Klein-Gordon Effective Equation for Yang-Mills SU(2) Classical Theory

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Adnei Marinho

Universidade Federal Rural da Amazonia / International Center of Physics (UnB),
UFRA / ICP-UnB
Parauapebas/Brasilia - Brazil

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Outline

The Classical Theory

The pure gauge sector:

$$L = \int Tr[F_{\mu\nu} F^{\mu\nu}] d^4x \tag{1}$$

There, the movement equations arises from the sourceless covariant derivative of the strength tensor:

$$D^\mu F_{\mu\nu} = \partial^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0 \tag{2}$$

For SU(2) gauge group:

$$A_\mu = A_\mu^a \sigma^a \ , \tag{3}$$

$===>>$ Large family of configurations:

- **Wu-Yang Monopoles**
- **Instantons**
- **Other topological configurations**

Faddeev-Niemi Representation

- **A skyrme-inspired model**
- **Complete representation of YM gauge theory in Infra Red Regime**
- **n variable works as an order parameter for the Higgs condensates**
- **Tensor structure ease**
- **Interesting Topology feature** $n.n = 1$

Faddeev and Niemi. Phys.Lett.B 525 (2002) 195-200
Faddeev and Niemi. Phys.Lett.B 464 (1999) 90-93

Faddeev-Niemi Approach for the Infrared sector

The Yang-Mills gauge strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ , \tag{4}$$

may be rewritten by decompose the gauge fields in terms of scalar and vector fields

$$A_\mu = C_\mu \mathbf{n} + \rho \partial_\mu \mathbf{n} + \xi \partial_\mu \mathbf{n} \times \mathbf{n} \ , \tag{5}$$

This Skyrme-inspired approach appears to be complete in the infrared sector provided the unitarity of the 'order parameter' n:

$$\mathbf{n}.\mathbf{n} = 1 \tag{6}$$

Old-Fashioned Classical Solutions:

So far, the usual prescriptions for classical solutions would invoke the tensor structure

$$n_a = f x_a \tag{7}$$

with f some general scalar function.

This choice $===>>$ **Witten's ansatz**

Again: instantons, monopoles and so on...

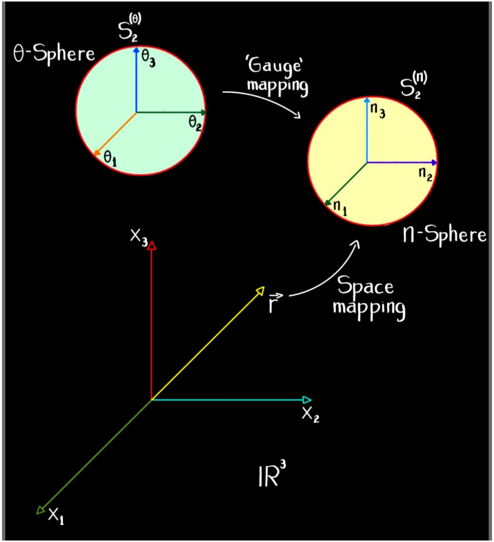
The θ parameters

We will summon another algebraic structure, defining the θ parameters as unit vector components

$$\{\theta_a = (\theta_1, \theta_2, \theta_3); \theta_a \in S_2 | \sum \theta_a \theta_a = 1\} \tag{8}$$

such that the prescription will employ the following variables

	t	
x_1		θ_1
x_2		θ_2
x_3		θ_3


θ-Sphere
$S_2^{(\theta)}$
θ_3
θ_2
θ_1
'Gauge' mapping
$S_2^{(n)}$
n_3
n_2
n_1
n-Sphere
X_3
$\vec{r}$
Space mapping
X_2
X_1
$\mathbb{R}^3$

• **We choose the following fairly complete tensor structure for the order parameter n:**

$$n^a = \theta.xF\theta^a + Gx^a + I\varepsilon^{abc}\theta^b x^c \tag{9}$$

The F,G, I functions being general scalar functions. The n's unitarity establishes the relations they must satisfy:

$$\begin{aligned} F^2 + 2FG - I^2 = 0 \\ G^2 + I^2 = \frac{1}{r^2} \end{aligned} \tag{10}$$

For the C-field, we choose a spherically symmetrical form not depending on θ;

$$C_0 = C_0(r,t); \qquad C_i = x_i f(t,r). \tag{11}$$

Censorship

- **It is possible, but not easy, to demonstrate the following theorem:**

$(x.\partial)n = 0 ===>>$ **no θ in the action**

This condition would means the commutation relation

$$[\hat{x}_i, \hat{p}_i]|n\rangle = -ih\delta_{ii}|n\rangle. \tag{12}$$

Some Considerations

- **Calculation performed in the Minkowsky space**
- $G = \frac{c_G}{r}$,$F = \frac{c_F}{r}$, $I = \frac{c_I}{r}$
- **Equations of motion in terms of the fundamental fields** ξ,ρ,f, C_0 **and the quantities**

  i) "Number" Operator $\mathcal{N}^2 = \rho^2 + \xi^2$
  ii) Pseudo-Casimir invariant $\mathcal{K} = \partial_0(rf) + \partial_r C_0$.

- **It will appear the super differential operator:**

$$\mathcal{D}_{[\hat{\mathcal{O}}_1,\hat{\mathcal{O}}_2]}\{A, B\} = \hat{\mathcal{O}}_1\{\hat{\mathcal{O}}_2 A + \hat{\mathcal{O}}_1 B\} + \hat{\mathcal{O}}_2\{\hat{\mathcal{O}}_1 A - \hat{\mathcal{O}}_2 B\}$$

Time Component of the Equations of Motion

- $D^{\mu} F^{a}_{\mu 0} = 0$:

$$A_M^{[C_0,\partial_0]} - \frac{1}{2}\partial_r(r^2\mathcal{K}) = 0 \tag{13}$$

Space Component of the Equations of Motion

• $D^{\mu} F^{a}_{\mu j} = 0$:

$$\begin{aligned}
&\left[A_T^{[\rho,\xi]} + A_S^{[\xi,-\rho]}\right]\left[\epsilon_{aej}\theta_e(\theta.x) + \theta_j x_a + \epsilon_{aej}x_e + \delta_{aj}(\theta.x) + \theta_j\epsilon_{abc}\theta_b x_c\right] \\
&+\left[A_T^{[\xi,-\rho]} + A_S^{[\rho,\xi]} + A_M^{[rf,-\partial_r]} - \frac{1}{2}\partial_0(r^2\mathcal{K})\right]\left[x_j\theta_a + x_j x_a\right] \\
&+\left[A_S^{[\rho,\xi]} + A_M^{[rf,-\partial_r]} - \frac{1}{2}\partial_0(r^2\mathcal{K})\right]\left[x_j\epsilon_{abc}\theta_b x_c\right] \\
&+A_C\left[\epsilon_{jbc}\theta_b x_c\epsilon_{aef}\theta_e x_f(\theta.x) + \epsilon_{jbc}\theta_b x_c\theta_a\right] \\
&+\left[A_T^{[\xi,-\rho]} + A_S^{[\rho,\xi]}\right]\left[\theta_j\theta_a + \epsilon_{aej}\theta_e\right] \\
&= \quad 0
\end{aligned}$$

Where the coefficients are

$$A_T^{[\rho,\xi]} = \mathcal{D}_{[C_0,\partial_0]}\{\rho,\xi\}$$
$$A_S^{[\rho,\xi]} = -\mathcal{D}_{[rf,\partial_r]}\{\xi,\rho\} - \frac{\rho}{r^2}\left[\mathcal{N}^2 - 1\right]$$
$$A_M^{[C_0,\partial_0]} = \xi(C_0\xi + \partial_0\rho) + \rho(C_0\rho - \partial_0\xi)$$
$$A_C = 3\partial_r\xi + \frac{6}{r}(\xi+1) + rf\xi$$

Let us make it simple!

$$
\begin{aligned}
&A_C = 0 \\
&A_S^{[\rho,\xi]} = A_T^{[\rho,\xi]} = 0; \qquad A_S^{[\xi,-\rho]} = A_T^{[\xi,-\rho]} = 0; \\
&A_M^{[C_0,\partial_0]} = \frac{\partial_r(r^2\mathcal{K})}{2}, \quad A_M^{[rf,-\partial_r]} = \frac{\partial_0(r^2\mathcal{K})}{2}
\end{aligned} \tag{14}
$$

•The result is a new system of higly coupled inhomogeneous non linear equations:

$$\begin{aligned}
&\partial_0\left[C_0\rho-\partial_0\xi\right]+C_0\left[C_0\xi+\partial_0\rho\right]=0\\
&\partial_0\left[C_0\xi+\partial_0\rho\right]-C_0\left[C_0\rho-\partial_0\xi\right]=0\\
&\partial_r\left[\partial_r\rho-rf\xi\right]-rf\left[\partial_r\xi+rf\rho\right]=\frac{\rho}{r^2}\left[\mathcal{N}^2-1\right]\\
&\partial_r\left[\partial_r\xi+rf\rho\right]+rf\left[\partial_r\rho-rf\xi\right]=\frac{\xi}{r^2}\left[\mathcal{N}^2-1\right]\\
&\xi\left[C_0\xi+\partial_0\rho\right]+\rho\left[C_0\rho-\partial_0\xi\right]=\partial_r\left(\frac{r^2\mathcal{K}}{2}\right)\\
&\xi\left[rf\xi-\partial_r\rho\right]+\rho\left[rf\rho+\partial_r\xi\right]=\partial_0\left(\frac{r^2\mathcal{K}}{2}\right)\\
&3\partial_r\xi+\frac{6}{r}(\xi+1)+rf\rho=0
\end{aligned}\tag{15}$$

• We could call them Procca-Stokes Weak Fluid equations (PSWFE). Do you have a better name? Pick your favorite one...

• These Differential Equations generalize the Witten's ansatz

Klein-Gordon Retrieve

- **The symmetrical pattern exhibited by the PSWFE inspire us to rearrange the terms.**

- **Let us define the Generalized derivative operators**

$$D_0 = \begin{pmatrix} \partial_0 & -C_0 \\ C_0 & \partial_0 \end{pmatrix} \quad D_r = \begin{pmatrix} \partial_r & rf \\ -rf & \partial_r \end{pmatrix} \tag{16}$$

- **They will act on the doublet state**

$$|u\rangle = \begin{pmatrix} \xi \\ \rho \end{pmatrix}$$

•**The first four equations of the PSFWE will then read**

$$D_0^2|u\rangle = 0\ ,\quad D_r^2|u\rangle = \frac{\mathcal{N}^2 - 1}{r}|u\rangle \tag{17}$$

•**The next step will be taken in the 1+1 space with**

$$\mathbf{D}_\mu := (D_0, D_r),\quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we may assemble the ingredients:

$$[\Box + M^2]|u\rangle = 0 \tag{18}$$

provided:

$$M = [\frac{\mathcal{N}^2 - 1}{r^2}]^{1/2} \ . \tag{19}$$

Therefore, the effective mass Operator M identification implies the theorem:

$$YM \to_{(GIV)} KGE \ . \tag{20}$$

Procca Equation

The B element of SU(2) and the current j,

$$B := \frac{rf}{e}\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \ , \quad |j\rangle := \begin{pmatrix} j_0 \\ j_1 \end{pmatrix}, \tag{21}$$

where the time component of the current (charge density) and the spatial component are

$$j_0 = \frac{4e}{r}, \quad j_1 = ie\left[\rho\left(-\frac{rf}{3}\right) + \xi\left(\frac{3}{r} - \frac{f'}{f}\right)\right] \tag{22}$$

turns possible to include the fifth equation in the (PSWFE) into our analisis.

•Making the identifications $\Psi_\alpha = |u\rangle$, and $J_\alpha = |j\rangle$

•taking the minimum coupling ($P_\mu = D_\mu - eB_\mu$)

The fifth equation inclusion reads

$$[P^2 + M^2]\Psi = BJ \tag{23}$$

Our Proca Equation

Screening Effect

j_0 **as charge density:**

$$q = 4e/r \tag{24}$$

The screening of the charge reflects in the Gauss law result:

$$E = \frac{1}{\pi\epsilon_0}\frac{e}{r^3} \tag{25}$$

that expands according to a 3-sphere surface in a 4-dimensional space.

The Mixing

There are two differential equations left for to treat: the last ones. They are rewritten as

$$\langle u|D_0|v\rangle = \mathcal{G}', \quad \langle v|D_R|u\rangle = \dot{\mathcal{G}} \ , \tag{26}$$

in which we have defined $\mathcal{G} = \frac{r^2\mathcal{K}}{2}$**, and the mix invariant number** $\mathcal{K}$ **is defined according to**

$$\mathcal{K} = [\partial_0(rf) + \partial_r C_0] = (det[D_0, D_R])^{1/2} \ . \tag{27}$$

Interpreting these mixes:

1) effective mass $<<===>>$ generalized derivative (Eq.17).

2) 0 (time) components of gauge fields $<<===>>$ Higgs modes. Julia and Zee, Phys.Rev.D 11, 2227 (1975)

CONJECTURAL CONCLUSION: mix equations are nothing but the Higgs modes exciting the generating of effective mass, hence corresponding to a mass generation mechanism.

3) K number $==$ time evolution and the space translations commutation.

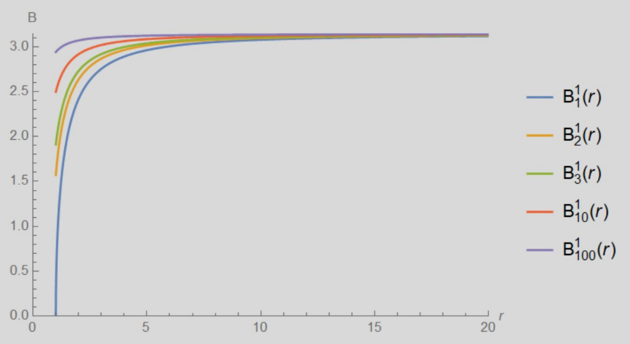
4) $[H, P] = ih\dot{P}$ vanishes for isolated system, without external "forces". $\mathcal{K} = 0$ would be a non dissipative system.

CONJECTURAL CONCLUSION: Higgs Mechanism does exist, the system interacts with itself. The K number is dissipation.

Analytical Solutions

The first solution

$$\rho = \cos\beta \;;\quad \xi = \sin\beta \;;\quad C_0 = -\dot{\beta};\quad f = \frac{\beta'}{r};\;\; \cos\beta_n^{(1)} = \frac{2}{n}\left(\frac{r_f}{r}\right)^3 - 1$$



Zabadal Method

$$\rho = w\cos\beta \qquad \xi = w\sin\beta \tag{28}$$

===>>

$$\beta = \arctan\left(\frac{-6+\sqrt{43-w^2}}{\sqrt{7-w^2}}\right). \tag{29}$$

Using definitions

$$Z_1 := -6+\sqrt{43-w^2}; \qquad Z_2 := \sqrt{7-w^2},$$

the fields solutions reads:

$$
\begin{aligned}
f &= -\frac{3}{2}\frac{\sqrt{7-w^2}}{r};\\
\rho &= \frac{w}{\sqrt{1+(\frac{Z_1}{Z_2})^2}};\\
\xi &= \frac{Z_1}{Z_2}\rho;\\
C_0 &= \frac{-3w\dot{w}}{Z_2(Z_1+6)^2}
\end{aligned}
\tag{30}
$$

The last functional solution found is the form for the w amplitude

$$
w = w(r,t) = \frac{c_1 t + c_2}{r^2}. \tag{31}
$$

Analysing the Causality

Let us construct the classical wave functions for the fields to understand the excitation propagation.

Setting up

$$\frac{\partial^2 \xi}{\partial r^2} - \frac{1}{v_\xi^2}\frac{\partial^2 \xi}{\partial t^2} = 0 \tag{32}$$

The identification of the velocity for the ξ field excitation modes follows

$$v_\xi = \frac{\frac{c_1 r}{2(c_1 t + c_2)}}{\sqrt{1 + \frac{A(A+6)\left(\frac{3}{A+6} - \frac{A(A+6)}{w^2}\right)}{12 + 13A - \frac{A(A+6)}{B} + \frac{9w^2}{(A+6)^2}\left(1 - \frac{4}{A} + \frac{A+6}{B}\right)}}}$$

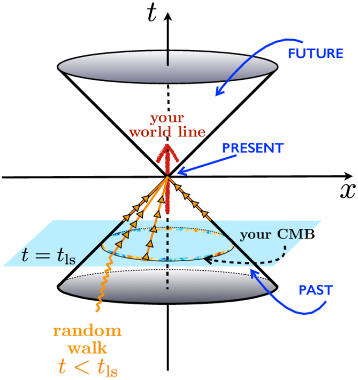
and for the ρ-field:

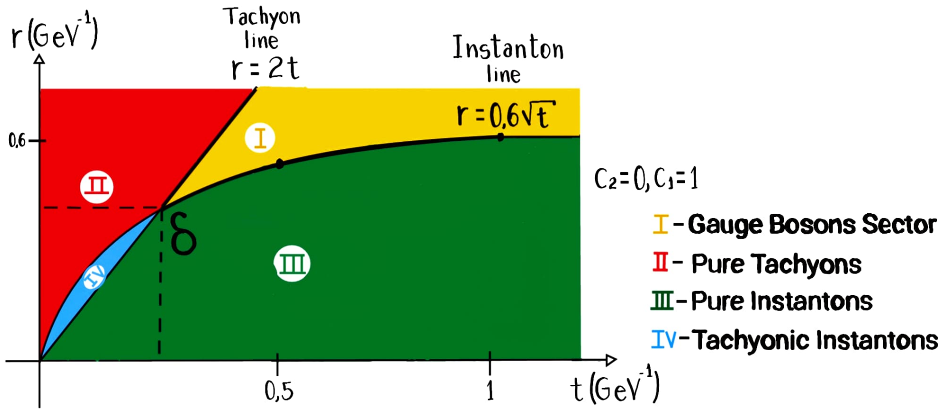
$$v_\rho = \frac{\frac{c_1 r}{2(c_1 t + c_2)}}{\sqrt{1 + \frac{B^2(A+6)^2}{18Aw^2\left(1 + \frac{w^2 A(A+8)}{B^2(A+6)^2 + 3Aw^2}\right)}}}$$

Some constraints:

•**Causality:** $v_\xi < c$, $v_\rho < c$. **The symmetries in the light cone!**

•**Reality: radicals positiveness**





"phase" diagram for the constraints of causality and reality. The instanton line comes from the radical positiveness $w^2 < 7$. The Tachyon line comes from the constraint $v < c$

Energy

• energy functional:

$$H = \frac{8}{r^2}\left[\dot{\xi}^2 + \dot{\rho}^2 + \frac{r^4}{2}\dot{f}^2 + \frac{r^3}{2}C_0'\dot{f} + C_0[\xi\dot{\rho} - \rho\dot{\xi}]\right] \qquad (33)$$

• the minimum will give us the spectrum:

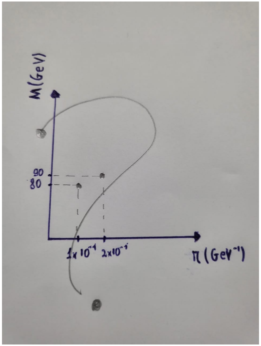
$$\frac{\partial H}{\partial r} = 0 \longrightarrow M(r_{min}, t_w) \qquad (34)$$

• As long as we have a mass!

$$M = \sqrt{\frac{\xi^2 + \rho^2 - 1}{r^2}} \qquad (35)$$

Spectrum

In progress...



Perspectives

•Spectrum

•SU(3)

•Some impact for Hadrons?...

Thank You