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Université de Strasbourg

Master Science, Mention Physique
Spécialité Physique Subatomique et Astroparticules

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Moduli spaces of 3d $\mathcal{N} = 4$ theories

Présentation du stage de Master
sous la direction de Ioannis Lavdas

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LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

Introduction

Introduction

- Study of a particular class of supersymmetric quantum field theories: **3d $\mathcal{N} = 4$ linear quiver gauge theories**.
- Emerging as dimensional reduction of the 4d $\mathcal{N} = 4$ super Yang-Mills theories.
- Play a very important role in the study of four dimensional quantum gravity.
- Flowing to super conformal fixed points at low energy.
- However, strongly coupled at low energy.
- Thus, no perturbative study possible in the IR.
- Fortunately, some specific configurations in Type-IIB String Theory can describe them in the IR.

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1. Theoretical context

1.1. String Theory, Supersymmetries and Field Theories

String Theory:

- Particles described by string modes.

- The 2 coordinates of the worldsheet are embedded in the space-time.

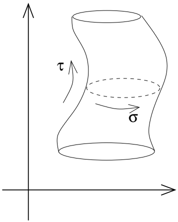


Figure 1: The worldsheet of a closed string

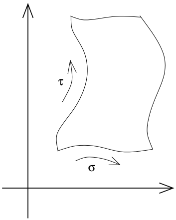


Figure 2: The worldsheet of an open string

Dp-branes:

- Different boundary conditions at the end points of the open string:
  - $\partial_\sigma X^\mu = 0$: Neumann boundary conditions.
    The end of the string can move freely.
  - $\delta X^\mu = 0$: Dirichlet boundary conditions.
    Constant position in space along this coordinate.

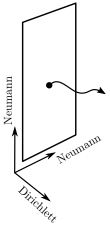


Figure 3: Boundary conditions of an open string

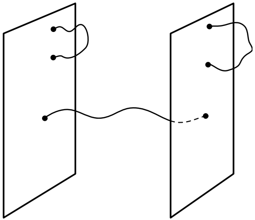


Figure 4: Open strings ending on different branes

1.1. String Theory, Supersymmetries and Field Theories

Supersymmetry:

- It is a symmetry whose charge is a fermionic operator.
- Noether's theorem: A symmetry implies a conserved current composed of charges.
- In supersymmetry, the charges Q satisfy a superalgebra and have the following action on fermionic and bosonic states:

$$Q\left|boson\right\rangle = \left|fermion\right\rangle, \; Q\left|fermion\right\rangle = \left|boson\right\rangle$$

- R-symmetry transforms different supercharges into each other. It can be introduced via commutation relations.

1.1. String Theory, Supersymmetries and Field Theories

Fields:

- Since we have quantum strings, oscillator modes acting on the ground state give rise to a quantized spectrum of oscillations.
- This spectrum reveals the presence of different kind of fields.
- To describe fermions, we require to add supersymmetries.
- For superstrings, the space-time needs to be D=10 dimensional to be consistent.
- There are massless fields from the bosonic string common to all string theories, but other kind of fields appears depending on how we add supersymmetry.

1.1. String Theory, Supersymmetries and Field Theories

- We can do different choices when we add fermions to the worldsheet.
- We can add fermions in the left-moving or in the right-moving sectors of the string giving rise to different forms of extra bosonic fields.
- Type-II superstrings: both left and right-moving worldsheet fermions.
- 2 kind of boundary conditions to make the theory consistent:
  Type-IIA or Type-IIB superstrings.

1.2. Type-IIB String Theory

- Type-IIB superstring spectrum gives rise to different fields than the Type-IIA superstring one.

- Type-IIB superstring exhibits $\mathcal{N} = (2, 0)$ supersymmetry.
  It is a chiral theory.

- Type-IIB has stable Dp-branes only with p odd.
  In 10d, the only possibilities are $(D_1, D_3, D_5, D_7, D_9)$.

- **3d $\mathcal{N} = 4$ linear quiver gauge theories**:
  Theories as a product of unitary groups, i.e. with gauge groups of the form:

$$G = \prod_{i=1}^{k} U(N_i)$$

- Can be represented as:

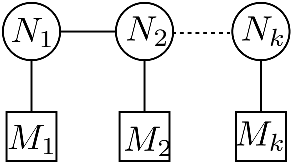


Figure 5: General quiver diagram

1.3. 3d $\mathcal{N} = 4$ Quiver theories

- In 3d, the field content of $\mathcal{N} = 4$ gauge theories is organized in multiplets with 4 real bosonic fields.

<table>
<tr><th>$\mathcal{N} = 4$</th><th>$\mathcal{N} = 2$ (superfield)</th><th>Components</th><th>$SU(2)_C \times SU(2)_H$</th><th>G</th></tr>
<tr><td rowspan="2">Vector multiplet</td><td>Vector multiplet (V)</td><td>A_μ
σ
fermions and aux.</td><td rowspan="2">$\{\sigma, Re\varphi, Im\varphi\}$ in $(1, 0)$</td><td rowspan="2">adjoint</td></tr>
<tr><td>Chiral multiplet (Φ)</td><td>φ
fermions and aux.</td></tr>
<tr><td rowspan="2">Hyper multiplet</td><td>Chiral multiplet (ϕ)</td><td>Q
fermions and aux.</td><td rowspan="2">$\{Q, \tilde{Q}\}$ in $(0, \frac{1}{2})$</td><td>$\mathcal{R}$</td></tr>
<tr><td>Chiral multiplet ($\tilde{\phi}$)</td><td>$\tilde{Q}$
fermions and aux.</td><td>$\mathcal{R}^*$</td></tr>
</table>

Table 1: Field content and R-charges of the $\mathcal{N} = 4$ supermultiplet

- $Q = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$, $\tilde{Q} = \frac{\tilde{\phi}_1 + i\tilde{\phi}_2}{\sqrt{2}}$, σ a real scalar.

- These Quantum Field Theories have a really well-known Lagrangian description.

1.3. 3d $\mathcal{N} = 4$ Quiver theories

- The Moduli space is the geometrical space parametrised by the vacuum expectation values (vevs) of the scalar fields present in the theory.
- The scalar fields of the vector-multiplet parametrize the Coulomb branch of the Moduli space.
- The scalar fields of the the hypermultiplet parametrize the Higgs branch.
- In 3d $\mathcal{N} = 4$ supersymmetry, we have an $SO(4)_R \simeq SO(3) \times SO(3) \simeq SU(2)_C \times SU(2)_H$ R-symmetry.
- The fields from the vector-multiplets are rotated by the $SU(2)_C$ factor of the R-symmetry $SU(2)_C \times SU(2)_H$.
- The fields from the hypermultiplets are rotated by the $SU(2)_H$ factor.
- The couplings of the theory have the dimension of a mass, the theories are strongly coupled at low energies.

1.4. Brane realization of 3d $\mathcal{N} = 4$ Quiver theories

- The ends of the open string are charged under gauge fields.
- We have U(1) gauge field on the worldvolume of the branes.
- With N parallel Dp-branes on top of each other, the worldvolume gauge symmetry is enhanced to $U(N)$.
- Thus we can find a description of 3d $\mathcal{N} = 4$ theories using branes in Type-IIB String theory.

1.4. Brane realization of 3d $\mathcal{N} = 4$ Quiver theories

- We start by considering D3-branes presenting a 4d theory.
- To make it a 3d theory, we do a compactification along one of its coordinates.
- To do so, we can introduce other branes in the theory on which the D3s will end.
- We can use two different kind of fivebranes: D5-branes or NS5-branes presenting different boundary conditions.
- Taking care of not breaking all the supersymmetries, we have to add them spanning different directions.

1.4. Brane realization of 3d $\mathcal{N} = 4$ Quiver theories

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	-	-	-	X	X	X	-	X	X	X
D5	-	-	-	X	X	X	X	-	-	-
NS5	-	-	-	-	-	-	X	X	X	X

Table 2: Our brane system realizing a 3d $\mathcal{N} = 4$ theory, where "X" means that the brane is point-like in that direction and "-" means it is extended in that direction.

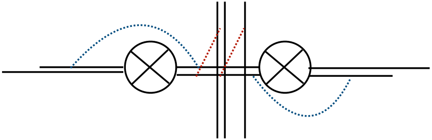


Figure 6: A brane configuration example

1.4. Brane realization of 3d $\mathcal{N} = 4$ Quiver theories

- The correspondence between quiver diagrams and brane configurations:

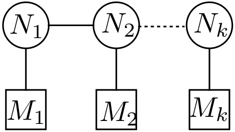


Figure 7: General quiver diagram

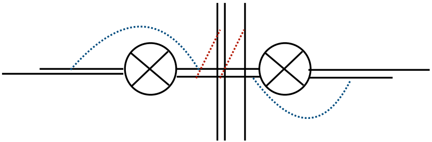


Figure 8: A brane configuration example

1.4. Brane realization of 3d $\mathcal{N} = 4$ Quiver theories

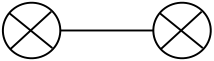
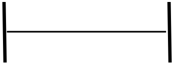
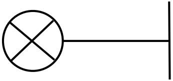
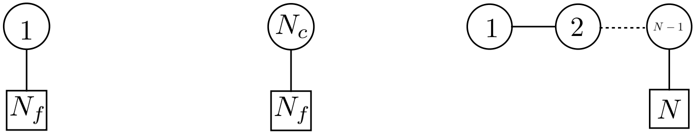
Brane configuration	Fixed transverse coordinates of the D3	Hypermultiplet scalars	Vector multiplet scalars
	(x_7, x_8, x_9)	$\{0\}$	$\{\varphi_i, \sigma\}$
	(x_3, x_4, x_5)	$\{Q, \tilde{Q}\}$	$\{0\}$
	$(x_3, x_4, x_5, x_7, x_8, x_9)$	$\{0\}$	$\{0\}$

Table 3: The different boundary conditions of the D3 and its fields associated

2. Study of the 3d $\mathcal{N} = 4$ theories

- We can deduce the analog brane configuration of a quiver via the correspondence we found.

- In the IR, we can reorder all the branes along x^6. During this procedure, we have to take care of:
  - The S-rule: only one D3 stretched between an NS5 and a D5.
  - The Hanany-Witten transitions.



(a) $SQED_{N_f}$ **(b)** $SQCD_{N_f}$ with N_c colors **(c)** T[SU(N)]

Figure 9: Quiver diagram of some special theories

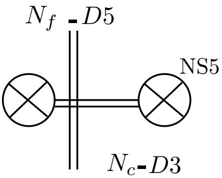
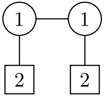
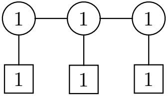


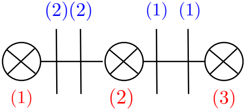
Figure 10: The SQCD picture with branes



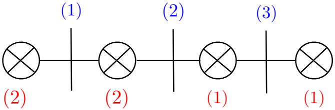
(a) Quiver of a random theory



(b) Quiver of an other random theory



(c) Its linking numbers



(d) Its linking numbers

Figure 11: Determination of ρ and $\hat{\rho}$ for two different quivers

$$\rho = (l_1, l_2, l_3, ...),\ \hat{\rho} = (\hat{l}_{\hat{1}}, \hat{l}_{\hat{2}}, \hat{l}_{\hat{3}}, ...)$$

2.2. Mirror symmetry

- 3d mirror symmetry predicts the existence of 3d $\mathcal{N} = 4$ pairs of theories that flow to the same conformal theory in the infrared.

- An easy way to visualize it is the following:

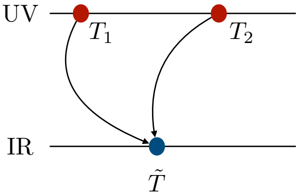


Figure 12: Representation of two different theories in the UV flowing to the same one in the IR

2.2. Mirror symmetry

- Mirror symmetry appears in the study of the Moduli spaces of the dual gauge theories.
- The space of vacua parametrised by the scalar fields in the vector multiplet is called the Coulomb branch.
- The space of vacua parametrised by the scalar fields in the hypermultiplet is known as the Higgs branch.
- One of the possible representation is the following:

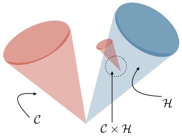


Figure 13: A possible representation of the Coulomb and Higgs branches of a Moduli space as well as a Mixed branch appearing at some subloci

2.2. Mirror symmetry

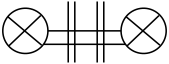
- Mirror symmetry predicts that the Coulomb branch of a certain gauge theory corresponds to the Higgs branch of a dual theory, and vice-versa.
- Coulomb branch can be affected by quantum corrections, however the Higgs branch is not.
- With mirror symmetry, possibility to compute the Coulomb branch of a given theory via the computation of the Higgs branch of its dual.
- Two theories flow in the IR if they respect the following inequalities for each gauge node:

$$M_i + N_{i-1} + N_{i+1} \geq 2N_i$$

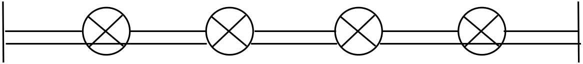
- They are called good theories and always flow in the IR.

2.2. Mirror symmetry

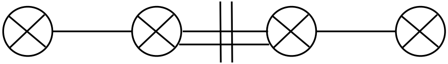
- There are several ways to obtain the dual Mirror of a theory.
- In field theory, via Electric-magnetic duality.
- In Type-IIB String theory, via S-duality.
- In Type-IIB, S-transformations exchange D5-branes and NS5-branes but keep D3-branes unchanged.
- We just have to reorder all the branes to determine which is the new theory obtained, the dual.
- The theory we obtained in the IR present a flavor group which is the product of the flavor froups of the two Mirror theories.



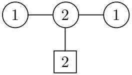
(a) Brane construction of $SQCD_4$ with 2 colors



(b) $SQCD_4$ brane construction with 2 colors after a S-transformation



(c) $SQCD_4$ mirror brane construction with 2 colors after Hanany-Witten transitions



(d) Quiver of the Mirror of $SQCD_4$ with 2 colors

Figure 14: The process to obtain the Mirror theory of $SQCD_4$ with 2 colors

2.2.2. $SQED_{N_f}$

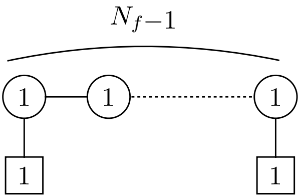
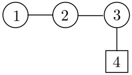
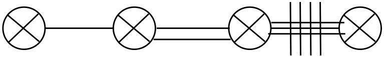


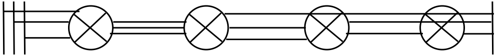
Figure 15: The general quiver diagram of the $SQED_{N_f}$ mirrors



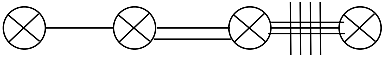
(a) Quiver of T[SU(4)]



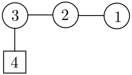
(b) Brane construction of T[SU(4)]



(c) T[SU(4)] brane construction after a S-transformation



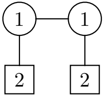
(d) T[SU(4)] mirror brane construction after Hanany-Witten transitions



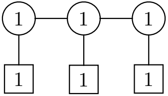
(e) Quiver of the Mirror of T[SU(4)]

Figure 16: The process to obtain the Mirror theory of T[SU(4)]

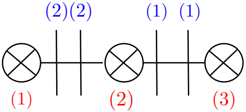
- $T[SU(N)]$ theories are the self-dual theories of Mirror symmetry.



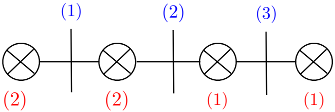
(a) Quiver of a random theory



(b) Quiver of the mirror of the random theory



(c) Determination of its linking numbers



(d) Determination of the linking numbers of the mirror

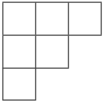
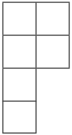
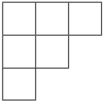
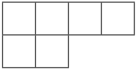
Figure 17: Example of the determination of ρ and $\hat{\rho}$ for a quiver and its mirror

- The condition for the theory to respect all the supersymmetries is:

$$\hat{\rho}^T > \rho$$

- The condition for the theory to be good and flow in the IR has become:

$$\rho^T \geq \hat{\rho}$$

We have ρ:  , $\hat{\rho}$:  , ρ^T:  , and $\hat{\rho}^T$:  .

2.3. The Moduli space

- The scalar vevs parametrize the Moduli space of vacua of the theory.
- To have a simple visualization, we can look at the following:

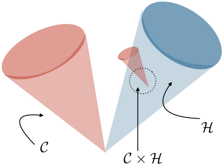
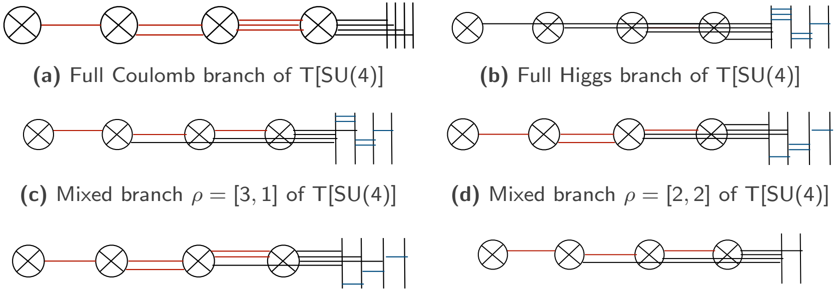


Figure 18: A possible representation of the Coulomb and Higgs branches of a Moduli space as well as a Mixed branch appearing at some subloci

2.3.1. Brane realization of Mixed Branches of $T[SU(N)]$ theories

- The vevs of the vector-multiplets are non zero when the D3s are between two NS5s and free to move along (x^3, x^4, x^5).
- The vevs of the hypermultiplets are non zero when the D3s are between two D5s and free to move along (x^7, x^8, x^9).
- The same theory can have different configurations of moving D3s.
- To manipulate our theory and making it more simpler, we first start by putting all the NS5s of our theory on the left and all the D5s to the right.
- Then, we can choose different ways of linking the D5s and the NS5s by keeping the same number of D3s and always respecting the S-rule.



(a) Full Coulomb branch of T[SU(4)]

(b) Full Higgs branch of T[SU(4)]

(c) Mixed branch $\rho = [3,1]$ of T[SU(4)]

(d) Mixed branch $\rho = [2,2]$ of T[SU(4)]

(e) Mixed branch $\rho = [2,1,1]$ of T[SU(4)]

(f) $T^{[3,1]}[SU(4)]$ theory

Figure 19: Different branches of T[SU(4)]

2.3.1. Brane realization of Mixed Branches of $T[SU(N)]$ theories

- We can classify these Mixed branch with a partition of N:

  $\rho = [a_1, a_2, \cdots, a_n]$ with $a_1 \geq a_2 \geq \cdots \geq a_n$, $\sum_{i=1}^{n} a_i = N$, and a_i is the number of D3s on the i-th D5 from the left to the right that are link to an NS5 with $i = 1, \cdots, n$.

- It is different from the ρ used to describe the theory $T^{\rho}_{\hat{\rho}}[SU(N)]$.

- We can also introduce an other parameter:

  $\rho' = [b_1, b_2, \cdots, b_{n'}]$ with $b_1 \geq b_2 \geq \cdots \geq b_{n'}$, $\sum_{k=1}^{n'} b_k = N$, and b_k is the number of D3s on the k-th NS5 from the right to the left that are link to an D5 with $k = 1, \cdots, n'$.

- However, ρ' is always completely determined by ρ.

- ρ is the only parameter we need to describe the configuration of our Mixed branch.

2.3.1. Brane realization of Mixed Branches of $T[SU(N)]$ theories

- We can observe that ρ is the dual partition of ρ', which we can then call ρ^D.

- The full Moduli space expression corresponds to:

$$\mathcal{M} = \bigcup_{\rho} \mathcal{C}_{\rho} \times \mathcal{H}_{\rho}$$

- The Full Coulomb branch is thus: $\mathcal{C}_{\rho} \times \{0\}$, and the Full Higgs branch is: $\{0\} \times \mathcal{H}_{\rho}$.

- For good theories, the Coulomb branch and the Higgs branch are swapped for the dual theory, we have $\mathcal{C}_{\rho} \simeq \mathcal{H}_{\rho^D}$ and $\mathcal{H}_{\rho} \simeq \mathcal{C}_{\rho^D}$. We can thus rewrite:

$$\mathcal{M} = \bigcup_{\rho} \mathcal{C}_{\rho} \times \mathcal{C}_{\rho^D}, \text{ or: } \mathcal{M} = \bigcup_{\rho} \mathcal{H}_{\rho^D} \times \mathcal{H}_{\rho}$$

2.3.2. Dimensions of Coulomb and Higgs branches via Brane configurations

- We can find the dimension of the Coulomb and Higgs branch factor of all Mixed branches by looking at the brane configurations.

- The dimensions of the Coulomb and Higgs branch are the most important parameters because they count the number of scalars that can take non-zero vevs in each part of the Moduli space.

- The dimension of the Coulomb branch factor $\mathcal{C}_\rho$ and of the Higgs branch factor $\mathcal{H}_\rho$ formula are the following:

$$\dim(\mathcal{C}_\rho) = \frac{1}{2}(N^2 - \sum_{i=1}^{n} a_i^2), \text{ and: } \dim(\mathcal{H}_\rho) = \frac{1}{2}(N^2 - \sum_{i=1}^{n'} b_i^2)$$

denoted by d_C and d_H respectively.

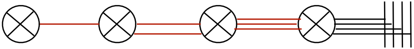
2.3.2. Dimensions of Coulomb and Higgs branches via Brane configurations

- We can thus easily compute d_C and d_H just by looking at the Brane configurations realizing these Mixed branches.

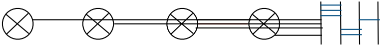
- The dimension of the Coulomb branch corresponds to the number D3s that are free to move between two NS5s

- The dimension of the Higgs branch corresponds to the number D3s that are free to move between two D5s.

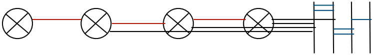
2.3.2. Dimensions of Coulomb and Higgs branches via Brane configurations



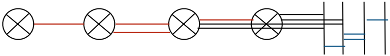
(a) Full Coulomb branch of T[SU(4)]



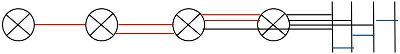
(b) Full Higgs branch of T[SU(4)]



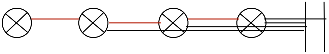
(c) Mixed branch $\rho = [3, 1]$ of T[SU(4)]



(d) Mixed branch $\rho = [2, 2]$ of T[SU(4)]



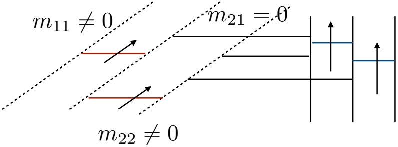
(e) Mixed branch $\rho = [2, 1, 1]$ of T[SU(4)]



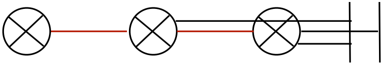
(f) $T^{[3,1]}[SU(4)]$ theory

Figure 20: Different branches of T[SU(4)]

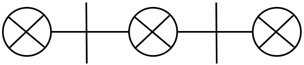
- I have also done these computations for $T[SU(5)]$ and $T[SU(6)]$, they can be seen in the Appendices.



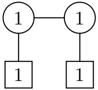
(a) Mixed branch $\rho = [2,1]$ of T[SU(3)]



(b) Coulomb branch of the $T^{[2,1]}[SU(3)]$ theory



(c) $T^{[2,1]}[SU(3)]$ theory after Hanany-Witten transitions



(d) Quiver of the $T^{[2,1]}[SU(3)]$ theory

Figure 21: Process to obtain the $T^{[2,1]}[SU(3)]$ theory

$$\mathcal{C}_{[2,1]}(T[SU(N)]) \simeq \mathcal{C}([1] - U(1) - U(1) - [1])$$

2.3.4. Hilbert series

- We can reconstruct $\mathcal{M}$ by counting all the gauge invariant chiral operators, and grade them by their charges under all the different global symmetries of the theory.

- This is what do the Hilbert series.

- If we suppose the chiral operators are charged under N global symmetries, for each one of them, we can choose x_i, with $i = 1, \ldots, N$ as a grading parameter called fugacity. We have the formula:

$$HS(x_i) = \sum_{k_1} \sum_{k_2} \ldots \sum_{k_N} a_{k_1,k_2,\ldots,k_N} \prod_{i=1}^{N} x_i^{k_i}$$

  $a_{k_1,k_2,\ldots,k_N}$ is the number of chiral operators having respectively charges $k_1, k_2, \ldots, k_N$ under the N symmetries.

2.3.4. Hilbert series

- We will now present the general formula that can be derived for the Hilbert series of the Coulomb branches.

- Unlike the Higgs branch, the Coulomb branch receives quantum corrections.

- Thus, to study it, we can use monopole operators, which can be defined as a disorder operator in the infrared superconformal field theory.

- The general formula is:

$$HS(t) = \sum_{m} t^{\Delta(m)} P_G(m, t)$$

where t is a fugacity keeping track of the conformal dimension of the monopole operators and m their magnetic charge.

2.3.4. Hilbert series

- $P_G(m,t)$ is a correction factor taking care of the different dressings, which can take the form:

$$P_G(t,m) = \prod_{i=1}^{r} \frac{1}{1 - t^{d_i(m)}}$$

where r is the rank of the subgroup in which the gauge group is broken when the vev of a bare monopole operator is turned on and $d_i(m)$ are all the degrees of the r Casimir operators of this subgroup.

- Then the conformal dimension of a bare monopole operator is given in terms of the magnetic charge m by the following dimension formula:

$$\Delta(\mathbf{m}) = -\frac{1}{2}\sum_{j=1}^{k}\sum_{\alpha,\beta=1}^{N_j} |m_{j,\alpha} - m_{j,\beta}| + \frac{1}{2}\sum_{j=1}^{k} M_j \sum_{\alpha=1}^{N_j} |m_{j,\alpha}| + \frac{1}{2}\sum_{j=1}^{k-1}\sum_{\alpha=1}^{N_j}\sum_{\beta=1}^{N_{j+1}} |m_{j,\alpha} - m_{j+1,\beta}|$$

where N_j corresponds to gauge nodes on the Figure 1 and M_j to the flavors.

2.3.4. Hilbert series

- In the case of linear quiver theories we have considered, we can further refine the counting by including other fugacities z_i.

- We can relate, in the brane picture, moving D3s between NS5s positions with the magnetic charges of monopole operators m.

- We now consider the simple case of the $T[SU(3)]$.
- Our brane configurations are then associated in the following way.

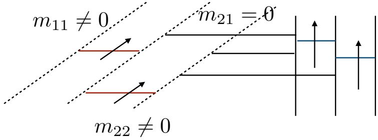


Figure 22: Mixed branch $\rho = [2, 1]$ of T[SU(3)]

- m_1 corresponds to the magnetic charge for U(1), and m_{21} and m_{22} are the magnetic charges for U(2).

- We also have the Weyl chamber condition stating: $m_{21} \geq m_{22}$. The Hilbert series for the full Coulomb branch is thus given by the general formula:

$$HS(t, z_1, z_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_{21} \geq m_{22}} t^{\Delta(m_1, m_{21}, m_{22})} z_1^{m_1} z_2^{m_{21}+m_{22}} P_{U(1)}(m_1, t) P_{U(2)}(m_{21}, m_{22}, t)$$

with: $\Delta_1(m_1, m_{21}, m_{22}) = -|m_{21} - m_{22}| + \frac{1}{2}(|m_1 - m_{21}| + |m_1 - m_{22}| + 3|m_{21}| + 3|m_{22}|)$,

and the classical factors:

$$P_{U(1)}(m_1, t) = \frac{1}{1-t},$$

and:

$$P_{U(2)}(m_1, m_{21}, m_{22}) = \begin{cases} \frac{1}{(1-t)(1-t^2)}, \text{ for } m_{21} = m_{22} \\ \frac{1}{(1-t)^2}, \text{ for } m_{21} > m_{22} \end{cases}$$

- Then we can focus on the Mixed branch $\rho = [2,1]$.

- We either need $m_{21} = 0$ or $m_{22} = 0$ to respect this configuration. Thus, we deduce the following expression for the Hilbert series of the Coulomb branch factor of the Mixed branch $\rho = [2,1]$:

$$\begin{aligned} HS(t, z_1, z_2) = & \sum_{m_1=-\infty}^{\infty} \sum_{m_{22}\leq 0}^{\infty} t^{\Delta_1(m_1, m_{22})} z_1^{m_1} z_2^{m_{22}} P_{U(1)}(m_1, t) P_{U(1)}(m_{22}, t) \\ & + \sum_{m_1=-\infty}^{\infty} \sum_{m_{21}\geq 0}^{\infty} t^{\Delta_2(m_1, m_{21})} z_1^{m_1} z_2^{m_{21}} P_{U(1)}(m_1, t) P_{U(1)}(m_{21}, t) \end{aligned}$$

with: $\Delta_1(m_1, m_{22}) = \Delta(m_1, 0, m_{22}) = \frac{1}{2}(|m_1| + |m_1 - m_{22}| + |m_{22}|)$,

and: $\Delta_2(m_1, m_{21}) = \Delta(m_1, m_{21}, 0) = \frac{1}{2}(|m_1| + |m_1 - m_{21}| + |m_{21}|)$.

- The Hilbert series of the Full Coulomb branch of the [1]-U(1)-U(1)-[1] theory is:

$$HS(t, z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} t^{\Delta(n_1,n_2)} z_1^{n_1} z_2^{n_2} P_{U(1)}(n_1, t) P_{U(1)}(n_2, t)$$

  with: $\Delta(n_1, n_2) = \frac{1}{2}(|n_1| + |n_1 - n_2| + |n_2|)$, and where n_1, n_2 are the magnetic charges of the two U(1) gauge groups.

- The full Coulomb branch of [1]-U(1)-U(1)-[1] and the the Coulomb branch factor of the Mixed branch $\rho = [2, 1]$ of $T[SU(3)]$ are indeed isomorphics.

2.3.4. Hilbert series

- By computing these Hilbert series, we obtain:

$$\begin{aligned}
HS(t, z_1, z_2) = 1 &+ t\left[2 + z_1 z_2 + \frac{1}{z_1 z_2} + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2}\right] \\
&+ t^2 \left[3 + 2\left(z_1 z_2 + \frac{1}{z_1 z_2} + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2}\right)\right. \\
&\left. + z_1^2 z_2^2 + \frac{1}{z_1^2 z_2^2} + z_1^2 z_2 + \frac{1}{z_1^2 z_2} + z_1^2 + \frac{1}{z_1^2} + z_1 z_2^2 + \frac{1}{z_1 z_2^2} + \frac{z_1}{z_2} + \frac{z_2}{z_1} + z_2^2 + \frac{1}{z_2^2}\right] \\
&+ t^3 \left[4 + 3\left(z_1 z_2 + \frac{1}{z_1 z_2} + z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2}\right)\right. \\
&+ 2\left(z_1^2 z_2^2 + \frac{1}{z_1^2 z_2^2} + z_1^2 z_2 + \frac{1}{z_1^2 z_2} + z_1^2 + \frac{1}{z_1^2} + z_1 z_2^2 + \frac{1}{z_1 z_2^2} + \frac{z_1}{z_2} + \frac{z_2}{z_1} + z_2^2 + \frac{1}{z_2^2}\right) \\
&+ z_1^3 z_2^3 + z_1^3 z_2^2 + z_1^3 z_2 + z_1^3 + z_1^2 z_2^3 + \frac{z_1^2}{z_2} + z_1 z_2^3 + \frac{z_1}{z_2^2} + z_2^3 + \frac{1}{z_2^3} + \frac{z_2^2}{z_1} + \frac{1}{z_1 z_2^3} \\
&\left. + \frac{z_2}{z_1^2} + \frac{1}{z_1^2 z_2^3} + \frac{1}{z_1^3} + \frac{1}{z_1^3 z_2} + \frac{1}{z_1^3 z_2^2} + \frac{1}{z_1^3 z_2^3}\right] + O\left(t^4\right)
\end{aligned}$$

- In the end, we can gathered the terms in the the following expression, verified up to the 9-th order:

$$HS(t, z_1, z_2) = \sum_k \chi^{(k,k)}(z_1, z_2) t^k$$

  where $\chi^{(k,k)}(z_1, z_2)$ are the characters of the representation where (k, k) are the Dynkin labels of the $\mathfrak{su}(3)$ Lie algebra.

- We thus have fully computed the Hilbert series of the Coulomb branch factor of the Mixed branch $\rho = [2, 1]$ of the $T[SU(3)]$ theory.

- However, this is a whole different process to determine it for the Higgs branch.

- However, since $T[SU(3)]$ is self-dual under the Mirror Symmetry, the computation of the Higgs branch is not necessary.

Conclusion

Conclusion

- We have studied 3d $\mathcal{N} = 4$ quiver theories via Type-IIB Brane configurations.
- We have seen:
  - How to manipulate them.
  - How the branches of the Moduli space exchange in respect to the Mirror Symmetry, corresponding to S-duality in Type-IIB String Theory.
- We have been able to compute:
  - The dimensions of the Coulomb and Higgs branch parts of different Mixed branches of $T[SU(N)]$ theories.
  - The Hilbert series of $T[SU(3)]$ describing its Moduli space.

Perspectives

- Computation of dimensions of Coulomb or Higgs branch parts of mixed branches of more general unitary quiver theories, such as $T_{\rho}^{\hat{\rho}}[SU(N)]$ which is not yet found in the bibliography.

- Computation of Hilbert series for the Higgs branch.

  Computation of Hilbert series can be applied to all families of 3d $\mathcal{N} = 4$ theories (unitary and non-unitary).

- Study of the Type-IIB supergravity solutions in AdS_4 which are dual to the above 3d $\mathcal{N} = 4$ theories studied here.

  These theories play a central role in gauge/gravity correspondence or equivalently the AdS/CFT correspondence (Anti- de Sitter/Conformal Field theory correspondence).

  Conjecture stating that String Theory (or its low-dimensional supergravity) in $(d+1)-$dimensional Anti-de Sitter space is dual to a (super)conformal field theory living on its boundary in one dimension lower.

Appendices

ρ	Coulomb branch dimension d_C	Higgs branch dimension d_H
[5]	0	10
[4,1]	4	9
[3,2]	6	8
[3,1,1]	7	7
[2,2,1]	8	6
[2,1,1,1]	9	4
[1,1,1,1,1]	10	0

Table 4: Dimensions of all possible mixed branches of $T[SU(5)]$

ρ	Coulomb branch dimension d_C	Higgs branch dimension d_H
[6]	0	15
[5,1]	5	14
[4,2]	8	13
[3,3]	9	12
[4,1,1]	9	12
[3,2,1]	11	11
[3,1,1,1]	12	9
[2,2,2]	12	9
[2,2,1,1]	13	8
[2,1,1,1,1]	14	5
[1,1,1,1,1,1]	15	0

Table 5: Dimensions of all possible mixed branches of $T[SU(6)]$

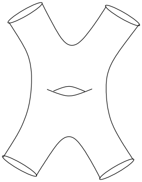


Figure 23: One-loop scattering of two closed strings

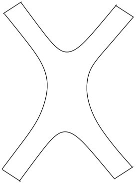


Figure 24: Scattering of two open strings

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