\frac{\alpha_s}{\pi}\right)^{4/9}\left[1 + 0.895062\left(\frac{\alpha_s}{\pi}\right)\right]$$

$$\beta_0 = 11 - 2N_f/3,\ \beta_1 = 51 - 19N_f/3,\ L = 2\ln\left(\bar{\Lambda}/\Lambda_{\overline{MS}}\right)$$

$$\Lambda^{2+1}_{\overline{MS}} = 343^{+18}_{-12}\ \mathrm{MeV}$$

$$\hat{m}_s^{2+1} \approx 248.7\ \mathrm{MeV}$$

Arbitrariness in choosing the renormalization scale:

- Thermal QCD: besides quark masses, the only scale is $T \gg m_f$ —> usual choice: Matsubara frequency $2\pi T$ with a band around it ($\pi T < \Lambda < 4\pi T$).
- In the present case, we have 3 mass scales: $(eB)^{1/2}$, T & m_f

In the literature, one can find a few different assumptions for the form of the running coupling.

• We show results for a few representative choices and discuss their implications for our observables.

• Although we have our preference for the most physical choice, we believe that, ultimately, this must be settled by direct comparison to lattice QCD simulations.

• We consider the following cases:

(i) A fixed value of $\alpha_s = 0.336$

[ignore running]

(ii) The running [Ayala et al (2018)]

$$\alpha_s(|eB|) = \frac{\overline{\alpha}_s(\bar{\Lambda}^2)}{1 + (\beta_0/4\pi)\overline{\alpha}_s(\bar{\Lambda}^2)\ln(\frac{\bar{\Lambda}^2}{\bar{\Lambda}^2+|eB|})}$$

$\overline{\alpha}_s(\bar{\Lambda}^2)$ (usual MSbar one-loop running)

$\bar{\Lambda} = 1.5\ \text{GeV}$

[Motivation: provide an understanding of inverse magnetic catalysis]

(iii) Same as previous, but with $\Lambda = 2\pi T$

[Karmakar et al (2019)]

(iv) QCD running with usual thermal choice ($\Lambda = 2\pi T$)

[ignore effect from B on the running scale]

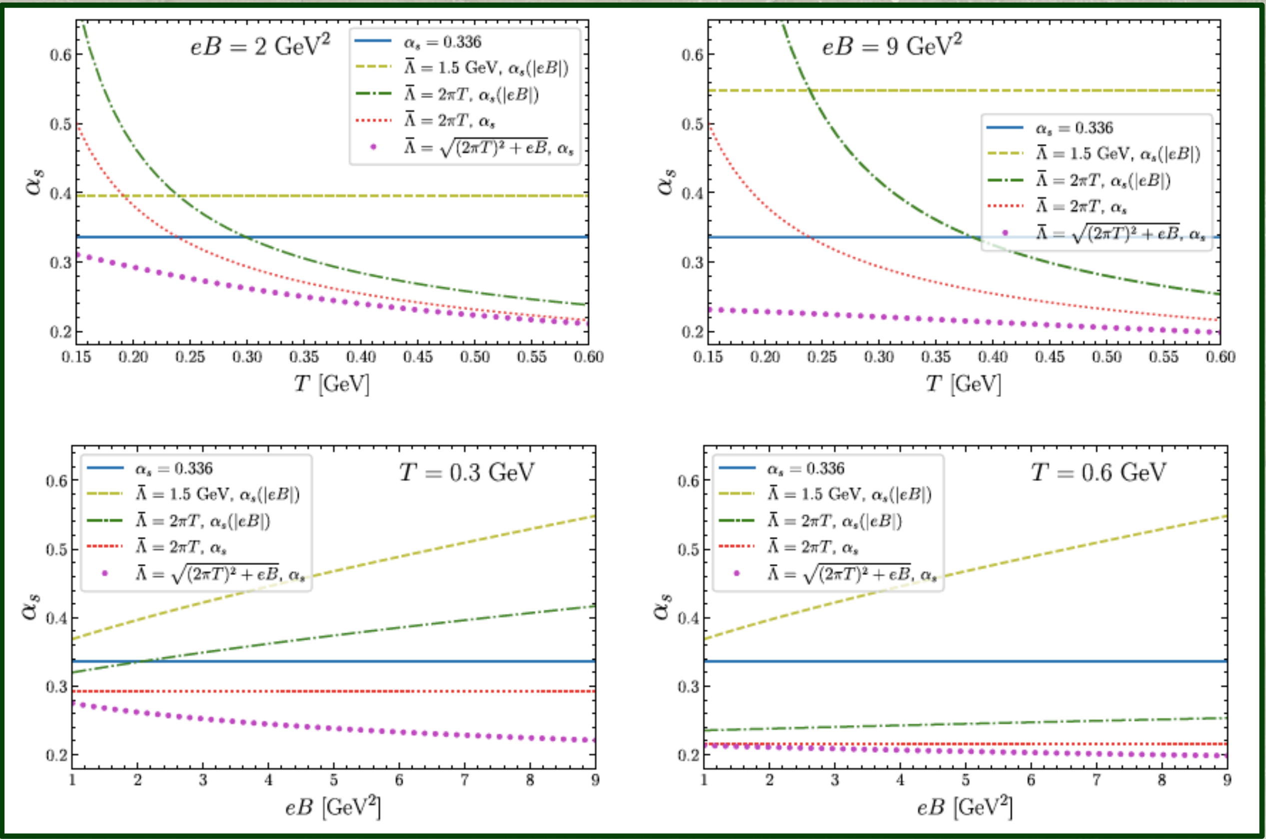
(v) QCD running with

$$\bar{\Lambda} = \sqrt{(2\pi T)^2 + eB}$$

[extension of what is done in in-medium field theory]

Results for the running coupling

[ESF, Palhares & Restrepo (2023)]



• Cases (ii) & (iii) display an unphysical behavior with increasing B: α_s simply grows while the energy density is also increasing. Then:

⋆ Perturbative calculations meaningless for high B.

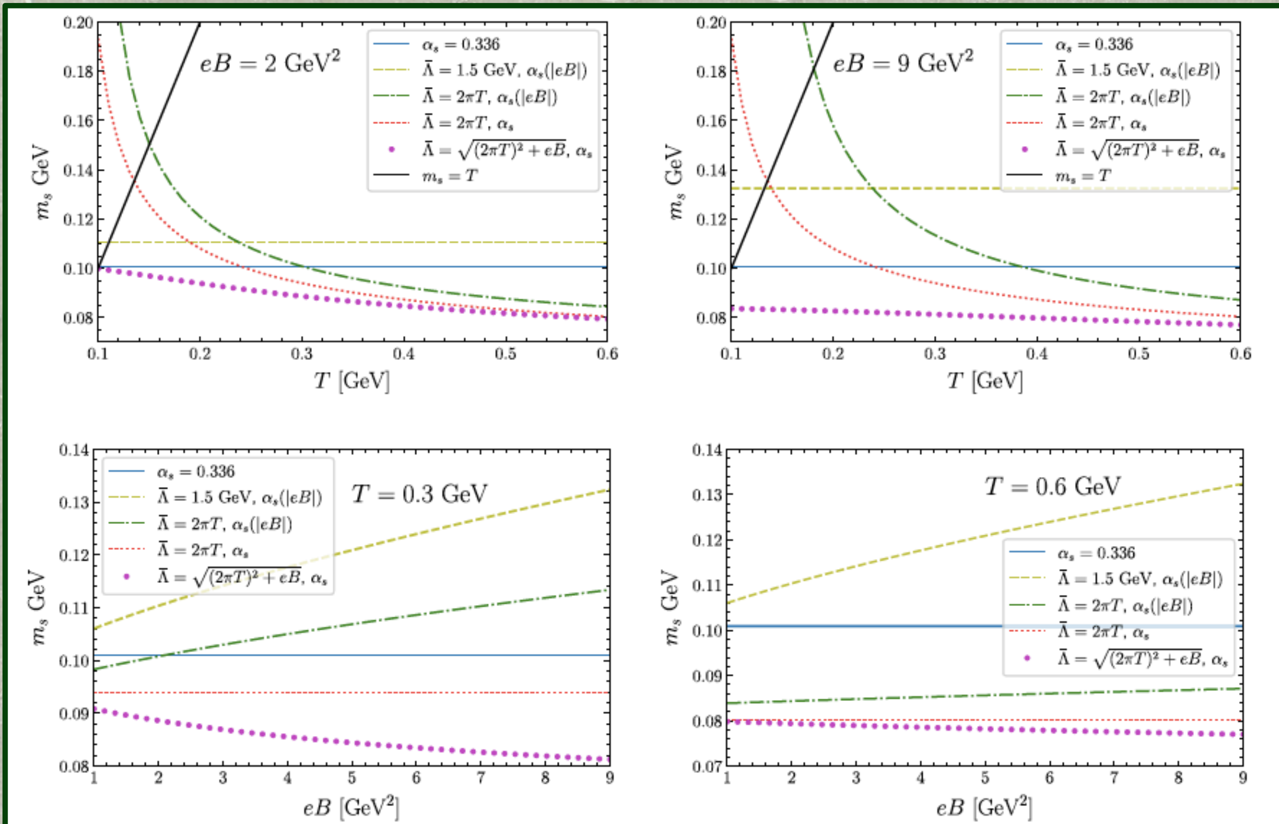
⋆ Incompatible with asymptotic freedom.

• In cases (iv) & (v), α_s exhibits the same qualitative (usual for QCD) behavior.

⋆ Quantitative difference because in case (v) B contributes to the running scale on an equal footing with respect to the temperature.

Results for the running strange quark mass

[ESF, Palhares & Restrepo (2023)]



• Black continuous line for m_s = T (reminder of the constraint $m_s \ll T$).

• Behavior of different running cases analogous to what has been discussed for α_s.

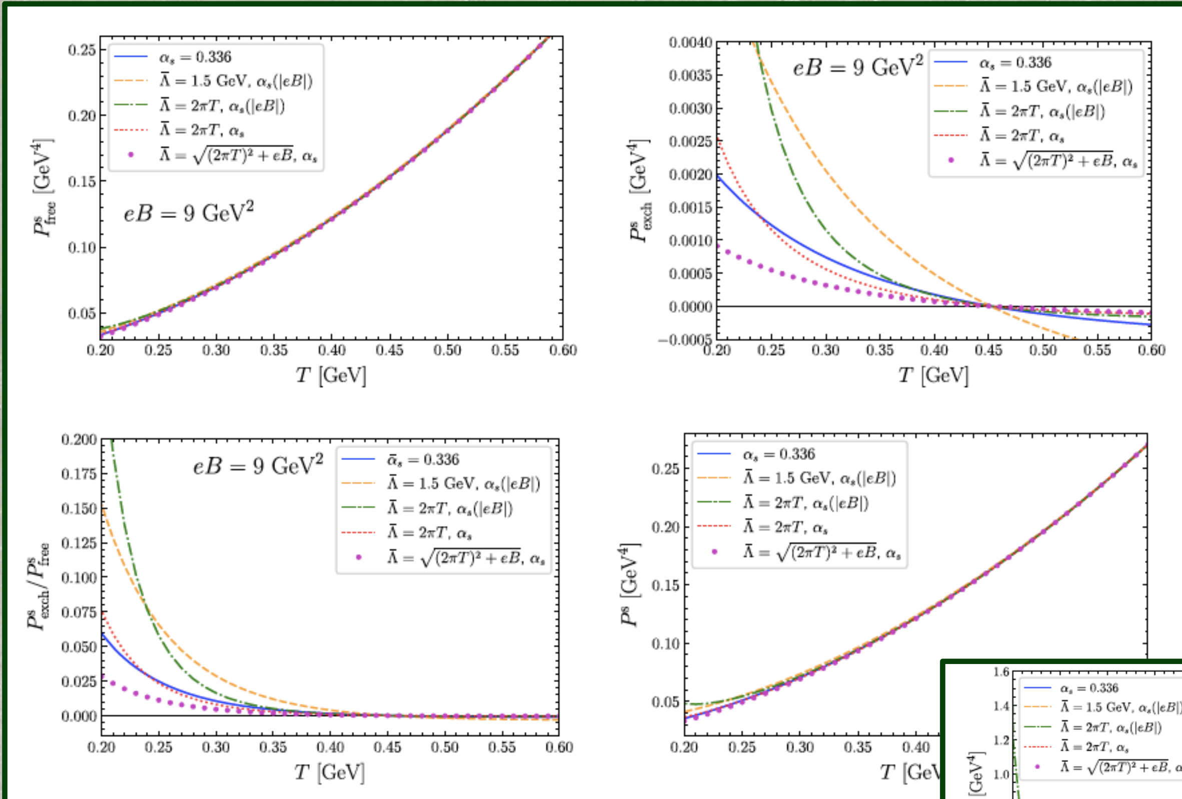
• Quark mass increase with B probably related to the original motivation of running choices like cases (ii) & (iii) — trying to encode magnetic catalysis & inverse magnetic catalysis in the properties of the running of the strong coupling.

• We believe that only cases (iv) & (v) provide a physical description of the running coupling & running quark mass.

• Since it can also be tested by direct comparison to lattice data, we keep all cases in our results for the pressure, chiral condensate & strange quark number susceptibility.

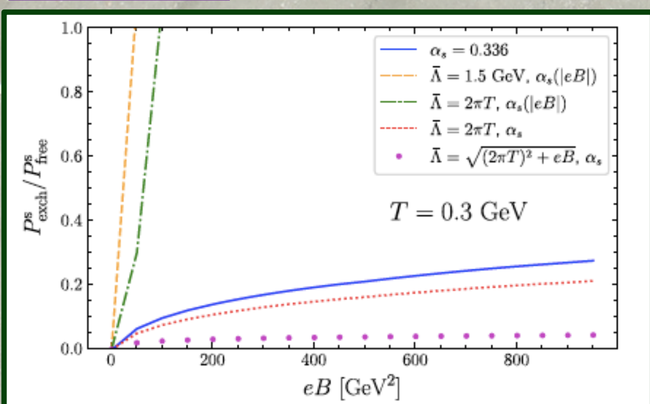
Results for the pressure

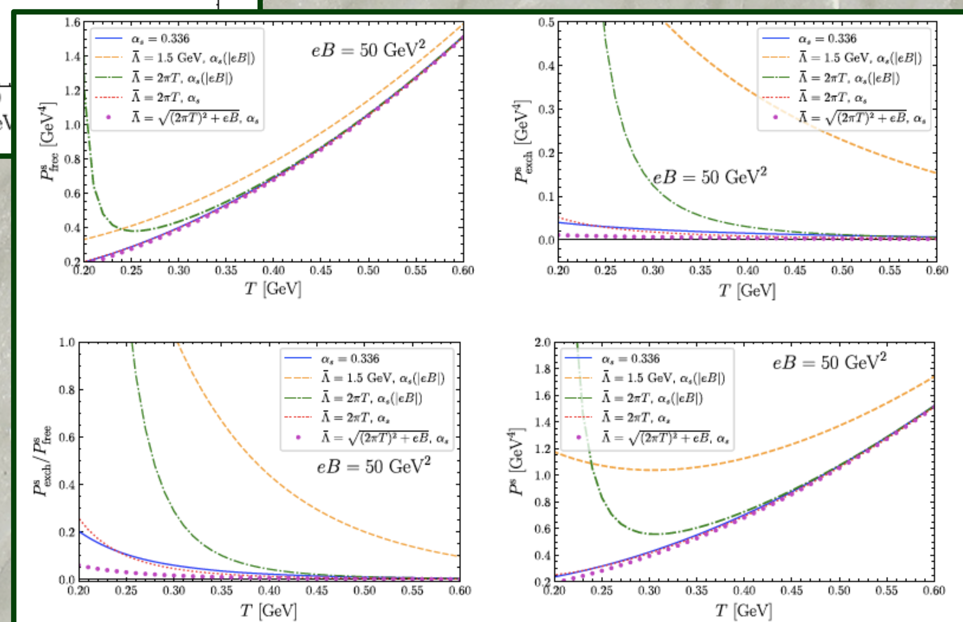
[ESF, Palhares & Restrepo (2023)]



- Free pressure, exchange contribution, ratio, full pressure.
- Results for strange quark because mass effects are more relevant.
- P_{exch}/P_{free}: measure of the reliability of perturbation theory (more well behaved than the case in the absence of a large magnetic field. [Blaizot, ESF & Palhares (2013)]

Huge B fields





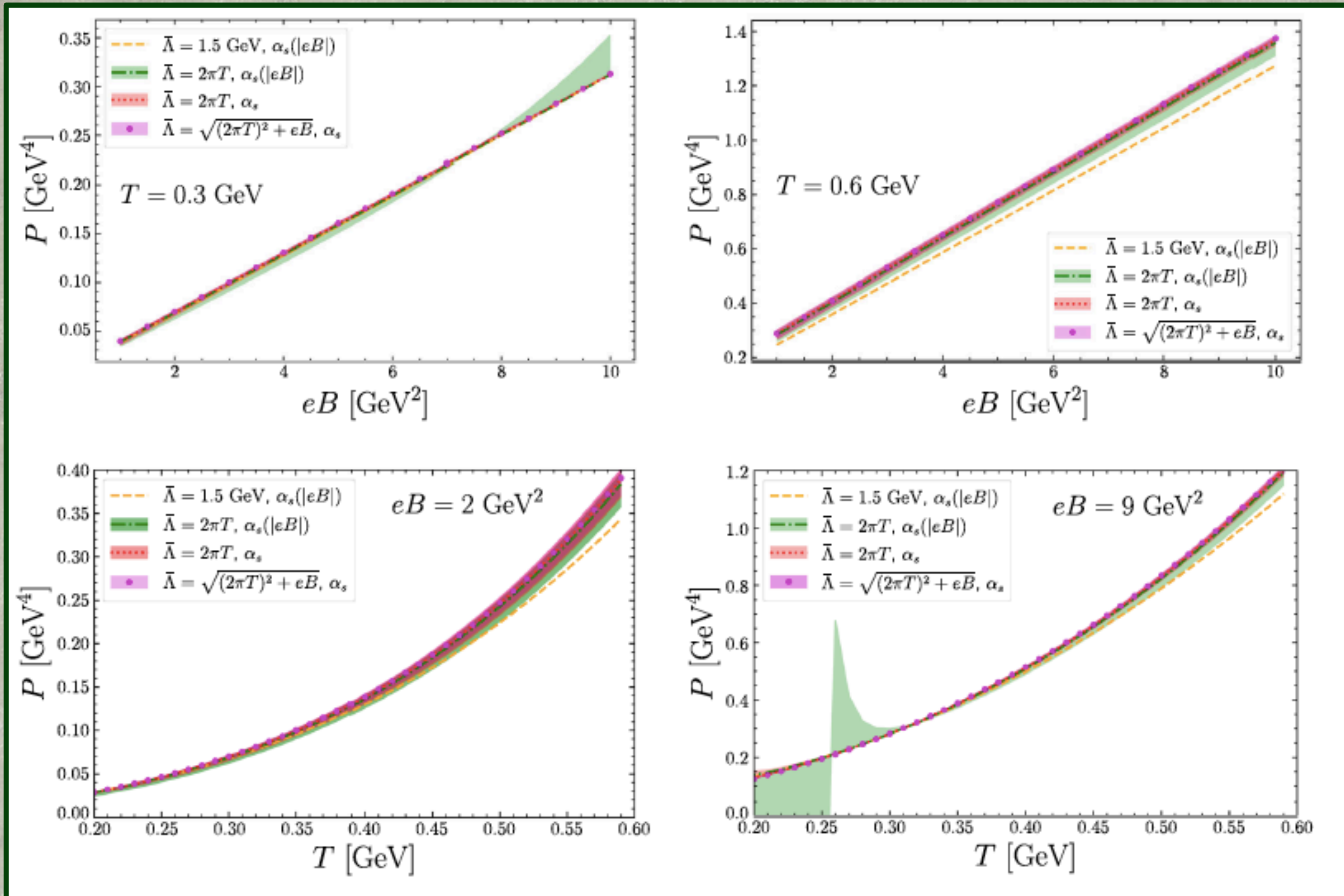
- Cases (ii) & (iii): much poorer convergence; becomes worse as one increases B.

- Compatible with the somewhat unphysical behavior observed in their running α_s & m_s.

- Cases (iv) & (v): well behaved.

- Contribution from the exchange very small for the physical cases, even for huge B fields.

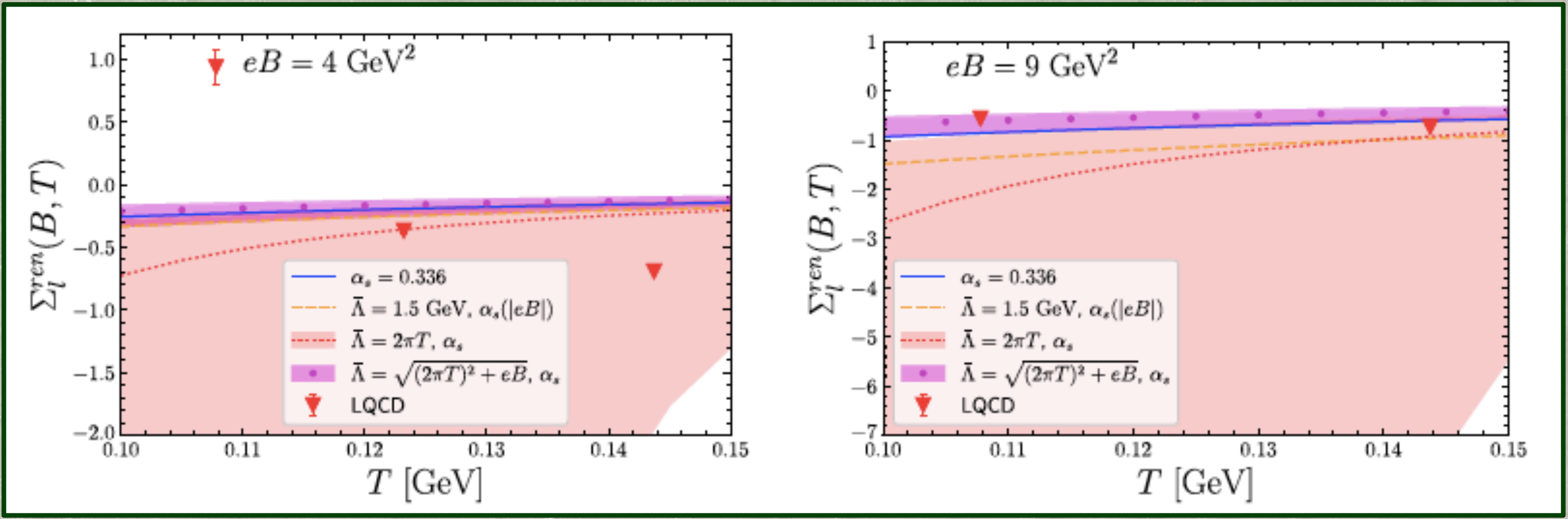
Full pressure with bands



- Bands: increasing/decreasing the central renormalization scale by a factor of 2.
- Bands ~ measure of the theoretical uncertainty of the perturbative series.
- Case (ii) has no band by construction, since Λ is fixed.

Results for the renormalized light chiral condensate

[ESF, Palhares & Restrepo (2023)]

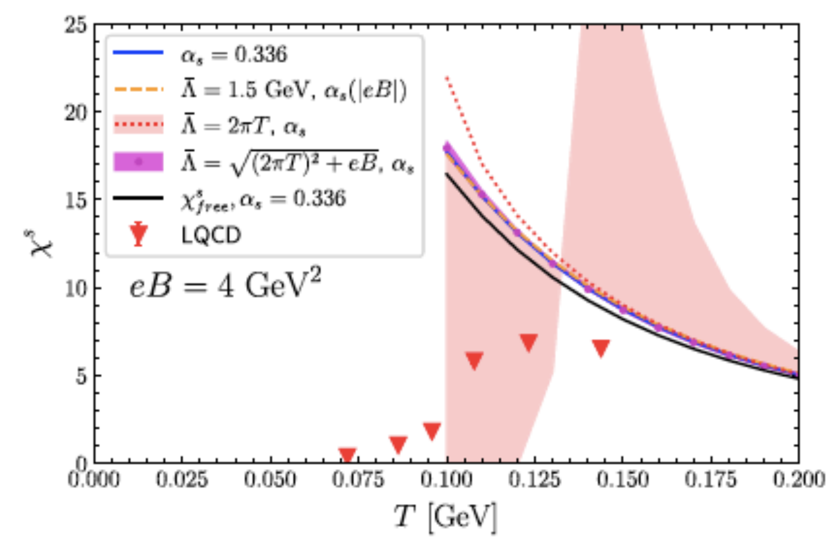


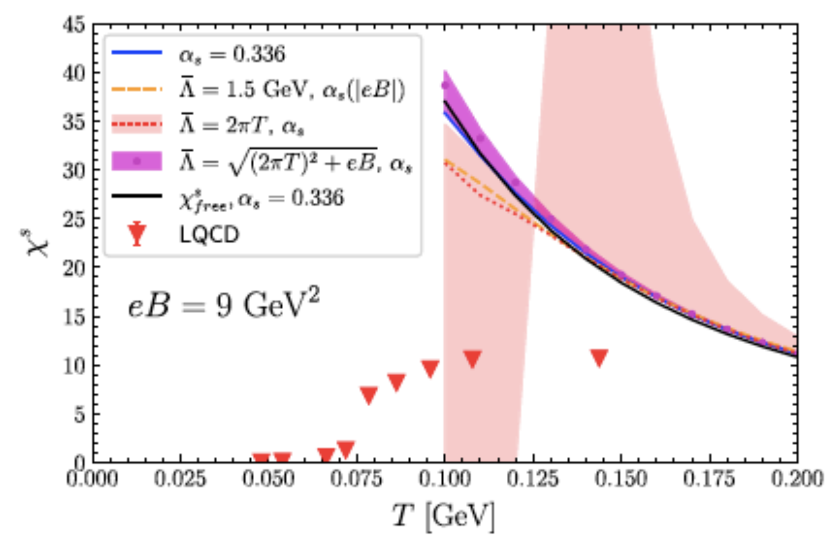
- Lattice data from D'Elia et al (2022).

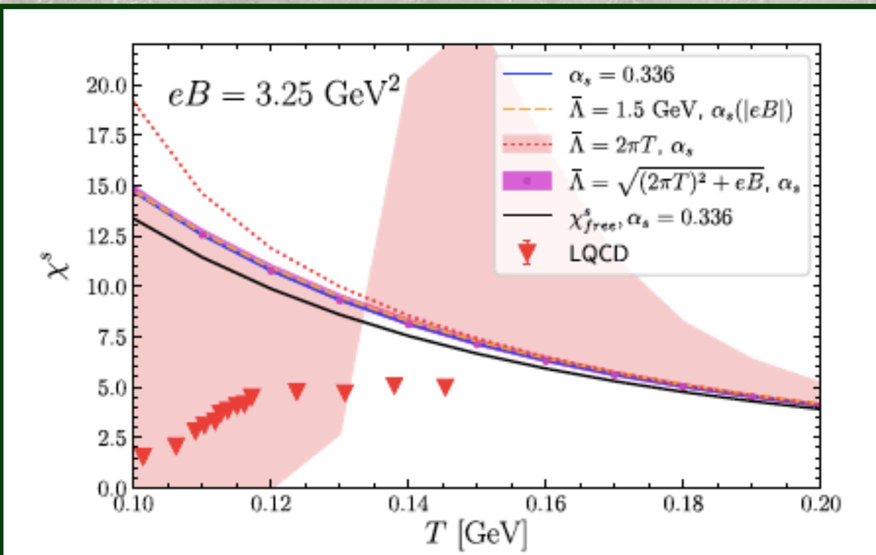
- The width of the band for case (iii) basically diverges (not shown in the figures), case (iv) has a wide band that also diverges at some point for the susceptibility, and case (v) is always well behaved.

Results for the strange quark number susceptibility

[ESF, Palhares & Restrepo (2023)]







- Lattice data from D'Elia et al (2022) & Endrodi (2014).
- Same qualitative behavior for the different running choices.
- Lattice data still far from

$$m_s \ll T \ll \sqrt{eB}$$

Final remarks

- Magnetic thermal pQCD with physical quark masses indicates that loop corrections are subdominant in the pressure for large B. The NLO term is comparatively very small.

- Window of applicability is still narrow, but results obtained from a clean first principle calculation that can be systematically improved & controlled.

- Convergence of the perturbative series for the pressure using the most commonly adopted choices for Λ & $\alpha_s(T,B)$: cases (ii) & (iii) show convergence problems; cases (iv) & (v) pass this criterion, case (v) being the most well behaved.

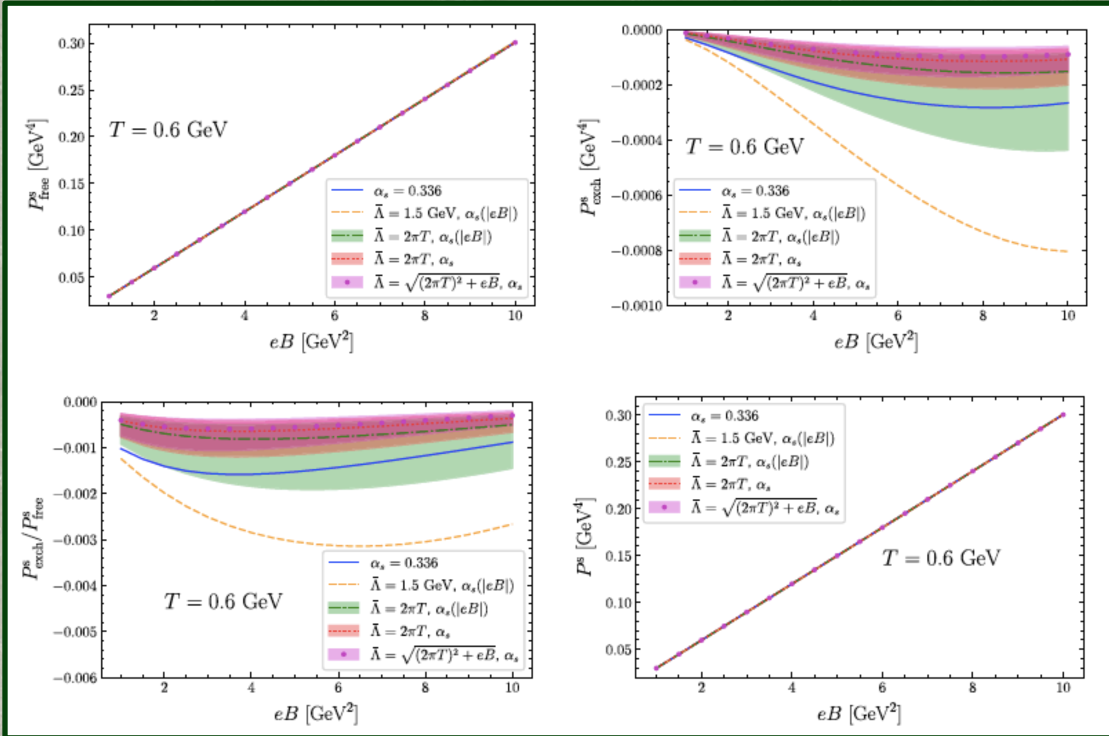
- It would be great to have lattice results that help clarifying this issue.

- Results for the chiral condensate & strange quark number susceptibility compared to recent lattice QCD data away from the chiral transition. Even if still out of the region of validity, pQCD results seem to be in the same ballpark, which is encouraging.

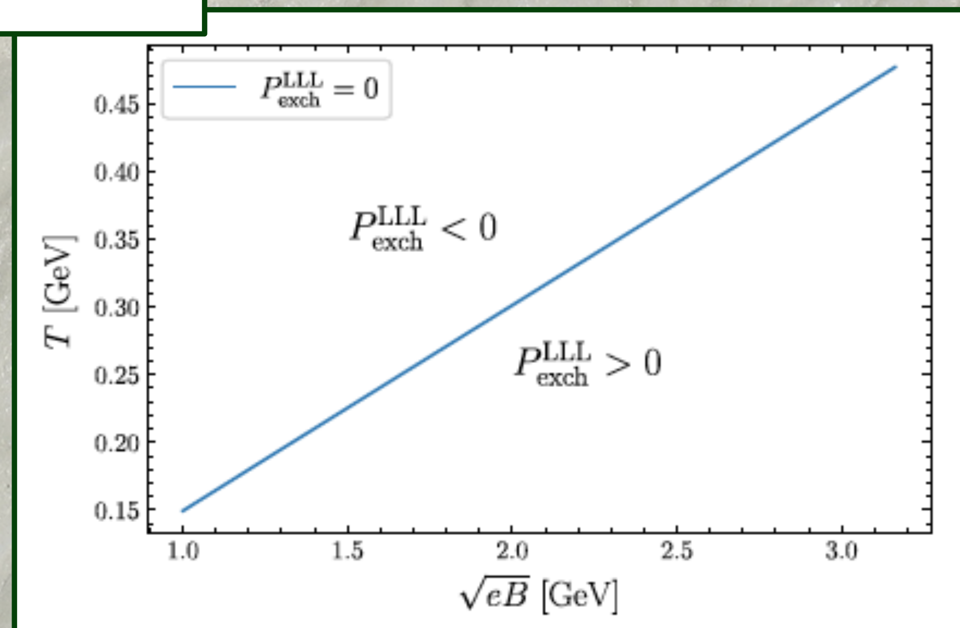
- It would be great to have lattice results at even higher B (and T)!

- For huge values of B, one can extract analytic results in the case of cold magnetic pQCD (IR logs, etc). Might be relevant for magnetar microphysics.

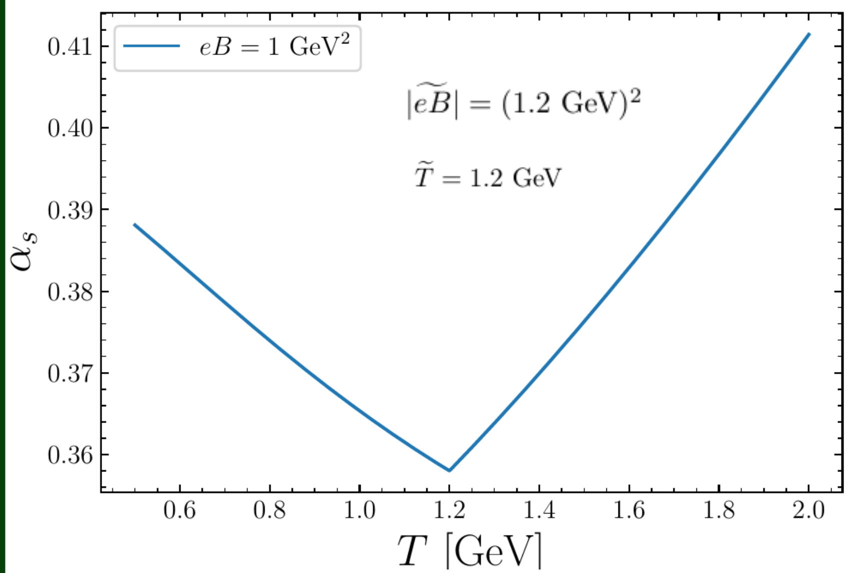
Back up slides



$T = 0.6$ GeV
$P^s_{\rm free}$ [GeV^4]
$P^s_{\rm exch}$ [GeV^4]
$P^s_{\rm exch}/P^s_{\rm free}$
P^s [GeV^4]
eB [GeV^2]
$\alpha_s = 0.336$
$\bar\Lambda = 1.5$ GeV, $\alpha_s(|eB|)$
$\bar\Lambda = 2\pi T$, $\alpha_s(|eB|)$
$\bar\Lambda = 2\pi T$, α_s
$\bar\Lambda = \sqrt{(2\pi T)^2 + eB}$, α_s



$P^{\rm LLL}_{\rm exch} = 0$
$P^{\rm LLL}_{\rm exch} < 0$
$P^{\rm LLL}_{\rm exch} > 0$
T [GeV]
$\sqrt{eB}$ [GeV]

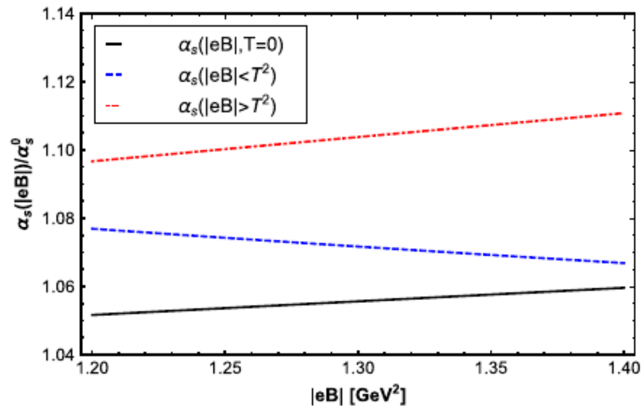


$|eB| < T^2$

$$\alpha_s(|eB|) = \frac{\alpha_s(Q^2 + |\widetilde{eB}|)}{1 + b_1\alpha_s(Q^2 + |\widetilde{eB}|)\ln\left(\frac{Q^2+|eB|}{Q^2+|eB|+T^2}\right)}$$

$|eB| > T^2$

$$\alpha_s(|eB|) = \frac{\alpha_s(Q^2 + \tilde{T}^2)}{1 + b_1\alpha_s(Q^2 + \tilde{T}^2)\ln\left(\frac{Q^2+T^2}{Q^2+T^2+|eB|}\right)}$$



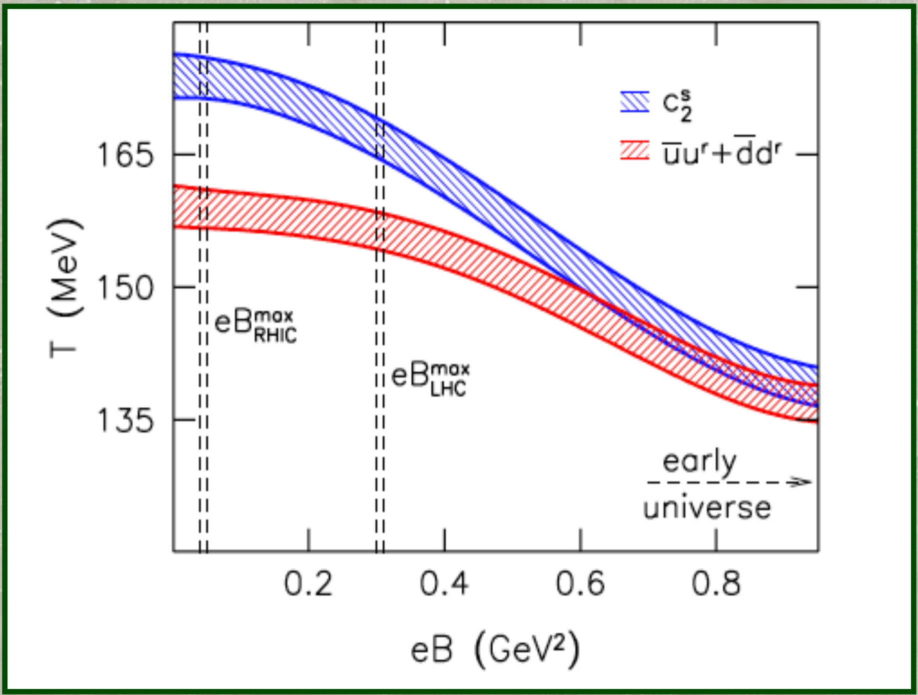
Usamos $\alpha_s(|eB|) = \dfrac{\alpha_s(\Lambda^2)}{1 + b_1 \ln(\frac{\Lambda^2}{\Lambda^2+|eB|})}$ com $\Lambda = 2\pi T$.

B. Karmakar et all PhysRevD.99.094002(2019)

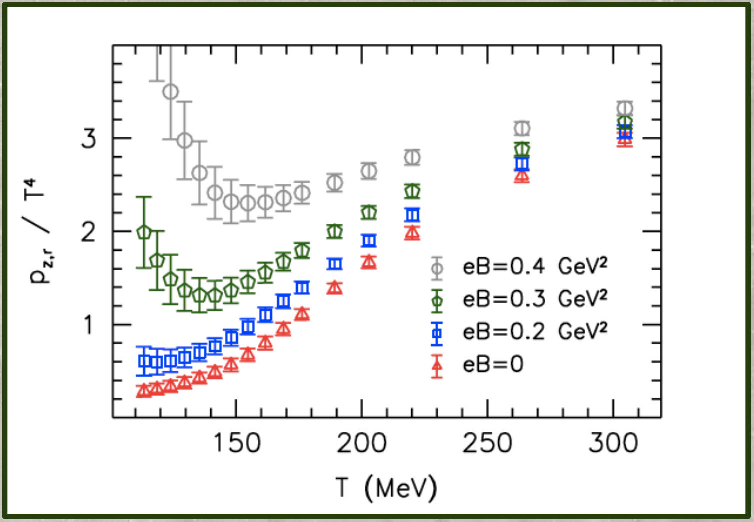
Ayala et all PhysRevD.98.031501(2018)

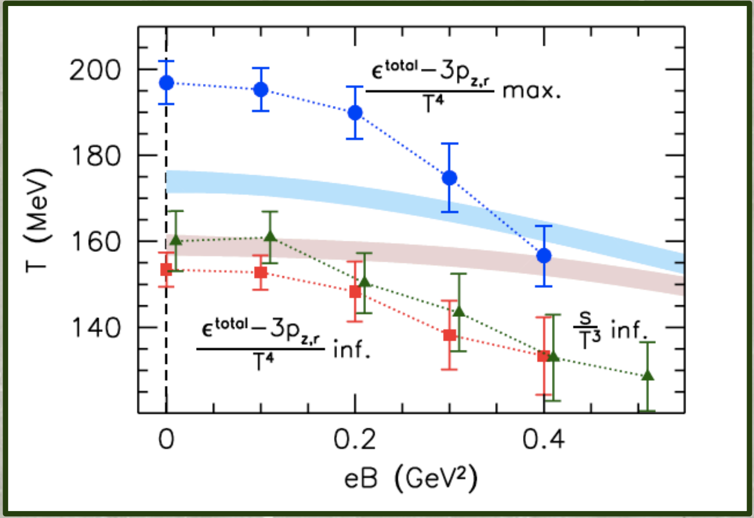
[From talk by T. Restrepo (2022)]

Lattice has already provided a great deal of info on the phase diagram (and essentially on all thermodynamic observables):



[Bali et al (2012)]





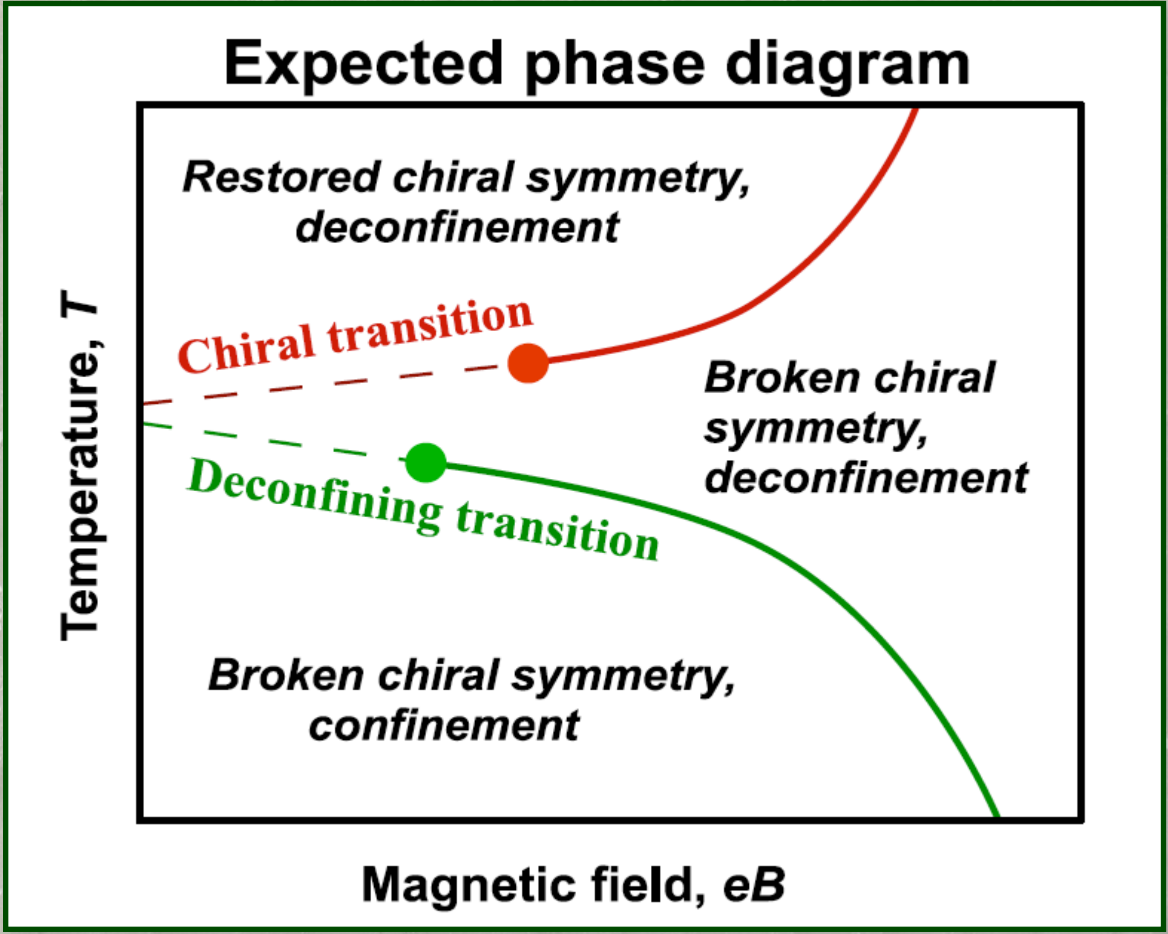
[Bali et al (2014)]

So, are we done??

Well, we would like to have some analytical understanding, too, which could provide new insights. Actually, it all started with effective models (before lattice QCD) but...

From the first papers:

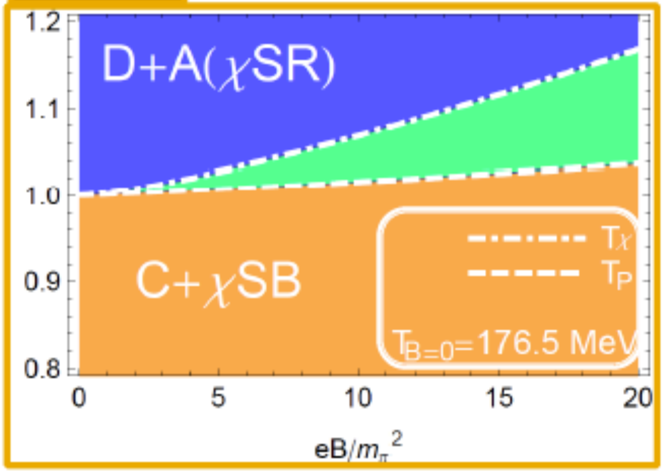
- Deconfining:
  Agasian & Fedorov (2008)
- Chiral:
  ESF & Mizher (2008)



[Mizher, Chernodub & ESF (2010)]

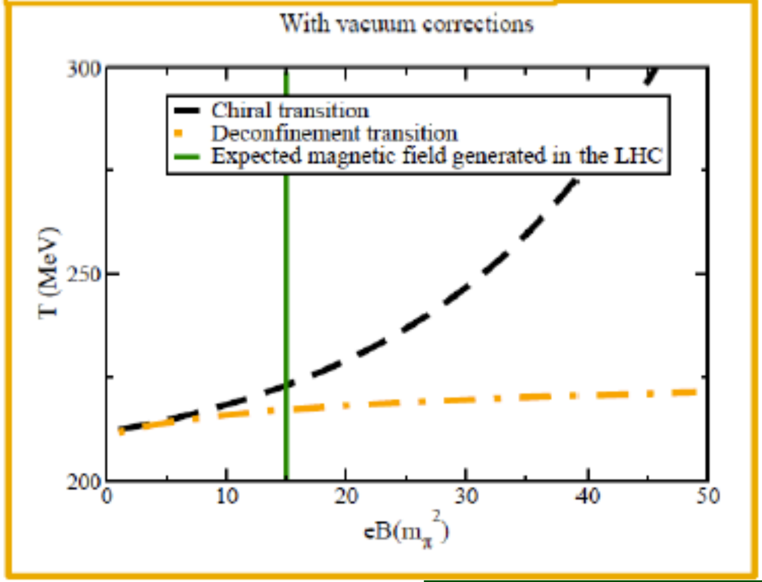
P-NJL

[Ruggeri & Gatto (2010)]



A. J. Mizher *et al*, **arXiv:1004.2712**

P-QM



Excellent agreement

So, all model predictions were basically wrong, except for T=0… (several came afterwards giving similar results, long list).

Clearly, effective chiral models were (are!) missing some crucial ingredient(s)!