Abnormal solutions of Bethe-Salpeter equation with massive exchanges

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PRELUDE

Bethe Salpeter equation deals with a - pre-existing - QFT object (Gell-Mann Low)

$$\Phi(x_1,x_2,P) =< 0 \mid T\{\phi(x_1)\phi(x_2)\} \mid P >$$

Its Fourier transform Φ(k,P)

$$\Phi(x_1,x_2,P) = \int \frac{dp_1}{(2\pi)^4}\frac{dp_2}{(2\pi)^4}\Phi(p_1,p_2)\, e^{-iPx}\, e^{-ikx} = \; e^{-iPx}\int \frac{dk}{(2\pi)^4}\Phi(k,P)\, e^{-ikx}$$

$$\begin{aligned} p_1 + p_2 &= P \\ p_1 - p_2 &= 2k \end{aligned}$$

satisfies a 4D equation. For bound states it reads :

$$\boxed{\Phi(k,P) = S_1(k,P)S_2(k,P) \int \frac{d^4k'}{(2\pi)^4}\; iK(k,k';P)\; \Phi(k',P)} \qquad (*)$$

$P^2=M^2$ with M the total mass of the two-body system

S_i = free propagators

$$\begin{aligned} \mathbf{S_1(k,P)} &= \frac{\mathbf{i}}{\left(\frac{\mathbf{P}}{2}+\mathbf{k}\right)^2 - \mathbf{m}^2 + \mathbf{i}\epsilon} \\ \mathbf{S_2(k,P)} &= \frac{\mathbf{i}}{\left(\frac{\mathbf{P}}{2}-\mathbf{k}\right)^2 - \mathbf{m}^2 + \mathbf{i}\epsilon} \end{aligned}$$

iK=Interaction kernel
- if K would contain all the IR graphs, solving **(*)** would be equivalent to solve the full QFT
- This is however a **wishful thinking**. In practice one uses a poor restriction: ladderwith simple kernels

PRELUDE

We will consider solutions of **(*)** with scalar massless exchange kernel (Wick-Cutkosky)

$$iK(k,k') = -\frac{g^2}{(k-k')^2+i\epsilon}$$

-> Coulomb potential $V(r) = -\frac{g^2}{4\pi}\frac{1}{r}$

and its natural extension to the massive case

$\alpha=g^2/4\pi$

$$iK(k,k') = -\frac{g^2}{(k-k')^2-\mu^2+i\epsilon}$$

-> Yukawa potential $V(r) = -\frac{g^2}{4\pi}\frac{e^{-\mu r}}{r}$

The **massless case** has the peculiarity to **accept solutions which have no counterpart in the non relativistic limit (Schrodinger eq)**…. even if they involve very small energies. Solved by **G.C. Wick and R.E. Cutkosky in 54**, our contribution to this field was presented by V.A.K. in several Conferences, e.g. LCM 2018 (USA) and LCM 2019 (France)

AIM OF THIS CONTRIBUTION

I. Briefly summarize the main results of the massless case

II. Present our (new) results for the massive exchanges

THE MASSLESS CASE (Wick-Cutkosky model)

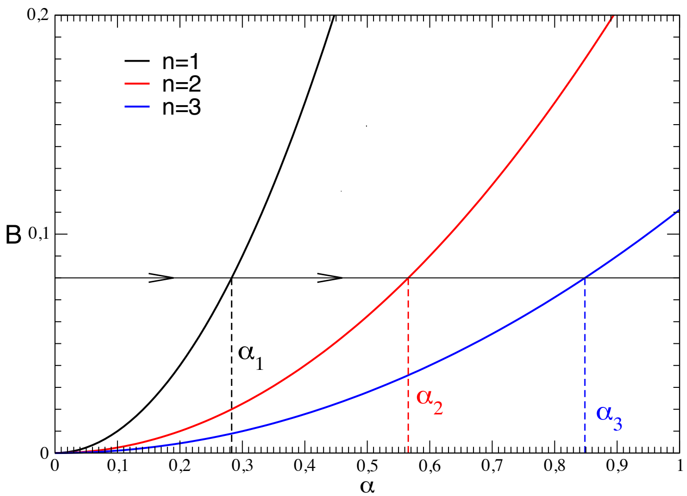
Since $P^2=M^2$, and $$iK(k,k') = -\frac{g^2}{(k-k')^2+i\epsilon}$$

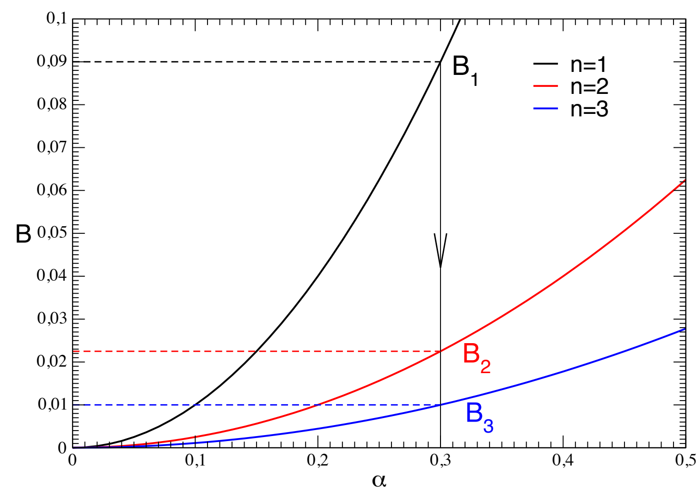
BS equation $$\Phi(k,P) = S_1(k,P)\, S_2(k,P) \int \frac{d^4k'}{(2\pi)^4}\, iK(k,k';P)\, \Phi(k',P)$$

is an **implicit e.v. equation** : $\quad \Phi_n = g^2_n\, O(M^2)\, \Phi_n$

For a given M^2, there is a **discret set** of g^2 a that « solve » the problem : $M^2(g^2)$
It is customary to introduce M=2m-B and $\alpha=g^2/4\pi$ and present rather $B(\alpha)$

In the non relativistic case : $B_n(\alpha)=m\alpha^2/4n^2$ **n**=1,2,3… (always $E_n=-B_n$)





W&C found that their model had indeed a familly of solution **(ϕ_n,B_n)** that, for small values of **α**, were « tangent » to the NR limit (logarithmic corrections)

$$B_n(\alpha) = \frac{m\alpha^2}{4n^2}\left[1 + \frac{4}{\pi}\alpha \ln \alpha + o(\alpha^2)\right] \qquad (1)$$

But for each **n**=1,2,3… they found **an additional infinite series** of ev labeled by a new quantum number **κ**=0,1,2,… (due to the SO4 symmetry of the problem)

$E_{n\kappa}$ (α) **n=**1,2,3,… **κ**=0,1,2,3,….

The subset **κ =0** correspond to the « Balmer series » (1)

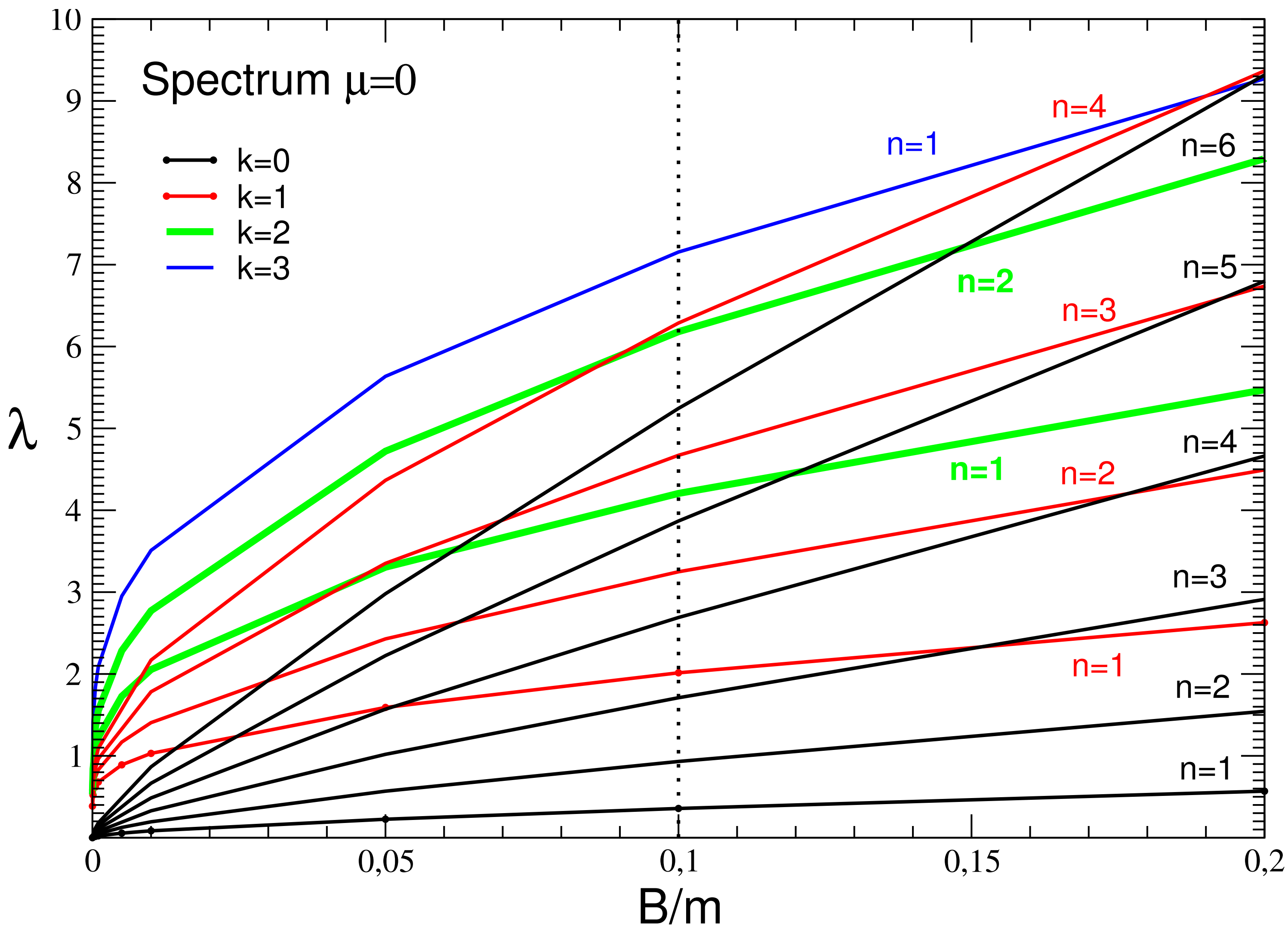
And the rest ????

- It was shown **(*)** that the odd values **κ =1,3,..**of do not contribute to S-matrix

- What about **κ=0,2,4....** ?

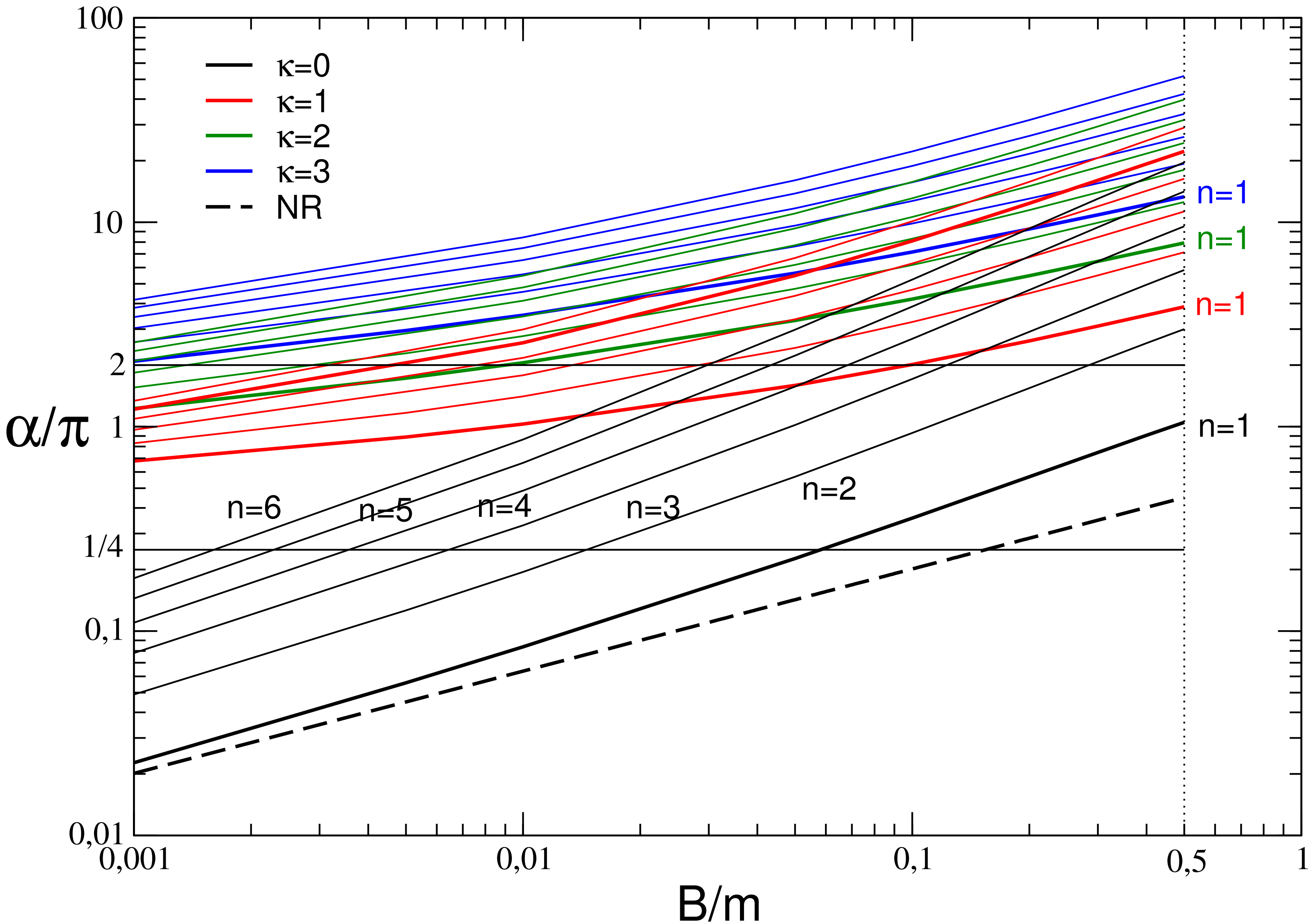
(*) M. Ciafaloni and P. Menotti, PR140 (1965), B929

Let us plot α(B) as before (rather $\lambda=\alpha/\pi$)



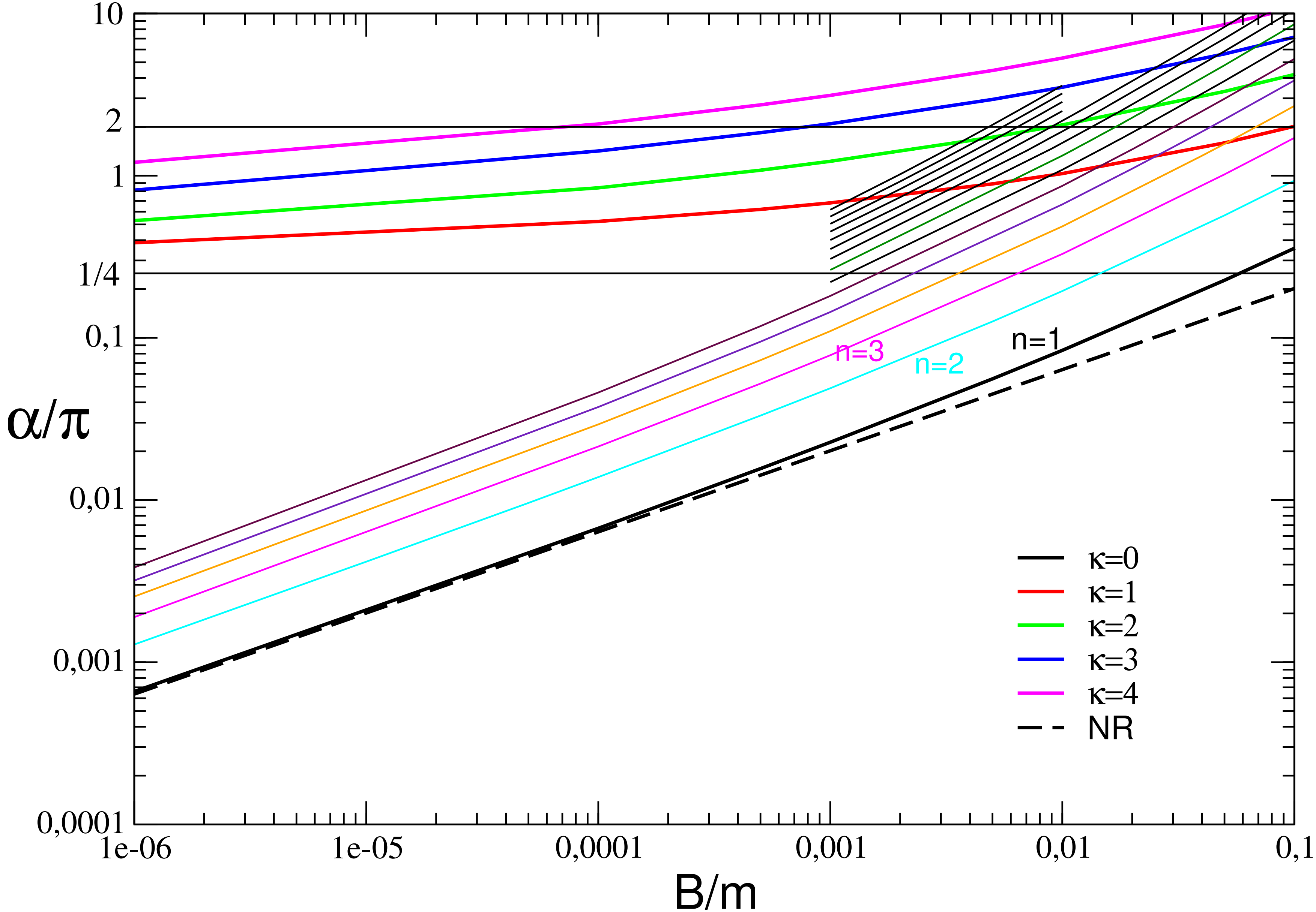
Black states (κ=0) have an « acumulation point » at (B=0, α=0) as in NR equations
All others go somewhere else, but « decoupled » from the NR solutions : **« ABNORMAL »**

A Log-Log zoom of the same picture… and comparison with the NR results (n=1)



Abnormal states require a non zero value of **α** in the limit B→0 (as in massive exchange!)

A zoom of the zoom….with two remarkable horizontal values **λ=2** and **λ=1/4**

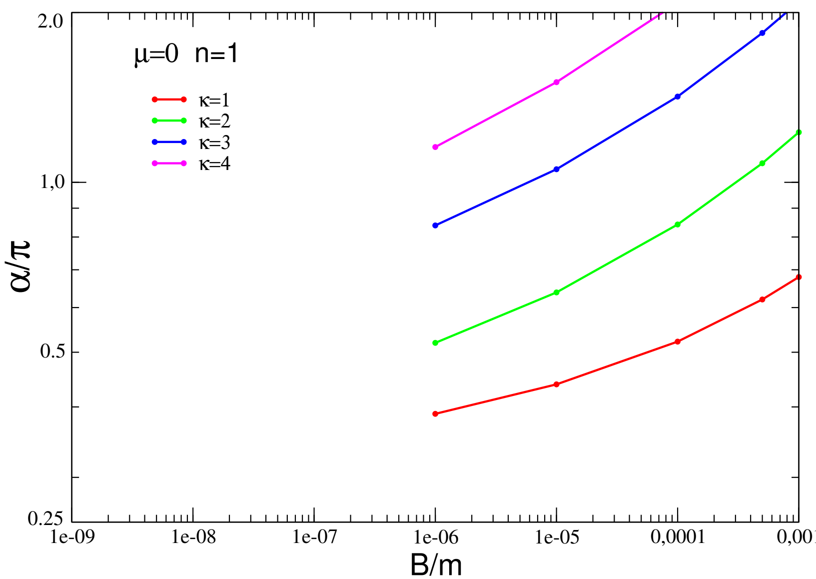


If normal states - all of them ! - exist for any value of **α**,
« building » an abnormal states requires large coupling contants …even if it has B=0 !

For abnormal states, **α/π** → ¼ (by above) with, for small B, the assymptotic relation

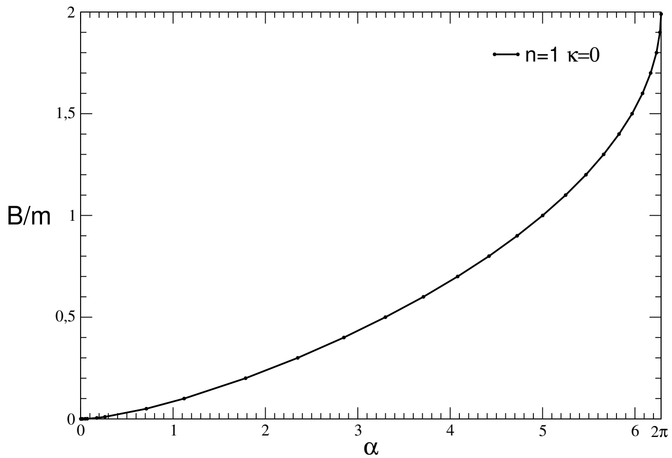
$$\alpha(B) \approx \frac{\pi}{4} + \frac{4\pi^3(\kappa-1)^2}{\ln^2\frac{B}{m}} \quad \Longrightarrow \quad \lambda \equiv \frac{\alpha}{\pi} \approx \frac{1}{4} + \frac{4\pi^2(\kappa-1)^2}{\ln^2\frac{B}{m}}$$

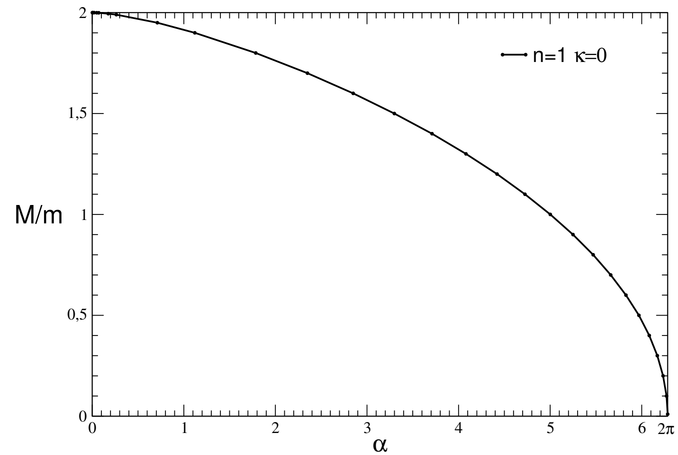
Result independent of n: **all abnormal states tend to the same point (B=0,α/π=1/4) !**
But very slowly…(for **κ=2** one still misses a factor 2 at B=10^{-6})



This is the relevance of the **α/π**=1/4 horizontal line in previous figures

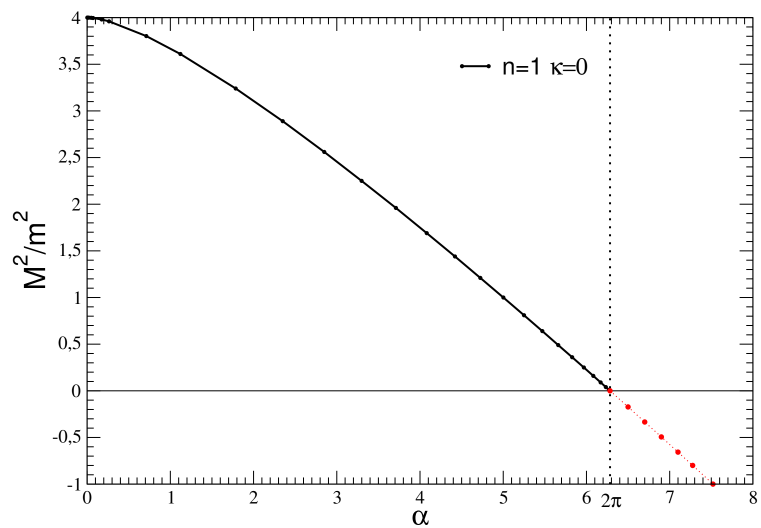
The value $\alpha/\pi=2$ has another meaning….
The ground state of WC model **(n=1,κ=0)** has M=0 (B=2) for $\alpha=2\pi$ ($\lambda=\alpha/\pi=2$)





Above that, $\alpha>2\pi$, the GS becomes « tachyonic » : $M^2<0$

Some people dont care,
we prefer avoid them !



SPECTRUM OF μ=0

If one does not take care about the tachyonic states :

- **Normal** solutions exist for any value of **α**

- **Abnormal** solutions only for **α>π**/2 (**λ>1/4**)

If one restrict to non–tachyonic solutions

- **Normal** solutions limited to $0 < \alpha < 2\pi \iff 0 < \lambda = \frac{\alpha}{\pi} < 2$

- **Abnormal** states limited to the range $\frac{\pi}{4} < \alpha < 2\pi \Leftrightarrow \frac{1}{4} < \lambda < 2$

Their energy is always very small B<0.009 !!!

NB: The existence of a minimal coupling constant for a bound state is typical from a massive exchange (Yukawa like).
Abnormal states behaves like if a « massive photon » with m/M=0.4

How to « characterize » the abnormal solutions ?

(apart from their behaviour at small B)

How to « characterize » the abnormal solutions ?

This was the main result of our recent work

The state vector |P> apprearing in the definition of BS amplitude Φ is a QFT state involve many body components (Fock expansion)

$$| P\rangle = \sum_{n\geq 2}^{\infty} \Psi_n(k_1, k_2, ...k_n) \mid n\rangle \qquad | n\rangle = a^\dagger_{k_1} a^\dagger_{k_2}, ...a^\dagger_{k_{n-2}} b^\dagger_{q_1} b^\dagger_{q_2} \mid 0\rangle$$

Its total norm results from the norms of

$$\langle P' \mid P\rangle = 1 = \int \Psi_2^2 + \int \Psi_3^2 + \int \Psi_4^2 + \ \ldots = N_2 + N_3 + N_4 + \ldots$$

The two-body contribution N_2 to the total norm can be obtained through the Light-Front projection of the BS amplitude

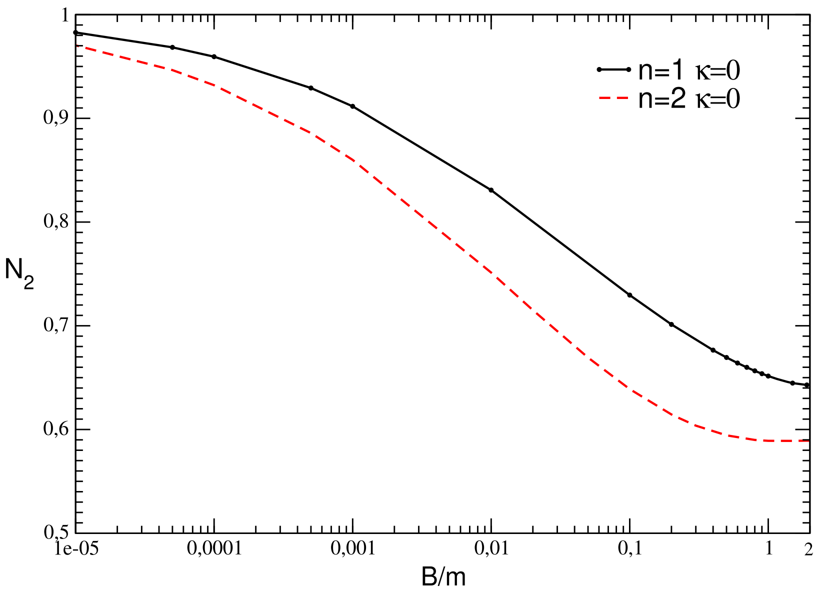
$$\Psi_2(k_1, k_2, P, \omega) = \frac{(\omega \cdot k_1)(\omega \cdot k_2)}{\pi(\omega \cdot P)} \int_{-\infty}^{\infty} \Phi(k + \beta\omega, P) d\beta$$

according to

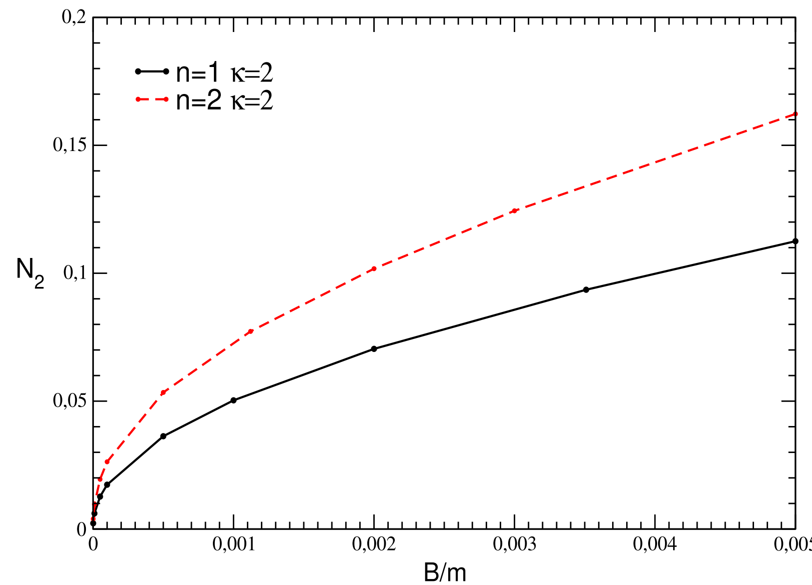
$$N_2 = \frac{1}{(2\pi)^3} \int \Psi_2^2(k_\perp, x) \, \frac{d^2 k_\perp dx}{2x(1-x)}$$

We found striking differences in the two-body contents of the WC solutions

« Normal states » are essentially two-body with N->1 when B->0



« Abnormal states », are genuine many-body sates with N_2->0 when B->0



Details of this and further calculations concerning « wave functions » and EM form factors can be found in our recent publication

Eur. Phys. J. C (2021) 81:50
https://doi.org/10.1140/epjc/s10052-021-08850-1

Hybrid nature of the abnormal solutions of the Bethe–Salpeter equation in the Wick–Cutkosky model

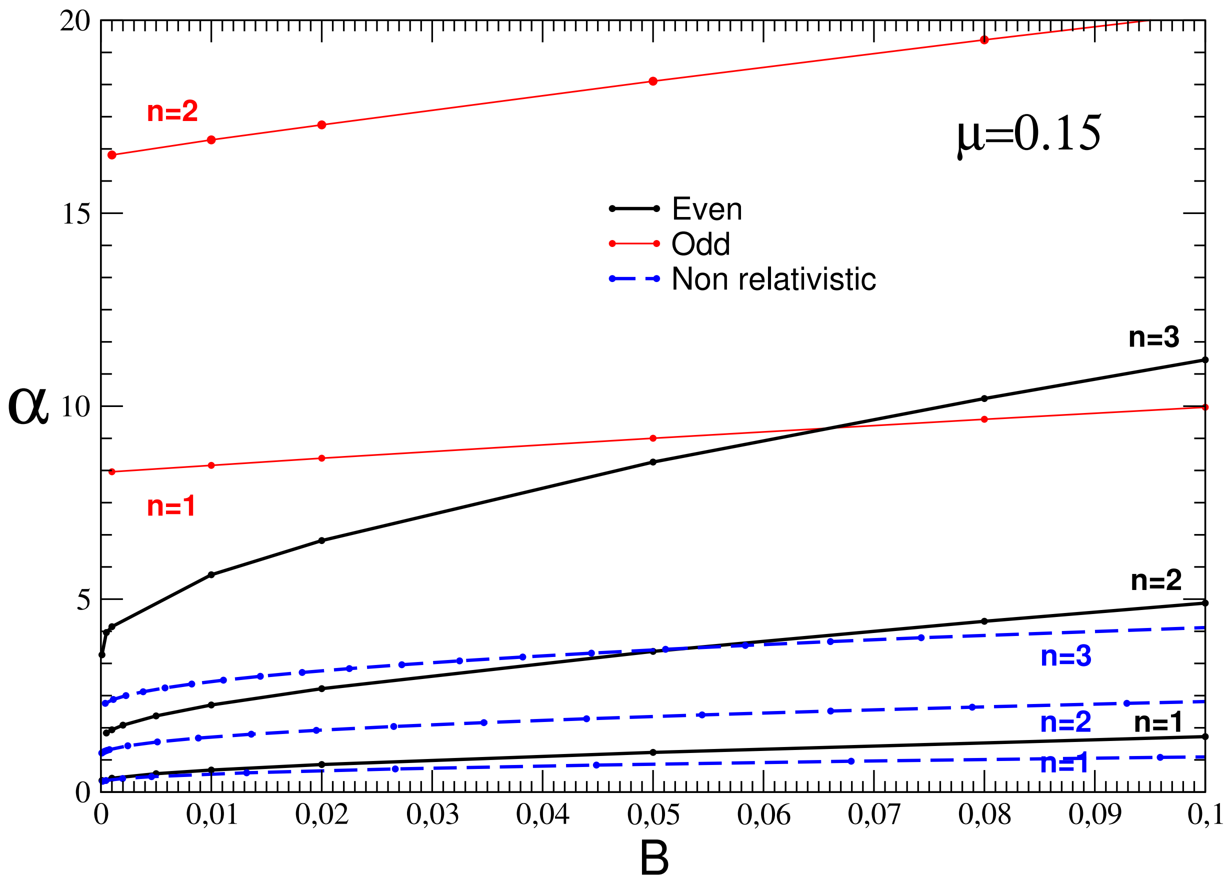
J. Carbonell[1,a], V. A. Karmanov[2,b], H. Sazdjian[1,c]

and **will be extended to unequal mass in the next talk by V.A. Karmanov**

DO THE ABNORMAL STATES SURVIVE IN THE MASSIVE CASE ?

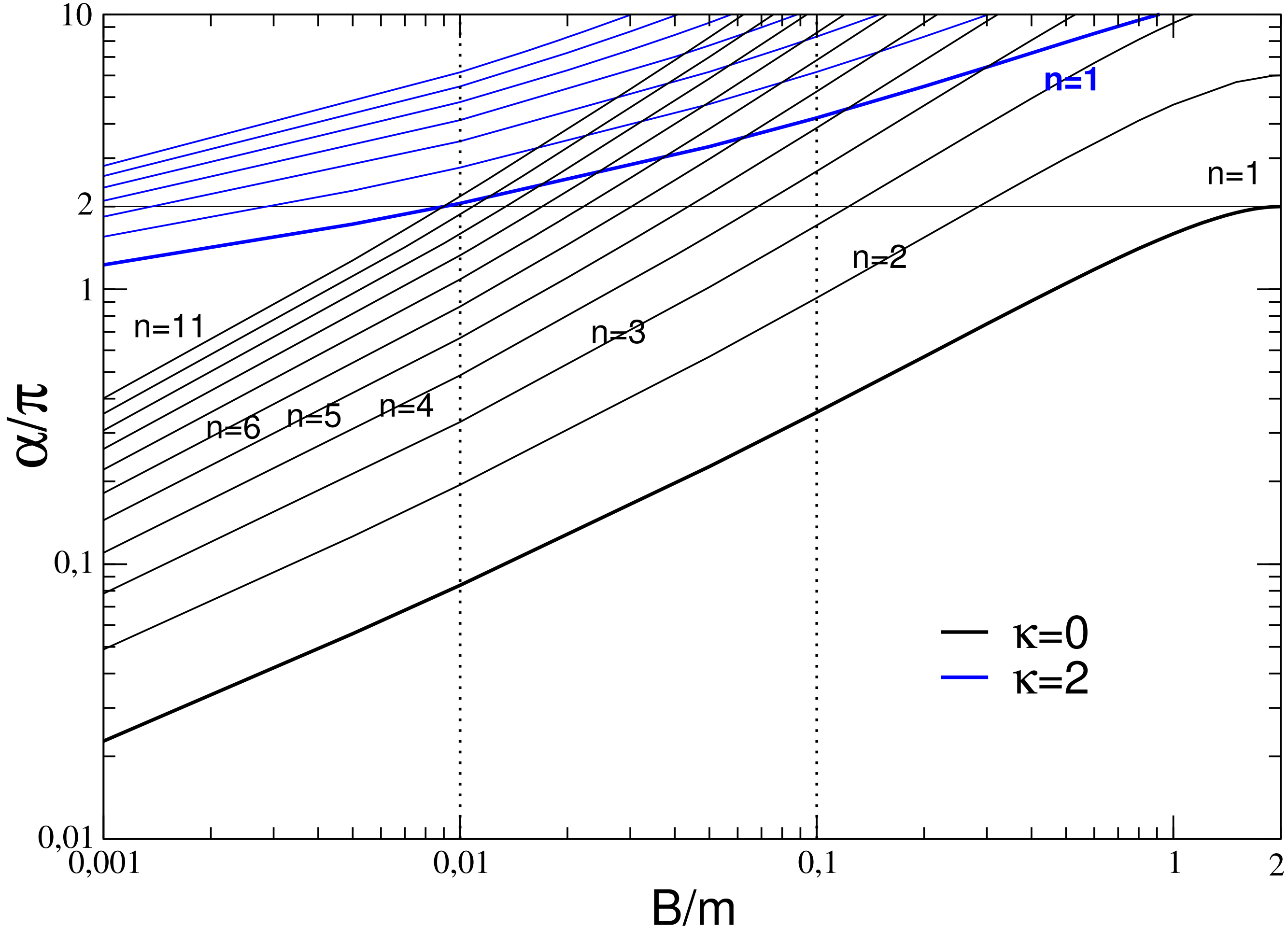
THE MASSIVE EXCHANGE CASE

When $\mu>0$, the SO4 symmetry is broken and the κ quantum number disappears
The (partial wave) of the 4d BS equation is bidimensional (instead of 1d for $\mu=0$)
Eigenstates are only labelled by E_n
However the parity $\phi_n(k,k4)$ is not destroyed by μ ! :
Normal and abnormal states are even on k_4



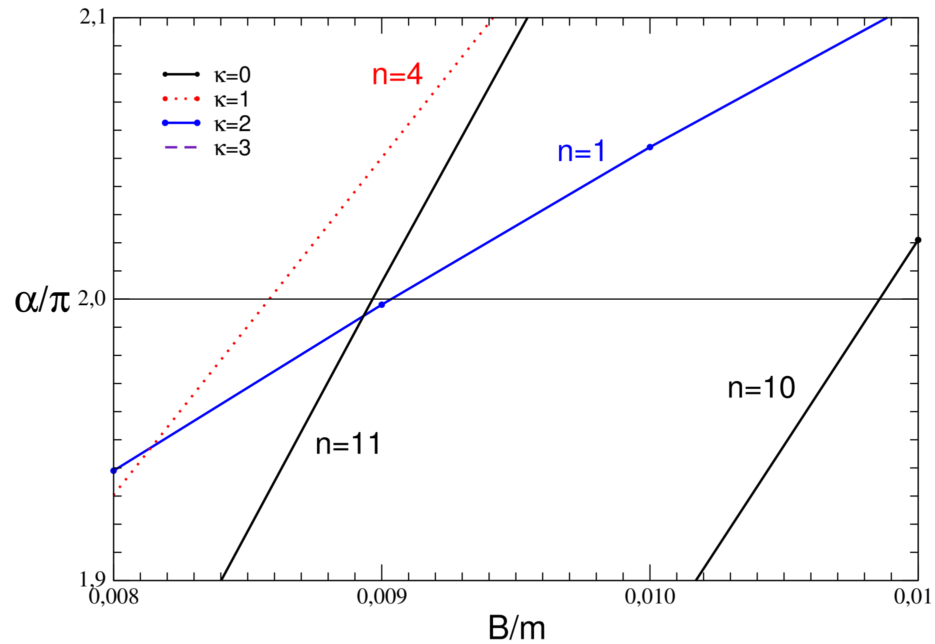
Where are hidden the abnormal states ?

Start from what we know in the **μ**=0 case



Maximum energy of the **Ground abnormal state (GAS)**
(blue state **κ=2,n=1**) is B≈0.00903

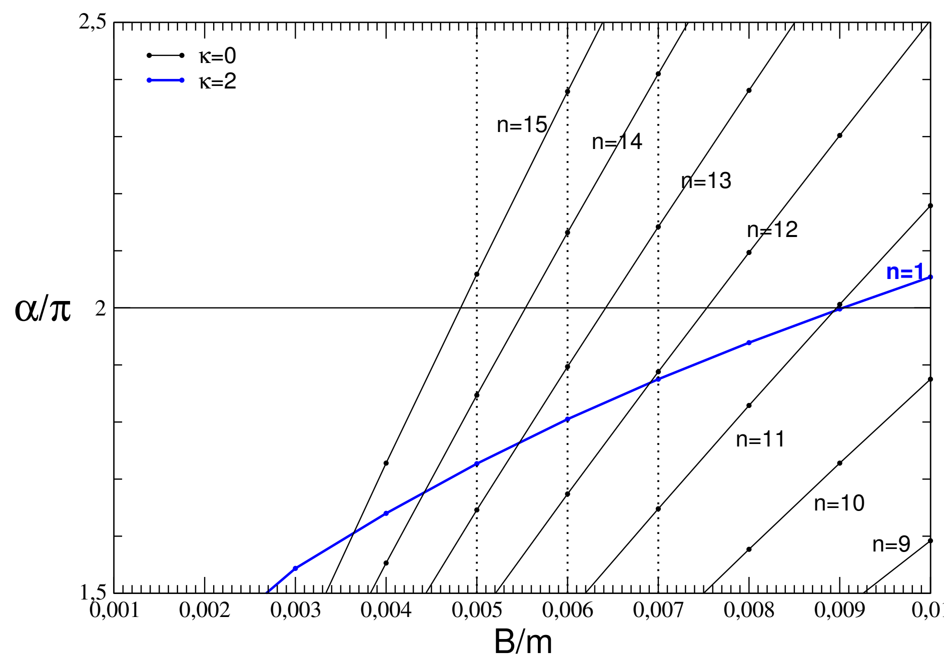
For B=0.009, **GAS** corresponds to the 11th state of the spectrum



But for smaller B, **GAS** corresponds to higher excitations :
B=0.003 n>15

If B→0, n → ∞ !

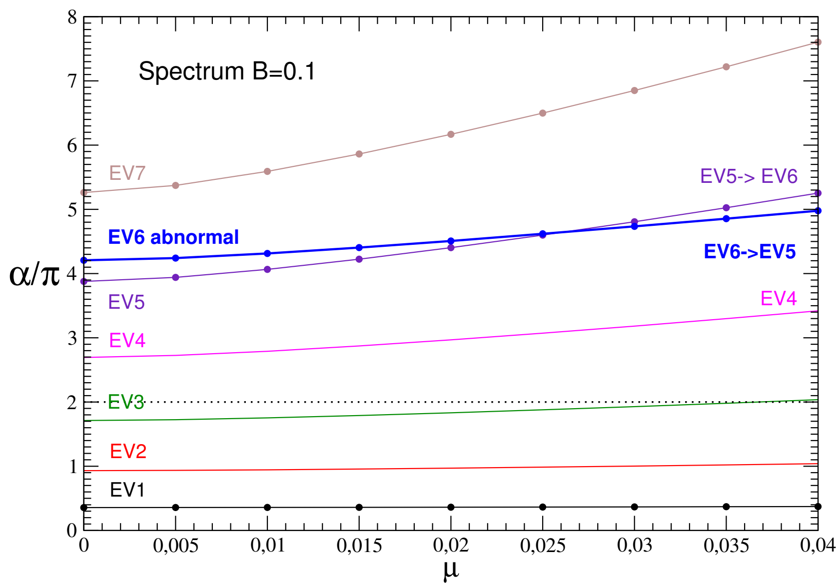
Level crossings: not fixed n !!!!

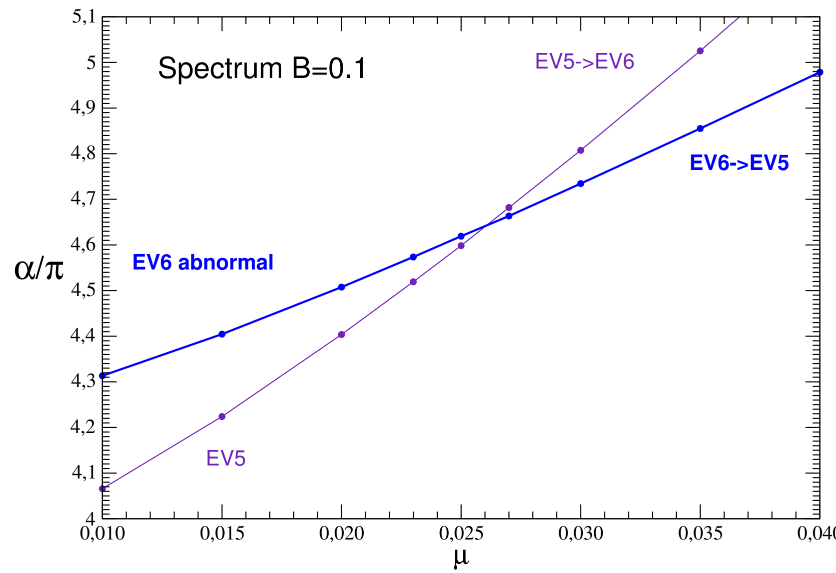


Our « fishing strategy » was

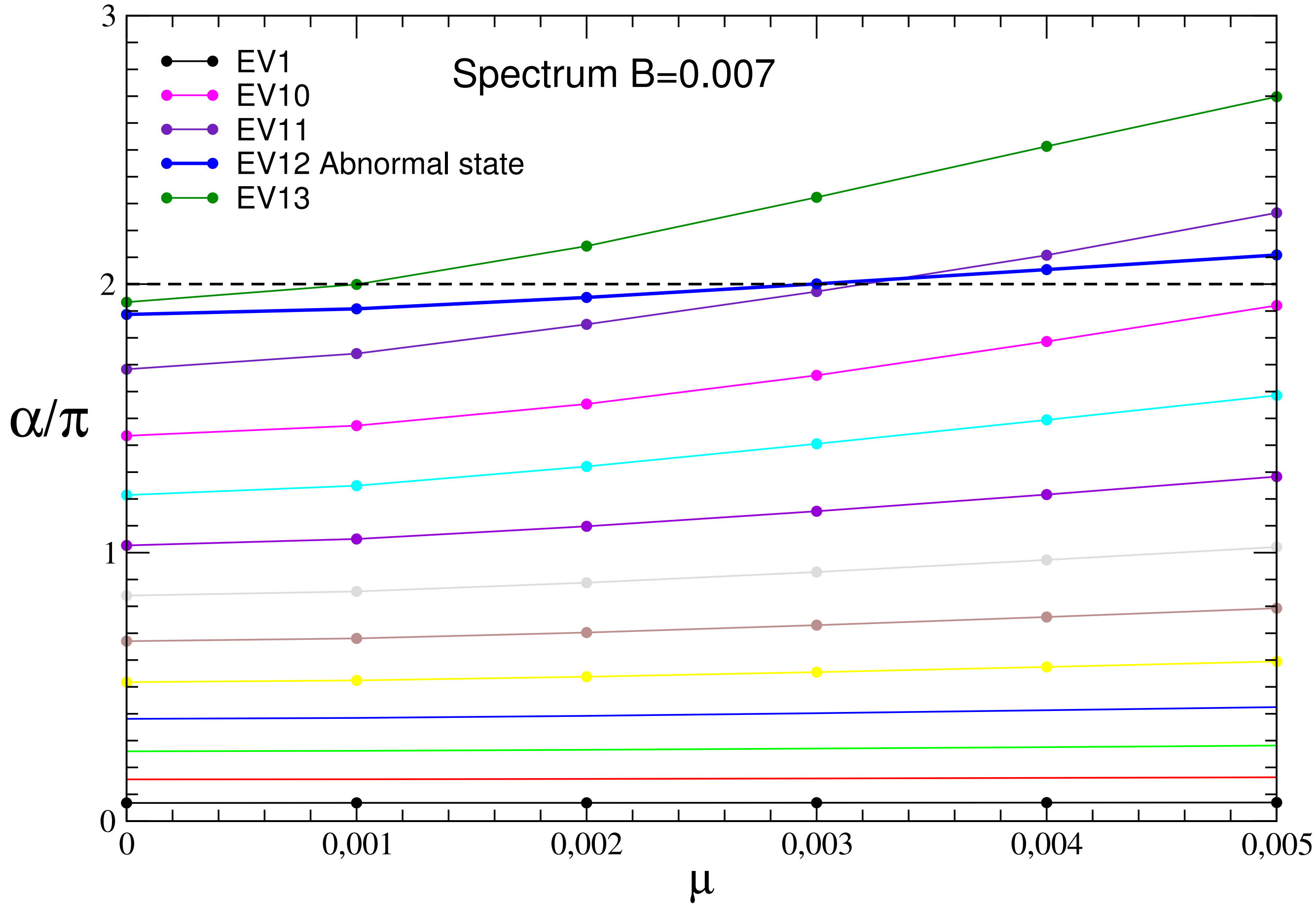
1. Identify the **GAS** at $\boldsymbol{\mu}$=0
2. follow its evolution as a function of **(B,α,μ)**
3. determine « the parameter domain of its existence » … if at all !!!

Exemple for B=0.1, just for illustrative purposes (tachyonic!)





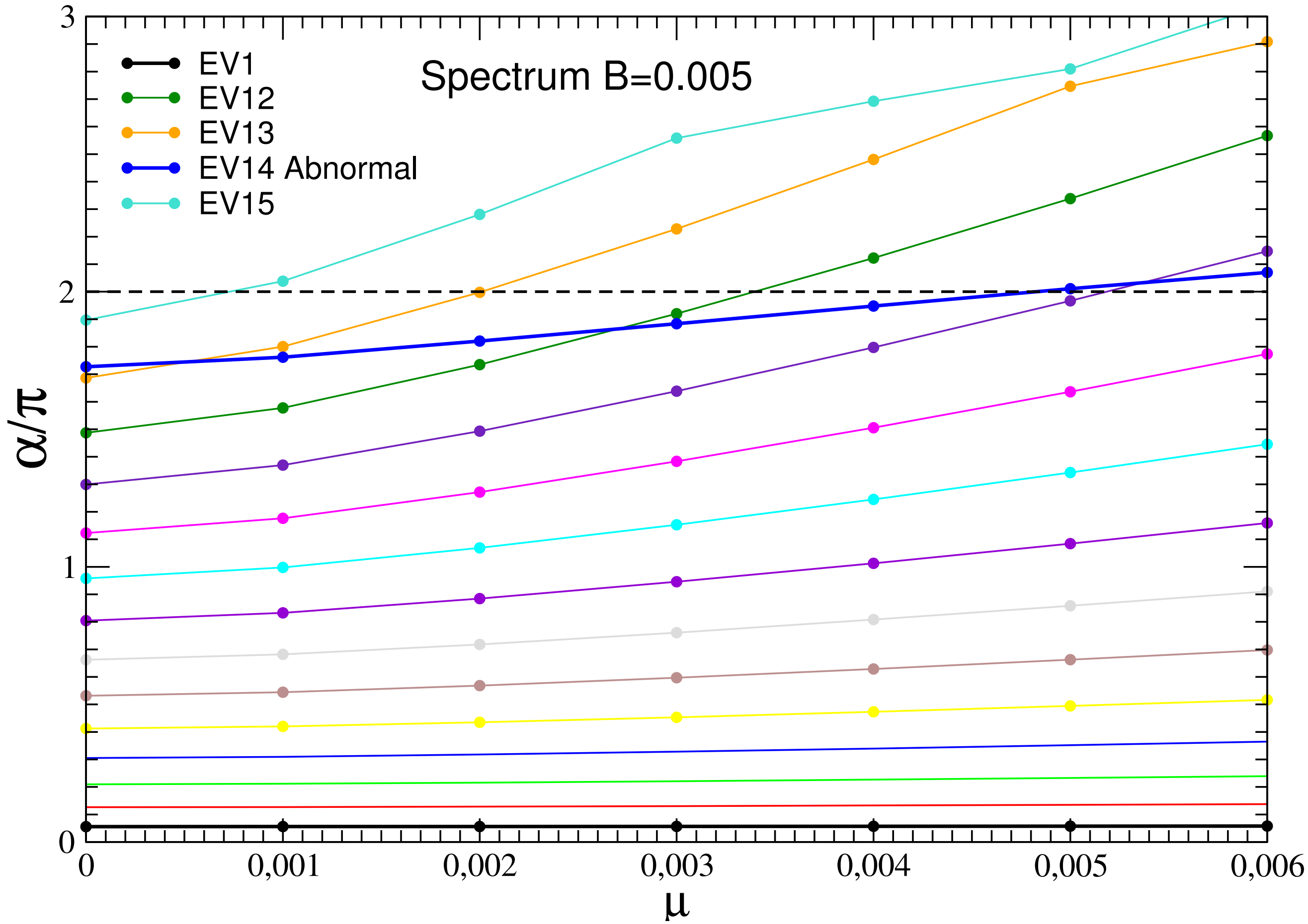
For **B=0.007** : the maximum value of μ for which $\alpha/\pi<2$ $\mu_{max}(B)=0.0034$



The **GAS** starts at 12th excitation, has one level crossing and becomes the 11th above $\mu=0.0030$

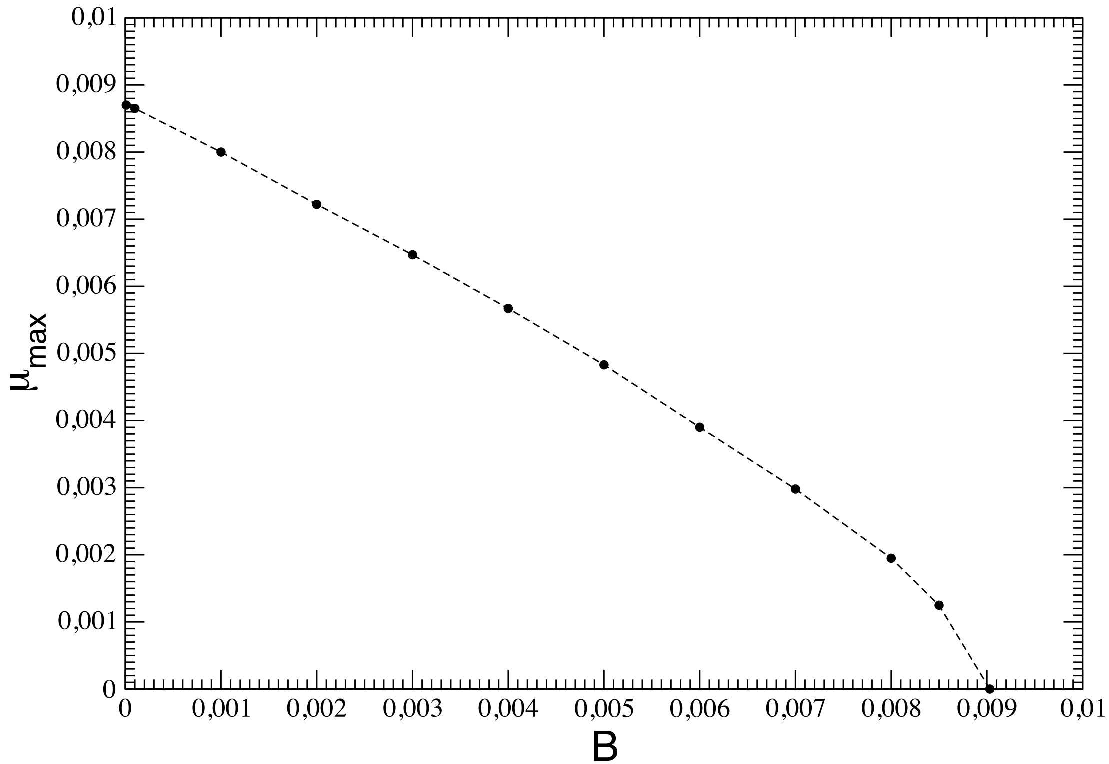
First evidence that (non tachyonic) abnormal states survive for $\mu>0$

For **B=0.005** : the maximum value of μ for which α/π<2 μ_{max}(B)=0.0050



The **GAS** starts as 14th excitation, has 2 level crossings and becomes the12th

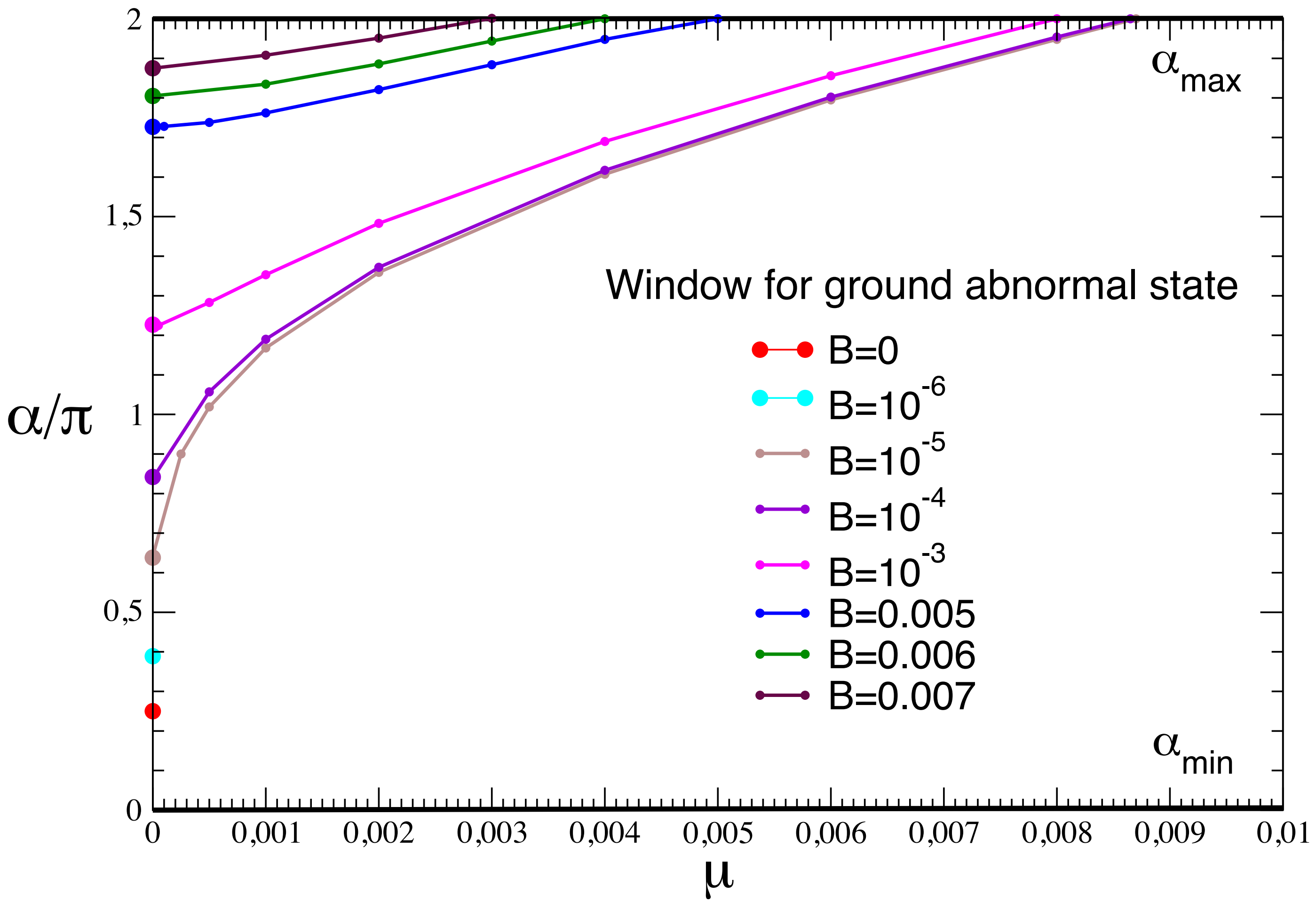
On can determine in this way the μ_{max}(B) dependence of the model



One can find non tachionic abnormal states in the energy range 0<B<0.0093

μ_{max}(B) decreases with B and has as maximim value μ_{max}(0)=0/0088

By collecting all previous results (accessible up to $B=10^{-5}$)
we have finally determined the (α,μ,B) « parameter window »



CONCLUSION

We have studied the normal and abnormal solutions of Bethe-Salpeter equation for a model consisting of two equal-mass (m=1) scalar particles interacting by a scalar exchange (**μ**)

$$iK(k,k') = -\frac{g^2}{(k-k')^2 - \mu^2 + i\epsilon}$$

$\alpha = g^2/4\pi$

For **$\mu=0$** (Wick-Cutkosky model) the **abnormal states** exist for **$\lambda=\alpha/\pi$** >1/4,
as for a “fictitious massive photon” of **μ** /M=0.4
If one restricts to “non tachyonic” ground state, the coupling constant is limited to

$$\frac{\pi}{4} < \alpha < 2\pi \quad \Leftrightarrow \quad \frac{1}{4} < \lambda < 2$$

and binding energies smaller than **B/m**=0.0093
They have a genuine many-body character with a two-body norm N_2 that vanishes in the limit $B\to 0$

For **$\mu>0$** the abnormal solutions remain but are limited to **$\mu/m=0.0088$** and **B/m<**0.0093

Work is in progress to study their $N_2(B)$ dependence as well as their existence in the case of different constituent masses