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Abnormal states with unequal constituent masses

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Spectrum of 2-body Coulomb system

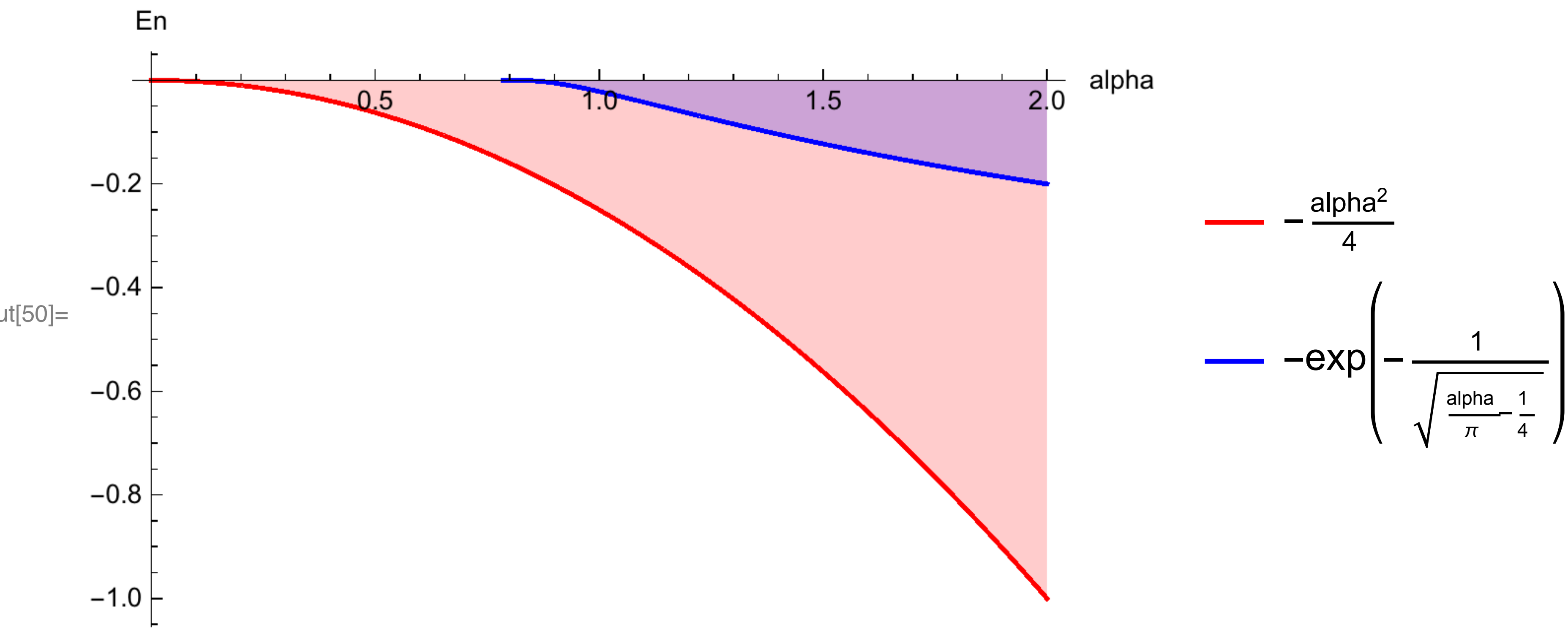
Nonrelativistic Schrödinger equation provides usual Balmer series:

$$E_n = -\frac{\alpha^2 m}{4n^2}$$

Relativistic Bethe-Salpeter equation reproduces the Balmer series (with a relativistic correction) and predicts another (abnormal) series (Wick & Cutkosky, 1954):

$$E_k = -m\exp\left(-\frac{2\pi k}{\sqrt{\frac{\mathcal{C}}{\pi}-\frac{1}{4}}}\right),\ \mathcal{C} = Z\alpha > \frac{\pi}{4} \to Z > 107$$

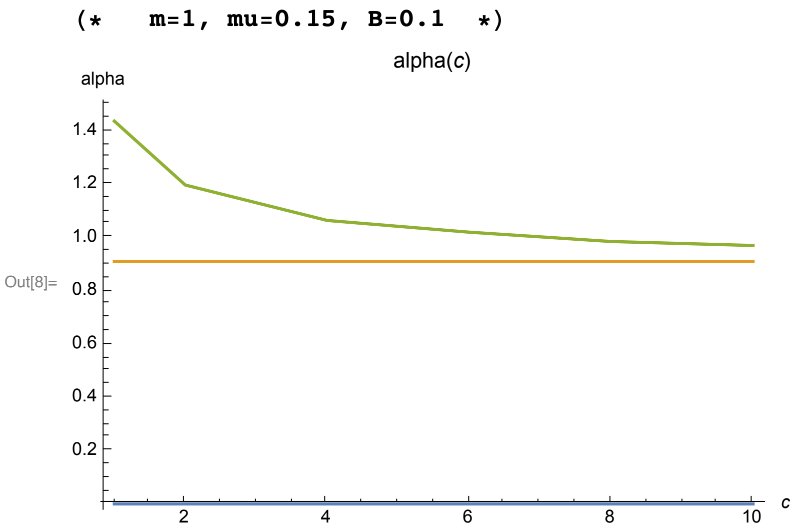
• Energy spectrum

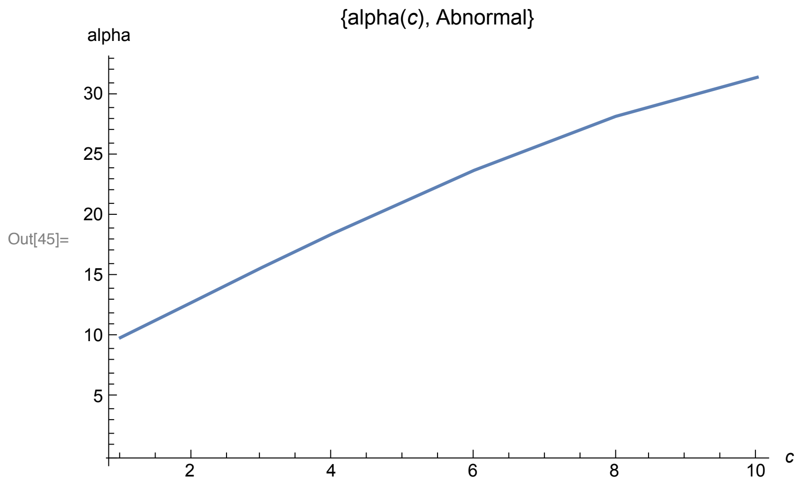


The binding energies for normal and abnormal states.

Abnormal states have purely relativistic origin!
They disappear in the nonrelativistic limit.
They are pushed out of the spectrum.

• Dependence $\alpha(c)$ vs. c for normal and abnormal solutions





Left panel: normal.

Right panel: abnormal

• Content of normal
and abnormal systems

J. Carbonell, V. A. Karmanov, H. Sazdjian,
Hybrid nature of the abnormal solutions of the Bethe-Salpeter equation
in the Wick-Cutkosky model,
Eur. Phys. J. C **81**, *50 (2021)*

The normal systems, almost for 100%,
consist of the two charged particles
+ small admixture ($\sim 1 \div 0.1\%$ or smaller) of massless
exchange particles.

On the contrary,
the abnormal systems, almost for 100%,
consist of the massless exchange particles
+ small admixture ($\sim 1 \div 0.1\%$ or smaller) of massive
charged particles.

• Experimental detection

The theoretical predictions put the experimental detection of the abnormal states on the agenda.

It would be ideal to deal with nucleus-antinucleus.

$$\mathcal{C} = (Ze)^2 = Z^2\alpha = \frac{1}{137}Z^2 > \frac{\pi}{4} \to Z = 11$$

(Natrium-Antinatrium) is enough. Unfortunately, heavy antinuclei are not available, at the present.

We should deal with electron and heavy ion ($Z > 107$). Therefore, the constituent masses are very different!

That's why this talk is devoted to the abnormal states with different constituent masses.

Do they exist in this case or not?

• BS equation with equal masses

$$\frac{1}{2}p + k = k_1, \qquad \frac{1}{2}p - k = k_2$$

$$\left[\left(\frac{1}{2}p + k\right)^2 - m^2\right]\left[\left(\frac{1}{2}p - k\right)^2 - m^2\right]\Phi_{eq}(k,p)$$
$$= \quad \frac{i\mathcal{C}m^2}{\pi^3}\int \frac{\Phi_{eq}(k';p)d^4k'}{(k-k')^2 + i\epsilon}, \quad \mathcal{C} = e_1e_2 = Ze^2 = Z\alpha.$$

$\mathcal{C}$ is the Coulomb constant: $\mathcal{C} = e_1e_2 = Ze^2 = Z\alpha$.
Integral representation (for the ground state $n = 1$):

$$\Phi_{eq}(k,p) = \int_{-1}^{1} \frac{-im^3 g_{eq}(z)dz}{[m^2(1-\eta_{eq}^2) - k^2 - p{\cdot}k\, z - \imath\epsilon]^3},$$

$$\eta_{eq}^2 = \frac{M^2}{4m^2}$$

• Equation for $g_{eq}(z)$ for the ground state $n = 1$

$$g''_{eq}(z) + \frac{\mathcal{C}}{\pi[1-\eta_{eq}^2(1-z^2)]}\frac{g_{eq}(z)}{(1-z^2)} = 0.$$

$-1 \leq z \leq 1,$ Boundary conditions: $g_{eq}(z = \pm 1) = 0.$

Principal quantum number $n = 1$ plays role of a parameter.

This is a homogeneous equation.

Normally, it has discrete spectrum.

It is indeed so for the Coulomb interaction.

The discrete levels and states are labeled by κ.

$\kappa = 0 \rightarrow$ normal states.

$\kappa = 1, 2, 3, \cdots \rightarrow$ abnormal states. And similarly for any n.

This is the mathematical origin of the abnormal states!

• BS equation with unequal masses

$$\mu_{1,2} = \frac{m_{1,2}}{m_1 + m_2}, \quad \mu_1 p + k = k_1, \qquad \mu_2 p - k = k_2$$

$$[(\mu_1 p + k)^2 - m_1^2][(\mu_2 p - k)^2 - m_2^2]\Phi_{un}(k,p)$$
$$= \quad \frac{i\mathcal{C}(1-\Delta^2)m_{12}^2}{\pi^3} \int \frac{\Phi_{un}(k';p)d^4k'}{(k-k')^2 + i\epsilon},$$

$$\Delta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{r-1}{r+1}, \quad r = \frac{m_1}{m_2}, \quad m_{12} = \frac{1}{2}(m_1 + m_2).$$

$\mathcal{C}$ is still the Coulomb constant: $\mathcal{C} = e_1 e_2 = Ze^2 = Z\alpha$.

If the particle 1 is the heavy ion, the particle 2 is electron, then

$$r \approx 1800{\cdot}10^2 \approx 2{\cdot}10^5 \gg 1.$$

• Integral representation

for the ground state $n = 1$

$$\Phi_{un}(k,p) = -i m_{12}^3 \int_{-1}^{1} dz\, g_{un}(z,\Delta) \times$$

$$\frac{1}{[m_{12}^2(1-\eta_{un}^2)(1+2z\Delta+\Delta^2) - k^2 - kp(z+\Delta) - i\epsilon]^3},$$

where

$$\Delta = \frac{m_1 - m_2}{m_1 + m_2}, \quad m_{12} = \frac{1}{2}(m_1 + m_2), \quad \eta_{un}^2 = \frac{M^2}{4m_{12}^2}.$$

• Equation for $g_{un}(z)$ for the ground state $n = 1$

$$g''_{un}(z,\Delta) + \frac{\mathcal{C}(1-\Delta^2)}{\pi Q(z,\Delta)} \frac{g_{un}(z,\Delta)}{(1-z^2)} = 0,$$

$$Q(z,\Delta) = (1+2z\Delta+\Delta^2)(1-\eta_{un}^2)+\eta_{un}^2(z+\Delta)^2, \quad \Delta = \frac{m_1 - m_2}{m_1 + m_2},$$

$-1 \leq z \leq 1,$ Boundary conditions: $g_{un}(z = \pm 1, \Delta) = 0.$

Dependence of $g_{un}(z,\Delta)$ and spectrum $\eta_{un}^2 = \frac{M^2}{m_{12}^2}$ on Δ is determined by this equation.

It can be found analytically.

• Reducing the unequal-masses problem to the equal-masses one

Cutkosky, 1954

$$g_{un}(z,\Delta) = \left(\frac{1+\Delta z}{1-\Delta^2}\right) g_{eq}\left(\frac{z+\Delta}{1+\Delta z}\right)$$

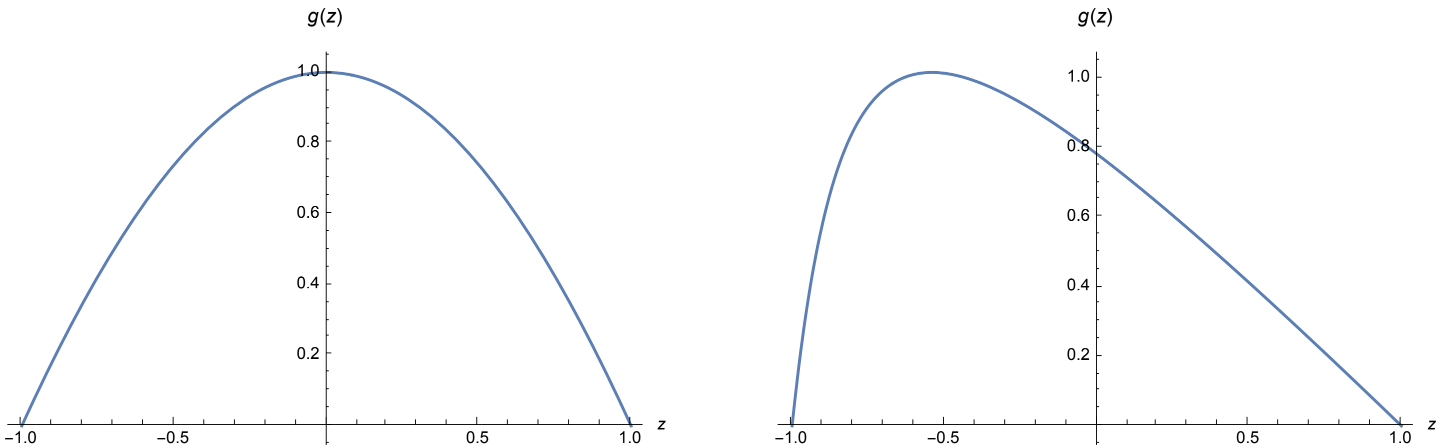
Then the unequal mass equation for $g_{un}(z,\Delta)$ is transformed to the equal one for $g_{eq}(z)$ and

$$\eta^2_{un} = \Delta^2 + (1-\Delta^2)\eta^2_{eq}.$$

Reminder: $\eta^2_{eq} = \frac{M^2}{4m^2}$, $\quad \eta^2_{un} = \frac{M^2}{4m^2_{12}}$, $\quad m_{12} = \frac{1}{2}(m_1+m_2)$.

To solve the unequal-masses problem, it is enough to solve the equal-masses one.

• Solutions
Normal states

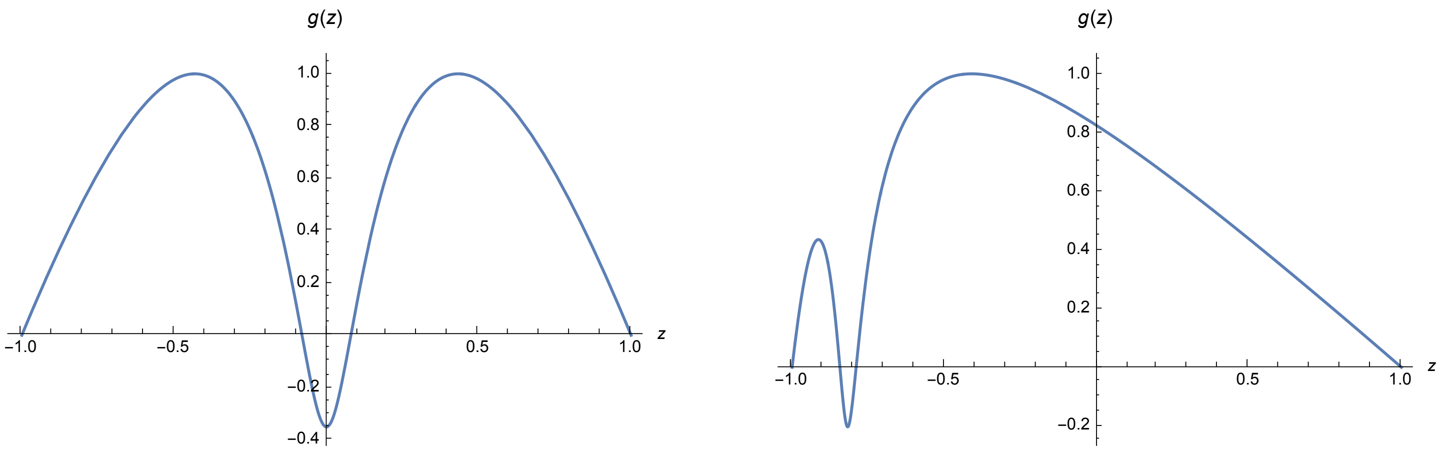


Left panel: $g_{eq}(z)$, equal masses ($r = 1$).

Right panel: $g_{un}(z, \Delta)$, non-equal masses ($r = 10 \to \Delta = \frac{9}{11}$).

• Solutions

Abnormal states, $\kappa = 2$



Left panel: $g_{eq}(z)$, equal masses ($r = 1$).

Right panel: $g_{un}(z, \Delta)$, non-equal masses ($r = 10 \to \Delta = \frac{9}{11}$).

• Binding energy vs. the ratio $r = \frac{m_1}{m_2}$

$$\eta_{un}^2 = \Delta^2 + (1 - \Delta^2)\eta_{eq}^2, \quad \Delta = \frac{m_1 - m_2}{m_1 + m_2} = \frac{r-1}{r+1}$$

$$\Longrightarrow$$

$$\frac{B_{un}}{m_2} = (r+1) - \sqrt{(r-1)^2 + 4r\left(\frac{2m - B_{eq}}{2m}\right)^2},$$

$$r \Longrightarrow \infty$$

$$B_{un} = 2\left(1 - \frac{B_{eq}}{4m_2}\right)B_{eq},$$

$$\Longrightarrow$$

If $\quad B_{eq} \ll m, \quad$ then $\quad B_{un} = 2B_{eq}$

Effect of different masses is attractive!

• Towards to finding balance: constituents – exchanged particles

Bethe-Salpeter amplitude in the coordinate state:

$$\Phi(x_1, x_2, p) = \langle 0 \,| T[\varphi(x_1)\varphi(x_2)] |\, p\rangle$$

State vector:

$$|p\rangle = \psi_2|2\rangle + \psi_3|3\rangle + \psi_4|4\rangle + \ldots$$

Normalization:

$$\langle p|p\rangle = 1 \;\rightarrow\; N_2 + N_3 + N_4 + \ldots = 1$$

where $N_2 = \int |\psi_2|^2 \ldots, \quad N_3 = \int |\psi_3|^2 \ldots$

Below we will calculate $N_2 = \int |\psi_2|^2 \ldots$

• Light-front wave function

Again BS amplitude in the coordinate state:

$$\Phi(x_1, x_2, p) = \langle 0 \,| T[\varphi(x_1)\varphi(x_2)] | \, p\rangle$$

$$\varphi(x) = \exp(iHt)\varphi(t=0, \vec{x}) \exp(-iHt)$$

is (very complicated!) Heisenberg operator.

On the quantization plane $t = 0$, $\varphi(x)$ becomes free.

Take the light-front quantization plane:

$t + z = 0 \to \omega{\cdot}x = 0, \omega = (\omega_0, \vec{\omega}),\ \omega^2 = 0$.

Then $\Phi(x_1, x_2, p)$ at $\omega{\cdot}x_1 = \omega{\cdot}x_2 = 0$ contains the free operators only and is determined by ψ_2:

$$\psi(\vec{k}_\perp, x) = \frac{x(1-x)}{\pi\sqrt{N_{tot}}} \int_{-\infty}^{\infty} \Phi\left(k + \frac{\beta\omega}{\omega{\cdot}p}, p\right) d\beta,$$

• Two-body contribution N_2

In terms of $g_{un}(z,\Delta)$:

$$\psi(\vec{k}_\perp,x)=\frac{(1-z^2)}{4\sqrt{N_{tot}}}\frac{m_{12}^3 g_{un}(z,\Delta)}{[\vec{k}_\perp^2+m_{12}^2 Q(z,\Delta)]^2},$$

$$Q(z,\Delta)=(1+2z\Delta+\Delta^2)(1-\eta_{un}^2)+\eta_{un}^2(z+\Delta)^2,\; z=1-2x.$$

$$\begin{aligned}
N_2 &= \frac{1}{(2\pi)^3}\int|\psi^2(R_\perp,x)|^2\frac{d^2R_\perp dx}{2x(1-x)}\\
&= \frac{1}{3{\cdot}2^7\pi^2 N_{tot}}\int_{-1}^{1}\frac{(1-z^2)g_{un}^2(z,\Delta)dz}{Q^3(z,\Delta)}\\
&= \frac{r^4}{3{\cdot}2^{15}\pi^2 N_{tot}}\int_{-1}^{1}\frac{(1-\bar{z}^2)g_{eq}^2(\bar{z})d\bar{z}}{[1-(1-\bar{z}^2)\eta_{eq}^2]^3}\propto\frac{r^4}{N_{tot}}
\end{aligned}$$

• Total normalization N_{tot}

Normalization condition: $\langle p|p\rangle = 1 \leftrightarrow F_{em}(0) = 1$

$$N_{tot} = F_{em}(0)$$

$$\begin{aligned} F_{em}(0) &= \frac{p\cdot J}{2M^2} = \frac{i}{M^2}\int \frac{d^4k}{(2\pi)^4}(\mu_2 M^2 - pk) \\ &\times (\mu_1^2 M^2 + 2\mu_1 pk + k^2 - m_1^2)\bar{\Phi}\,(k,p)\,\Phi\,(k,p)\,. \end{aligned}$$

This gives:

$$N_{tot}(r\to\infty) = F_{em}(0)|_{r\to\infty} = const\; r^4.$$

Hence:

$$N_2(r\to\infty) \propto \left.\frac{r^4}{N_{tot}}\right|_{r\to\infty} \to const$$

• Numerical results

No.	κ	r	B/m_2	N_2
1	0	1	0.99926	0.65
2	0	10	1.45983	0.65
3	0	∞	1.99852	0.65
4	2	1	$3.51169 \cdot 10^{-3}$	0.094
5	2	10	$6.38114 \cdot 10^{-3}$	0.102
6	2	∞	$7.02338 \cdot 10^{-3}$	0.093
7	4	1	$1.54091 \cdot 10^{-5}$	$6.19 \cdot 10^{-3}$
8	4	10	$2.80165 \cdot 10^{-5}$	$6.86 \cdot 10^{-3}$
9	4	∞	$3.08182 \cdot 10^{-5}$	$6.67 \cdot 10^{-3}$

• Conclusions

- The effect of unequal masses is "attractive" – the binding energy increases when $r = \frac{m_1}{m_2}$ increases.
  At $r \to \infty$ the binding energy increases up to the factor two.
- The two-body constituent contribution N_2 is changing insignificantly and remains small.
- The abnormal states with unequal masses are still dominated by the massless exchanges!

• Remarks

1. The abnormal states appear due to strong Coulomb forces. The spin effects change these forces:
   - less stronger for the parallel spins
   - more stronger for the antiparallel spins.

   Therefore, they can exist in the realistic case – heavy ions and electrons with spins.

2. To provide $\mathcal{C} = Z\alpha > \frac{\pi}{4}$, we need the ions with $Z > 107$.
   On the other hand, $Z < 137$ (pointlike) $\rightarrow$ $Z < 170$ (finite size of nuclei). Therefore $107 < Z < 170$.
   Transuranic nuclei with $Z > 107$ are created in laboratory, not very long living, but enough for experiments.
   For example, Flerovium: $Z = 114,\ T_{1/2} \approx 2$ sec.

It would be interesting (but not easy!) to study this problem in experiment! (To detect the abnormal states!)

Electron-ion collider is a tool for this search.

Thank you for your attention!