# A Theory of Condensed Dark Matter

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# Outline

Introduction

Non-interacting Dark Matter at finite density

First interactions at the Fermi surface

Superfluid Phases of Condensed Dark Matter

Focus on the BCS regime

# Introduction

# Dark Matter keywords - Choose your own adventure

- Strongly
- Self
- Weakly
- Feebly

- Bosonic
- Fermionic
- Composite
- Compact

- Superheavy
- Heavy
- Light
- Ultra-light

- Symmetric
- (partially)
- Asymmetric

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# Not a vacuum

Dark matter can reach high densities in $\neq$ regions of the Universe

- At the center of haloes
- Inside compact objects following DM capture
- DM spike near SMBH ?

One can expect collective quantum effects, such as superfluidity, to emerge.

## Non-interacting Dark Matter at finite density

# Free degenerate Dark Matter

$$H \to H - \mu N, \mu \text{ the chemical potential}$$

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\psi}\mu\gamma^0\psi$$

$$\langle \bar{\psi}\gamma_0\psi \rangle = \langle \psi^\dagger \psi \rangle = \int \frac{d^3k}{(2\pi)^3} f_{\text{Fermi-Dirac}}(k) = \text{Number density}$$

$$\text{At zero temperature : } \mu = \sqrt{m^2 + k_F^2},\ n = k_F^3/3\pi^2$$

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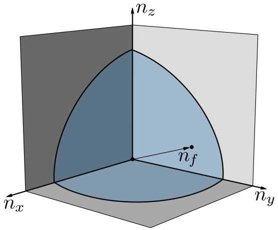


- Fermi Pressure
- Self-gravitating configurations
- Cores of dwarf spheroidal galaxies [Domcke, 2014 ; Randal 2017]

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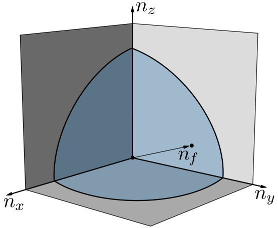


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- Cores of dwarf spheroidal galaxies [Domcke, 2014 ; Randal 2017]

**Self-interactions can significantly change this picture.**

# First interactions at the Fermi surface

# The scalar condensate and finite density

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ + \gamma^0\mu - m)\psi + \frac{1}{2}\left(\partial^2\phi - m_\phi^2\phi^2\right) + g\,\bar{\psi}\psi\phi$$

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$$\rightarrow\bar{\psi}(i\partial\!\!\!/ + \gamma^0\mu - [m - g\left\langle\phi\right\rangle])\psi \text{ with } \left\langle\phi\right\rangle = \frac{g}{m_\phi^2}\left\langle\bar{\psi}\psi\right\rangle$$

# The scalar density condensate

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ + \gamma^0\mu - m)\psi + \frac{1}{2}\left(\partial^2\phi - m_\phi^2\phi^2\right) + g\,\bar{\psi}\psi\phi$$

Minimize the potential $V(\phi) = \frac{1}{2}m_\phi^2\phi^2 - g\left\langle\bar{\psi}\psi\right\rangle\phi$

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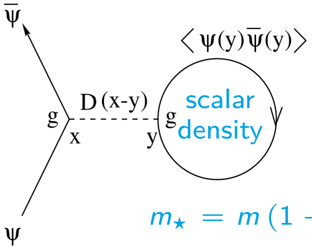



$m_\star = m(1 - \varphi)$

SIDM & finite density !

Phenomenological applications in :

Nuclear Physics [Walecka, 1974]

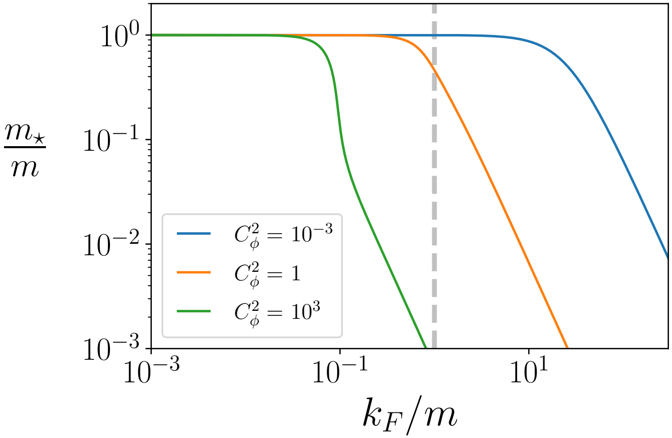
Neutrinos [Stephenson, 1996 ; Smirnov 2022]

Cosmology [Esteban, 2021]

Dark stars [Gresham, 2017, 2018]

**Cored haloes, avoid TG [Garani, Tytgat, JV 2022]**

# Large DM self-interactions → New phases !




- $C_\phi^2 = 4\alpha_\phi/3\pi \times (m_{\rm DM}/m_\phi)^2$ the DM$-\phi$ coupling
- Effective mass in the medium : $m_\star = m\,(1-\varphi) < m$
- Qualitatively different from a thermal mass !
- van der Waals matter : gas/liquid [Gresham, 2018 ; **Garani, Tytgat, JV 2022**]

# Superfluid Phases of Condensed Dark Matter

# Superbehaviours and their ingredients

Breaking of a global/local symmetry and free transport of mass/charge.

Recent interest in dark matter phenomenology :

- Bosonic DM [Berezhiani, 2015]
- Coloured dark sector (« STUMP ») [Alexander, 2020]
  - Imports results from the QCD community
  - Superfluid quark matter at large densities (« CFL »)

Emerges if, for example,

- Fermionic system
- Degeneracy
- Attractive interactions

The Yukawa theory is very economical and can exhibit also superfluidity ! [Pisarski, 1999 ; Alford 2017] & toy model for more complicated setup.

# Characterizing superfluid phases

Start by focusing on non-relativistic/low-density systems

- Superfluidity emerges via non-relativistic scattering between DM particles at the Fermi surface

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---

Qualitative understanding via scattering length $a$ [Bethe, 1949 ; Chu, 2020]

$$\mathcal{M}^{l=0} = \frac{1}{k\left(\cot\delta - i\right)}, \; k\cot\delta\big|_{\text{s-wave}} \approx \frac{-1}{a} + \frac{r_e}{2}k^2$$

$$\sigma_0 \approx \frac{4\pi a^2}{1 + k^2\left(a^2 - a r_e\right) + \frac{1}{4}a^2 r_e^2 k^4}$$

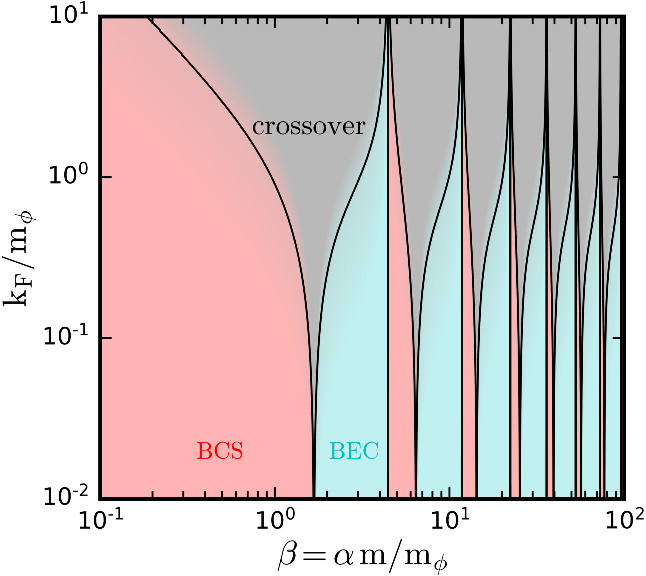
---

Allows for delimiting the different superfluid phases :

- $\left(k_F a\right)^{-1} > 1$ corresponds to bound state formation (BEC)
- $\left(k_F a\right)^{-1} < -1$ corresponds to attractive interactions (BCS)
- $\left(k_F |a|\right)^{-1} \to 0$, the unitary limit (crossover)

$\to$ Superfluid phase diagram of the Yukawa theory.

# The low-density phases of Condensed Dark Matter

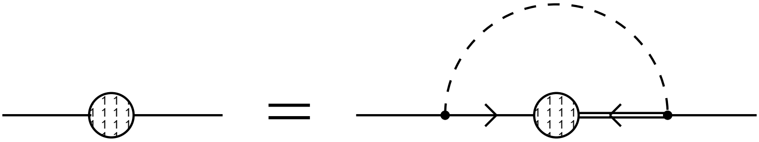



[Garani, Tytgat, JV 2022]

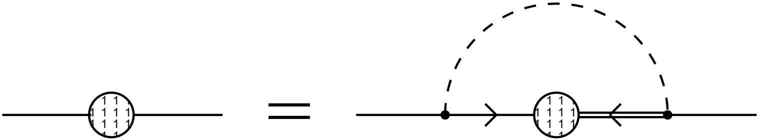
# Focus on the BCS regime

# The superfluid energy gaps or Cooper pairing

The blob is the « gap » $\langle\psi\psi\rangle$ : the number density of Cooper pairs.



# The superfluid energy gaps or Cooper pairing

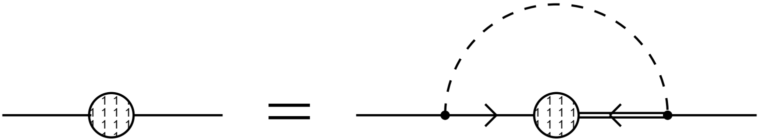


$$\tilde{\Delta}_{\pm}(p) = \frac{g^2}{32\pi^2} \sum_{\eta=\pm} \int_0^{\infty} dk \frac{k}{p} \left\{ \log \frac{m_\phi^2 + (p-k)^2}{m_\phi^2 + (p+k)^2} + \right.$$

$$\pm\eta \frac{kp}{\omega_p \omega_k} \left( -2 + \frac{m_\phi^2 + k^2 + p^2}{2kp} \log \frac{m_\phi^2 + (p-k)^2}{m_\phi^2 + (p+k)^2} \right)$$

$$\left. \pm\eta \frac{m^2}{\omega_p \omega_k} \log \frac{m_\phi^2 + (p-k)^2}{m_\phi^2 + (p+k)^2} \right\} \frac{\tilde{\Delta}_\eta(k)}{\epsilon_\eta(k)}$$

# The superfluid energy gaps or Cooper pairing

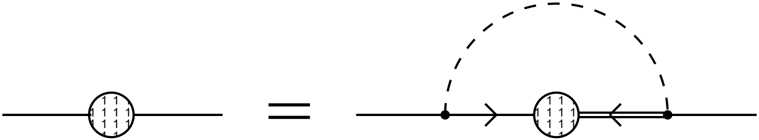


In the heavy mediator limit, $m_\phi \gg m, \mu$ :

$$\Delta_1 = \frac{g^2}{2m_\phi}\int \frac{d^3k}{(2\pi)^3}\frac{\Delta_1}{\sqrt{(\omega_k-\mu)^2+\Delta_1^2}}$$

*i.e.*, the textbook BCS gap equation

# The superfluid energy gaps or Cooper pairing



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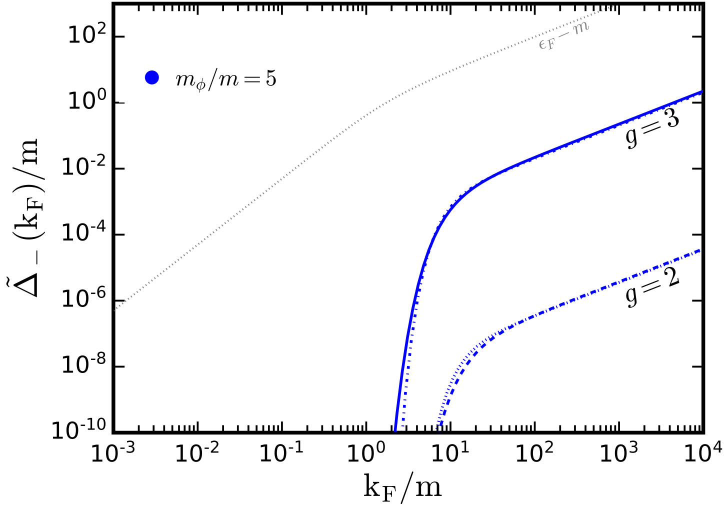
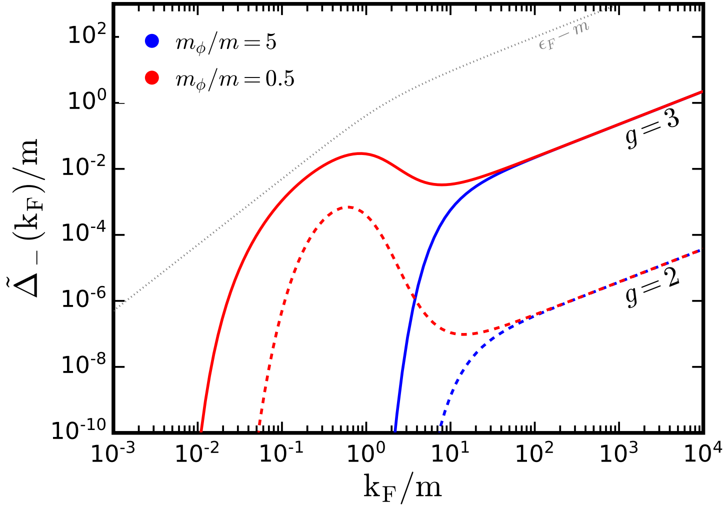
- General setup : heavy/light mediator, (non-)relativistic DM
- Not plagued by UV divergences
- Exactly what is required for doing phenomenology.
- Also sole phase for relativistic system : focus.

# Solutions to the gap equations : Heavy mediators




$$\tilde{\Delta}_- \approx \frac{4}{3} k_F \times \exp\left(-8\pi^2/g^2\right) \times \exp\left(-\frac{4\pi^2}{g^2}\frac{m_\phi^2}{k_F^2}\right) \quad \text{for } m < m_\phi$$

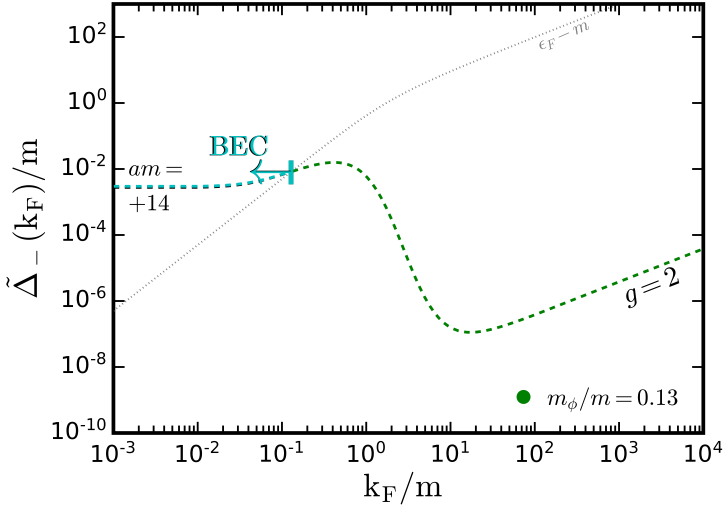
# Solutions to the gap equations : Beyond heavy mediators




Beyond the BCS assumptions.

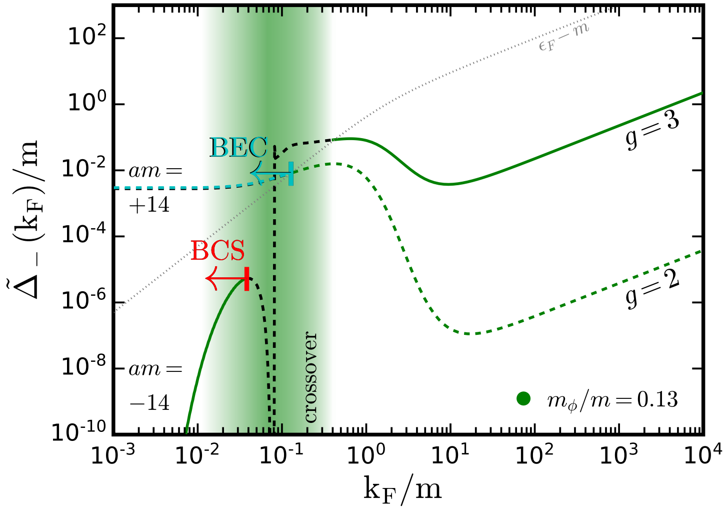
Larger gaps : Non-relativistic DM need light mediator !

# Solutions to the gap equations : Light mediators




- Change of regime at $k_F \sim a^{-1}$.
- Signature of the « phase transition »

# Solutions to the gap equations : Light mediators




- Limit not usually studied in other realization.
- Typical $g$ and $m_\phi$ for SIDM, $\sigma_{DM-DM}$ large but not excluded !
- Again, signature of the « phase transition ».

# Beyond the scalar interaction

The formalism we developed allows for other choices of mediators.
Some attractive choices would be :

Pseudo-scalar :

- No scalar density (zero momentum effect)
- Gap equations similar in spirit to the scalar mediator
- Requires a theory of non-relativistic scattering for full understanding.

Spin 2 :

- Work ongoing
- Many theoretical aspects to clear up.
- Graviton vs. Kaluza-Klein $\leftrightarrow$ extra dimension
- We expect new, qualitatively different density effects to appear.

# Conclusion

- DM could develop non-zero chemical potential in the Universe
- There exists a rich dark matter phase diagram at low densities
- Emerges due to strong, long-range DM$-$DM interactions
- I presented the thermodynamics of it. Dynamics is hard.
- We are at the crossroad of many areas of physics : Dark matter, condensed, nuclear, scattering physics
- Zero $T$, zero $\mu$ physics has been extensively chartered.

Fin

# Back up slides

# The BCS argument and electron superconductivity

$$\Omega_N = E - \mu N$$

Add a particle

$$\Omega_{N+1} = E + \mu - \mu\,(N+1)$$

with attractive interactions

$$\Omega_{N+1} < \Omega_N$$

Formation of many bosonic Cooper pairs which condensate $\sim \langle \psi\psi \rangle$
Inevitable

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For electronic superconductors :

- Photon exchanges between electrons are repulsive
- Attractive interaction sourced by the lattice (phonons), thanks to screening
- The $U(1)$ theory is the most complicated !

# Color superconductivity in QCD with 2 light flavours

QCD seems quite different than the Yukawa theory :

- How to handle the strong interaction ?
- Lattice can't help : numerical sign problem.
- If $\mu \gg \Lambda_{\mathrm{QCD}}$, probe short distance/large momentum exchange at the Fermi surface ✓

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How does super-behaviours arise ?

- $\mathbf{3} \times \mathbf{3} = \bar{\mathbf{3}} + \mathbf{6}$ : antisymmetric channel is attractive ✓
- Pairing is most attractive in ${}^1S$ : $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$
- Wavefunction has to be overall antisymmetric
- Need to be also antisymmetric in flavour ✓

# In the Yukawa theory

$$\mathcal{L} = \bar{\psi}(i\not{\partial} + \gamma^0\mu - m)\psi + \frac{1}{2}\left(\partial^2\phi - m_\phi^2\phi^2\right) + g\,\bar{\psi}\psi\phi + \frac{1}{2}m_\phi^2\phi^2$$

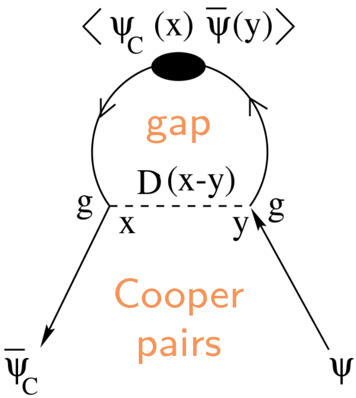
- There is enough in this very simple theory to accommodate what is needed for BCS superconductivity
-
  - Attractive particle−particle interactions ✓
  - $^1S$ pairing $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ ✓
  - Only need one specie ✓
- 1 colour, 1 flavour, spin−0 limit of QCD, share some properties

# BCS in QFT and parallels to BEH

Same Yukawa theory but heavy mediator : 4−fermion interaction

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ + \gamma^0\mu - m)\psi + G_\phi\bar{\psi}\bar{\psi}\psi\psi$$

$$\approx \begin{pmatrix}\bar{\psi} & \bar{\psi}_C\end{pmatrix}\begin{pmatrix} k\!\!\!/ + \mu\gamma^0 - m & \langle\psi\bar{\psi}_C\rangle \times G_\phi \\ \langle\psi_C\bar{\psi}\rangle \times G_\phi & k\!\!\!/ - \mu\gamma^0 - m\end{pmatrix}\begin{pmatrix}\psi \\ \psi_C\end{pmatrix}$$



# Dirac structure of the gap

$\Delta$ has fermionic indices, respects Fermi statistics,

$$\Delta_{\alpha\beta} \equiv \left\langle \psi_{C,\alpha}(x)\, \bar{\psi}_{\beta}(y) \right\rangle$$

The most general structure of the gap is

$$\begin{aligned}\Delta =& \Delta_1 \gamma_5 + \Delta_2 \boldsymbol{\gamma} \cdot \hat{\boldsymbol{k}} \gamma_0 \gamma_5 + \Delta_3 \gamma_0 \gamma_5 \\ &+ \Delta_4 + \Delta_5 \boldsymbol{\gamma} \cdot \hat{\boldsymbol{k}} \gamma_0 + \Delta_6 \boldsymbol{\gamma} \cdot \hat{\boldsymbol{k}} \gamma_5 \\ &+ \Delta_7 \boldsymbol{\gamma} \cdot \hat{\boldsymbol{k}} \gamma_5 + \Delta_8 \gamma_0\end{aligned}$$

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For the Yukawa theory, all gaps are zero or negligible except

$$\Delta = \Delta_1\gamma_5 + \Delta_2\boldsymbol{\gamma}\cdot\hat{\boldsymbol{k}}\gamma_0\gamma_5 + \Delta_3\gamma_0\gamma_5$$

# Dispersion relation in the medium

$$\Delta = \Delta_1\gamma_5 + \Delta_2\boldsymbol{\gamma}\cdot\hat{\boldsymbol{k}}\gamma_0\gamma_5 + \Delta_3\gamma_0\gamma_5$$

$$\mathcal{L} = \begin{pmatrix}\bar{\psi} & \bar{\psi}_C\end{pmatrix} \underbrace{\begin{pmatrix} \not{k} + \mu\gamma^0 - m & \Delta(k) \\ \Delta(k) & \not{k} - \mu\gamma^0 - m \end{pmatrix}}_{\text{inverse propagator}} \begin{pmatrix}\psi \\ \psi_C\end{pmatrix}$$

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$$\begin{aligned}\epsilon_\pm^2 =& \mu^2 + \omega^2 + \Delta_1^2 + \Delta_2^2 + \Delta_3^2 + \\ & \pm 2\left(\mu^2\omega^2 + 2\mu k\Delta_1\Delta_2 + m^2\Delta_2^2 + \Delta_1^2\Delta_2^2 + 2m\mu\Delta_1\Delta_3 \right. \\ & \left. -2mk\Delta_2\Delta_3 + k^2\Delta_3^2 + \Delta_1^2\Delta_3^2\right)^{1/2}\end{aligned}$$

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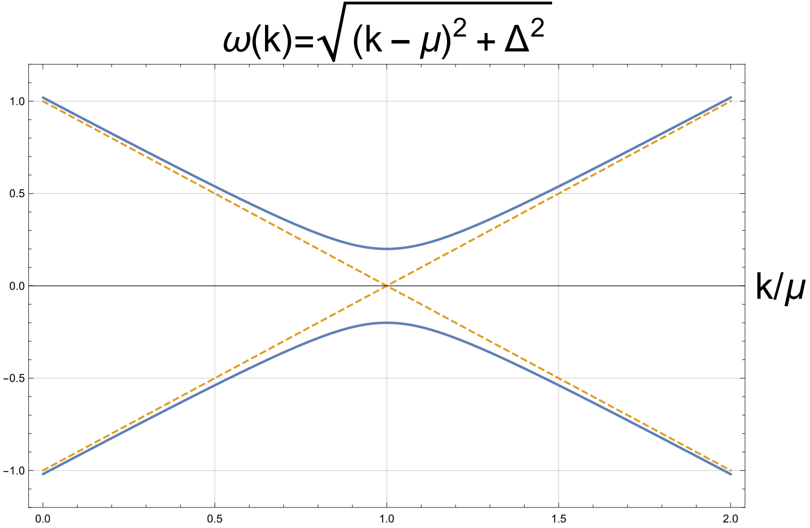
In BCS, $\Delta \ll \mu$

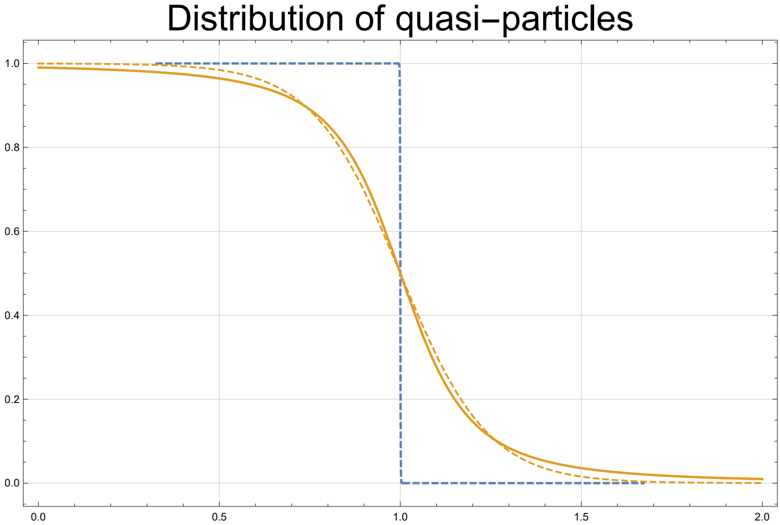
$$\epsilon_\pm^2 \approx (\omega \pm \mu)^2 + \left( \Delta_1 \pm \left( \frac{k}{\omega} \Delta_2 + \frac{m}{\omega} \Delta_3 \right) \right)^2$$

Effectively, there's one gap, as the standard BCS theory.

# Dispersion relation of massless particles with $\Delta = 0.2\mu$



# Impulsion distribution of massless particles with $\Delta = 0.2\mu$




- Dashed orange : Fermi-Dirac distribution with $\beta = 8$ (best fit)

# How to determine Δ ? The gap equations

- Starting from the action,
  $S = \int_{x,y} \left[\bar{\psi}(x)\, G_0^{-1}(x,y)\, \psi(y) - \frac{1}{2}\phi(x)\, D^{-1}(x,y)\, \phi(y)\right] - g \int_x \bar{\psi}(x)\, \psi(x)\, \phi(x)$
  do Hubbard-Strantanovich transformation to introduce gaps
  $1 \propto \int \mathcal{D}\bar{\Delta}\mathcal{D}\Delta \exp\{-\frac{1}{2}(\Delta - \psi_c\bar{\psi})\frac{D}{2}(\bar{\Delta} - \psi\bar{\psi}) - \frac{1}{2}(\bar{\Delta} - \psi\bar{\psi})\frac{D}{2}(\Delta - \psi_c\bar{\psi})\}$

# How to determine $\Delta$ ? The gap equations

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  $Z = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}\bar{\Delta} \mathcal{D}\Delta exp(-S)$
  Use mean-field approximation then compute path integrals

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  $g \int_x \bar{\psi}(x)\, \psi(x)\, \phi(x)$

  do Hubbard-Strantanovich transformation to introduce gaps
- $S = \int_{x,y} \bar{\psi}(x)\, G_0^{-1}(x,y)\, \psi(y) + \bar{\Delta} D \Delta + \psi \bar{\Delta} \psi + \bar{\psi} \Delta \bar{\psi}$
  $Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{\Delta}\mathcal{D}\Delta exp(-S)$

  Use mean-field approximation then compute path integrals
- Obtain partition function, free energy $\Omega$

  $\Omega = -\frac{T}{V} \log Z$

# How to determine $\Delta$ ? The gap equations

- Starting from the action,
  $S = \int_{x,y} \left[\bar{\psi}(x)\, G_0^{-1}(x,y)\, \psi(y) - \frac{1}{2}\phi(x)\, D^{-1}(x,y)\, \phi(y)\right] - g\int_x \bar{\psi}(x)\, \psi(x)\, \phi(x)$
  do Hubbard-Strantanovich transformation to introduce gaps
- $S = \int_{x,y} \bar{\psi}(x)\, G_0^{-1}(x,y)\, \psi(y) + \bar{\Delta} D \Delta + \psi \bar{\Delta} \psi + \bar{\psi} \Delta \bar{\psi}$
  $Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{\Delta}\mathcal{D}\Delta exp(-S)$
  Use mean-field approximation then compute path integrals
- Obtain partition function, free energy $\Omega$
  $\Omega = -\frac{T}{V}\log Z$
- Differentiate $\Omega$ w.r.t gaps, set to 0 :
  $\frac{\partial \Omega}{\partial \Delta} = 0$

# How to determine $\Delta$ ? The gap equations

- Starting from the action, do Hubbard-Strantanovich transformation to introduce gaps
- Use mean-field approximation then compute path integrals
- Obtain partition function, free energy $\Omega$
- Differentiate $\Omega$ w.r.t gaps, set to 0 :

$$\tilde{\Delta}_\pm(p) = \frac{g^2}{32\pi^2}\sum_\eta \int_0^\infty dk \frac{k}{p}\left\{\log\frac{m_\phi^2+(p-k)^2}{m_\phi^2+(p+k)^2} + \pm\eta\frac{kp}{\omega_p\omega_k}\left(-2+\frac{m_\phi^2+k^2+p^2}{2kp}\log\frac{m_\phi^2+(p-k)^2}{m_\phi^2+(p+k)^2}\right) \pm\eta\frac{m^2}{\omega_p\omega_k}\log\frac{m_\phi^2+(p-k)^2}{m_\phi^2+(p+k)^2}\right\}\frac{\tilde{\Delta}_\eta(k)}{\epsilon_\eta(k)}$$

- Solve by iterative numerical methods. **Very general setup !**

# The Schwinger-Dyson equation

The gap structure and the gap equations are

$$\Phi^{\pm} = \pm\Delta_1\gamma_5 + \Delta_2\vec{\gamma}\cdot\hat{\vec{k}}\gamma_0\gamma_5 + \Delta_3\gamma_0\gamma_5$$

$$\Phi^{+}(p) = g^2\frac{T}{V}\sum_k D(p-k)\, G_0^{\mp}\Phi^{\pm}G^{\pm}$$

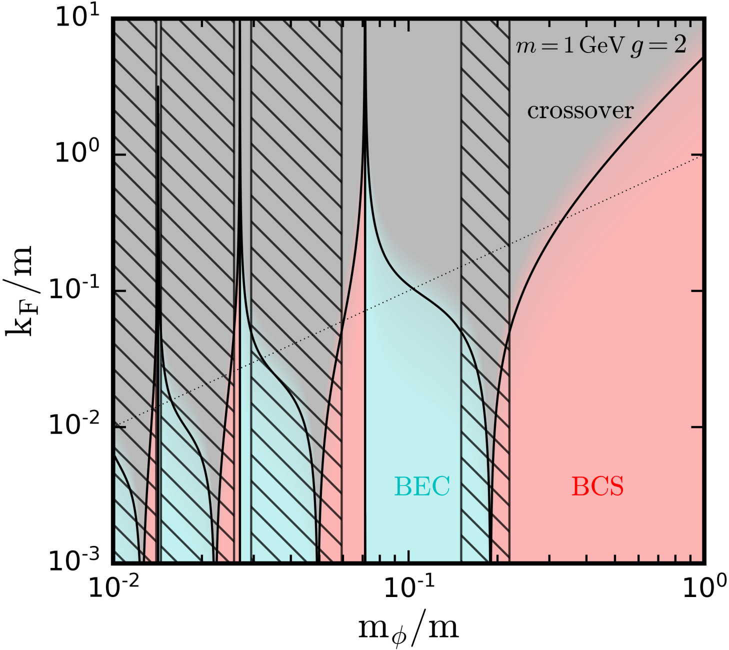
In a more transparent manner,

$$\Delta_1\gamma_5 + \Delta_2\vec{\gamma}\cdot\hat{\vec{p}}\gamma_0\gamma_5 + \Delta_3\gamma_0\gamma_5 =$$

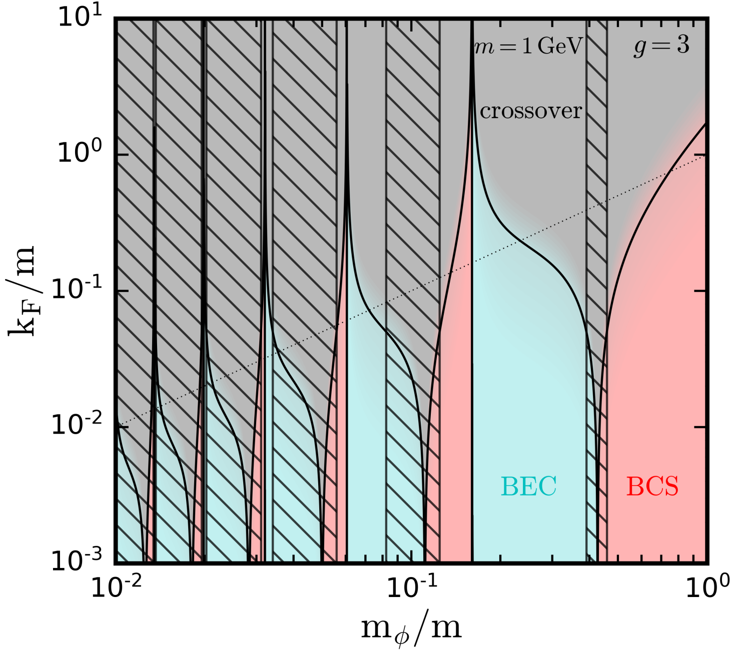
$$= \frac{-g^2}{\mathcal{V}/T}\sum_k D(p-k)\left[\frac{\Delta_1 - \Delta_2\frac{k}{\omega} - \Delta_3\frac{m}{\omega}}{k_0^2 - \left(\epsilon_k^-\right)^2}\Lambda^-\gamma_5 + \right.$$

$$\left. + \frac{\Delta_1 + \Delta_2\frac{k}{\omega} + \Delta_3\frac{m}{\omega}}{k_0^2 - \left(\epsilon_k^+\right)^2}\Lambda^+\gamma_5\right]$$

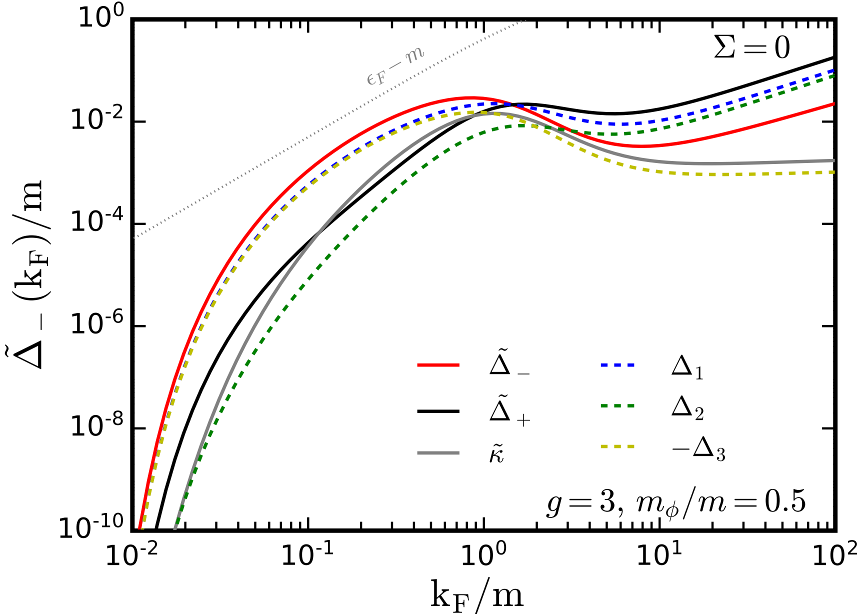
# SIDM cluster constraints for $m = 1$ GeV, $g = 2$



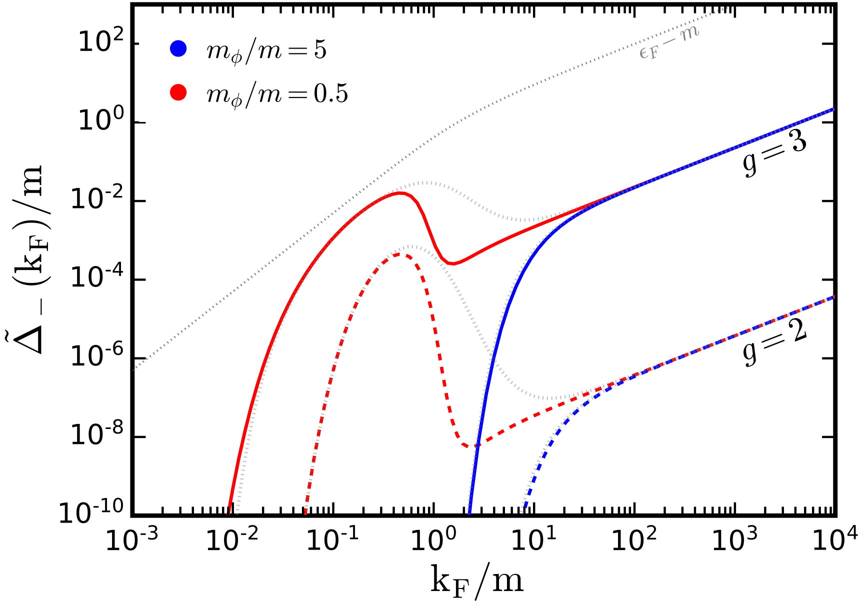
# SIDM cluster constraints for $m = 1$ GeV, $g = 3$



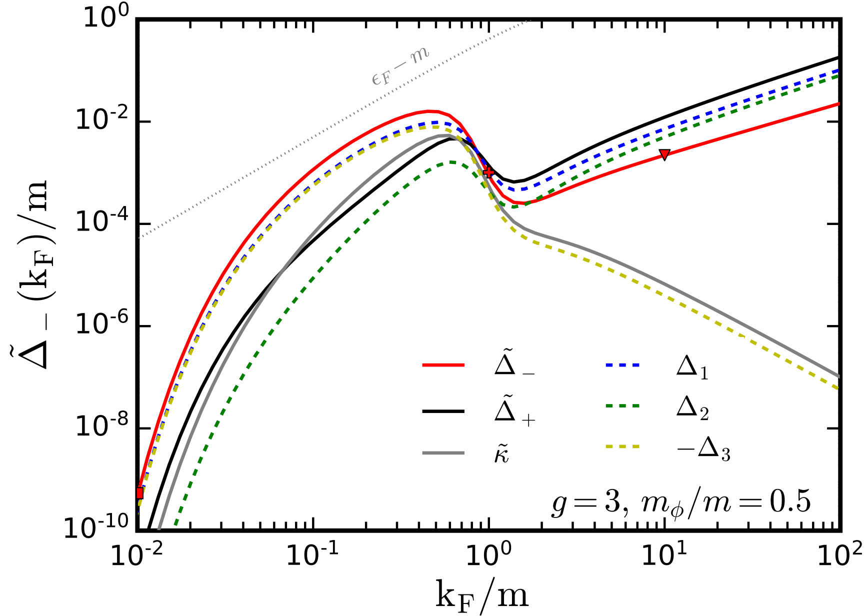
# The other gaps



# Gaps + condensate

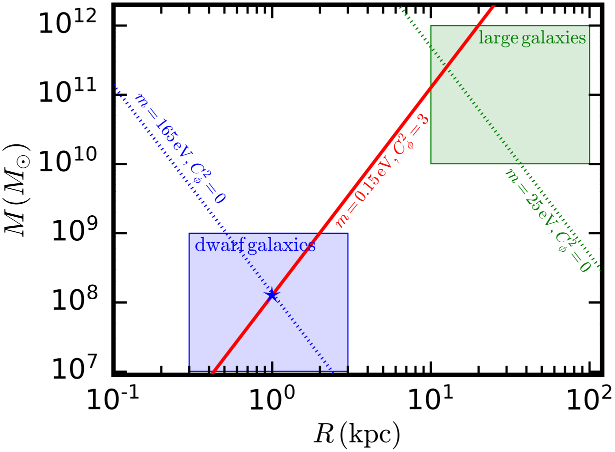


# Gaps + condensate



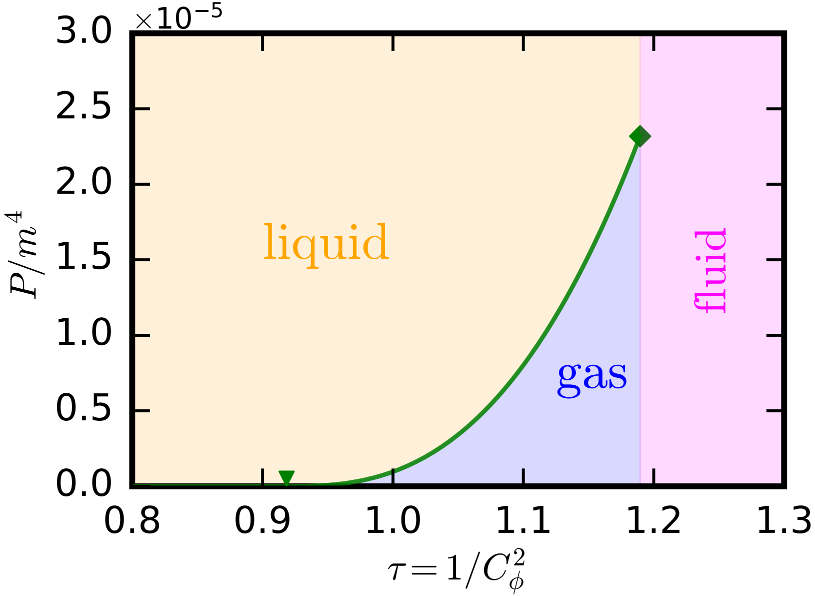
# One application : Scalar density dominated haloes

- Taking into account a gas−liquid phase transition
- Adding gravity through the TOV equations




- Dashed : Free fermions, $M \propto R^{-3}$. Dashed blue $m = 165$ eV
- Solid : Condensate, $M \propto R^3$. Very light. Viable w.r.t Bullet Cluster.

## Gas/liquid phase transition : $P - T$ phase diagram



# Gas/liquid phase transition : Coexisting phases