BPS dendroscopy on local $\mathbb{P}^2$

Bruno Le Floch, LPTHE
(Sorbonne Université and CNRS)

May 25, 2023, Annecy

arXiv:2210.10712
Pierrick Bousseau, Pierre Descombes, Boris Pioline



spectroscopy

BPS dendroscopy on local $\mathbb{P}^2$

Bruno Le Floch, LPTHE
(Sorbonne Université and CNRS)

May 25, 2023, Annecy

arXiv:2210.10712

Pierrick Bousseau, Pierre Descombes, Boris Pioline



spectroscopy **δένδρος** = trees

BPS dendroscopy on local $\mathbb{P}^2$

Bruno Le Floch, LPTHE
(Sorbonne Université and CNRS)

May 25, 2023, Annecy

arXiv:2210.10712

Pierrick Bousseau, Pierre Descombes, Boris Pioline

spectroscopy **δένδρος** = trees

of IIA strings

BPS dendroscopy on local $\mathbb{P}^2$

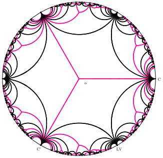
Bruno Le Floch, LPTHE
(Sorbonne Université and CNRS)

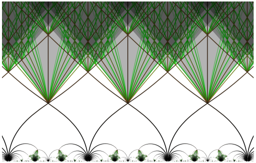
May 25, 2023, Annecy

arXiv:2210.10712
Pierrick Bousseau, Pierre Descombes, Boris Pioline

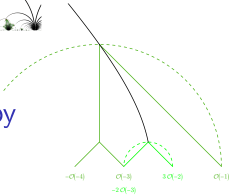


1. IIA strings and BPS spectrum
2. Local $\mathbb{P}^2$

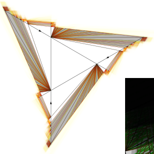




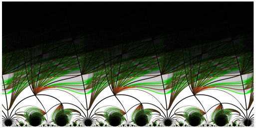
3. Scattering diagrams ($\psi = 0$)



4. Split attractor flow conjecture and dendroscopy



5. Initial data and orbifold region



6. Scattering diagrams for all values of ψ

1. IIA strings and BPS spectrum
2. Local $\mathbb{P}^2$

3. Scattering diagrams ($\psi = 0$)

4. Split attractor flow conjecture and dendroscopy

5. Initial data and orbifold region

6. Scattering diagrams for all values of ψ

IIA strings on a local Calabi–Yau 3-fold

IIA strings on X a compact CY_3	$\Longrightarrow$ 4d $\mathcal{N}=2$ SUGRA
D6-D4-D2-D0 bound states	$\Longrightarrow$ BPS black holes
IIA strings on X a non-compact CY_3	$\Longrightarrow$ 4d $\mathcal{N}=2$ QFT
D4-D2-D0 bound states	$\Longrightarrow$ BPS particles saturating Mass $\geq |Z|$ (where $\{Q^1_\alpha, Q^2_\beta\} = 2Z\epsilon_{\alpha\beta}$)

IIA strings on a local Calabi–Yau 3-fold

IIA strings on X a compact CY_3	$\Longrightarrow$ 4d $\mathcal{N} = 2$ SUGRA
D6-D4-D2-D0 bound states	$\Longrightarrow$ BPS black holes
IIA strings on X a non-compact CY_3	$\Longrightarrow$ 4d $\mathcal{N} = 2$ QFT
D4-D2-D0 bound states	$\Longrightarrow$ BPS particles
	saturating Mass $\geq \lvert Z\rvert$
	(where $\{Q^1_\alpha, Q^2_\beta\} = 2Z\epsilon_{\alpha\beta}$)

IIA strings on a local Calabi–Yau 3-fold

IIA strings on X a compact CY_3	$\Longrightarrow$ 4d $\mathcal{N}=2$ SUGRA
D6-D4-D2-D0 bound states	$\Longrightarrow$ BPS black holes
IIA strings on X a non-compact CY_3	$\Longrightarrow$ 4d $\mathcal{N}=2$ QFT
D4-D2-D0 bound states	$\Longrightarrow$ BPS particles saturating Mass $\geq |Z|$ (where $\{Q^1_\alpha, Q^2_\beta\} = 2Z\epsilon_{\alpha\beta}$)

More precisely

IIA strings on X

$\Longleftrightarrow$ M-theory on $X \times S^1$

$\Longrightarrow$ 5d $\mathcal{N}=1$ SCFT on S^1 (keeping KK modes)

Behaves like a 4d $\mathcal{N}=2$ Seiberg–Witten theory with an infinite spectrum

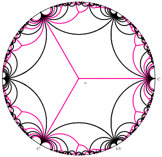
IIA strings on a local Calabi–Yau 3-fold

Goal: $\Omega_\tau(\gamma)$ counting BPS states of charge γ (weighted by y^{J_3})
for every value of τ (Kähler moduli of X)

$$\left.\begin{array}{l}\bullet\ \text{known in some limits}\\ \bullet\ \text{Kontsevich–Soibelman}\\ \quad \underline{\text{wall crossing}}\text{ when varying }\tau\end{array}\right\}\ \text{follow } \frac{\text{attractor flow}}{\text{to cross walls}}$$

New: fully doing it for the simplest $X = K\mathbb{P}^2$ (canonical bundle on $\mathbb{P}^2$)
Ideas should generalize to local del Pezzo

1. IIA strings and BPS spectrum
2. Local $\mathbb{P}^2$



3. Scattering diagrams ($\psi = 0$)
4. Split attractor flow conjecture and dendroscopy

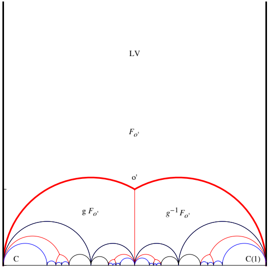
5. Initial data and orbifold region

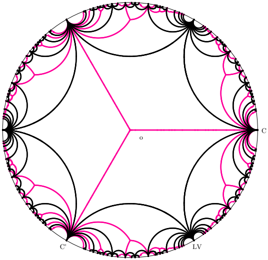
6. Scattering diagrams for all values of ψ

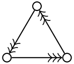
Local $\mathbb{P}^2$ moduli space

Kähler moduli space
$\mathcal{M}_K = \mathbb{H}/\Gamma_1(3)$

$\dim_{\mathbb{C}} \mathcal{M}_K = 1$





- Large volume point (cusp): geometric phase, $\mathbb{P}^2$ large
  BPS particle = Gieseker stable sheaf (complicated)
- Orbifold point: $K\mathbb{P}^2 \to \mathbb{C}^3/\mathbb{Z}_3$ limit
  BPS particle = stable representation of a quiver 
- Conifold point (cusp): no easy description

Local $\mathbb{P}^2$ charges

Electromagnetic charge $\gamma = [r, d, \chi) = [r, d, \mathrm{ch}_2]$

(Chern vector of homology class wrapped by D-branes)

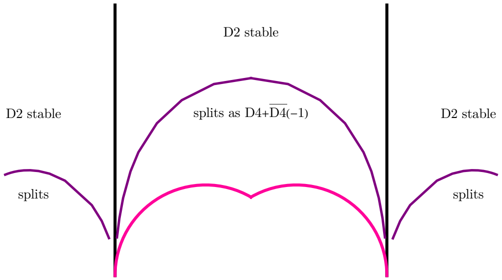
- $\gamma = [1, 0, 1)$ D4 brane $\mathcal{O}$ (structure sheaf of $\mathbb{P}^2$)
- $\gamma = [0, 1, 1)$ D2 brane $\mathcal{O}_C$ (structure sheaf of curve $C \subset \mathbb{P}^2$)
- $\gamma = [0, 0, 1)$ D0 brane (skyscraper sheaf at point of $\mathbb{P}^2$)

Central charge $Z_\tau([r, d, \mathrm{ch}_2]) = -rT_D(\tau) + dT(\tau) - \mathrm{ch}_2$

Picard–Fuchs equation (from susy localization or mirror symmetry) gives

$$\begin{pmatrix} T \\ T_D \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{3} \end{pmatrix} + \int_{\tau_o}^{\tau} \begin{pmatrix} 1 \\ u \end{pmatrix} \frac{\eta(u)^9}{\eta(3u)^3} \mathrm{d}u \qquad \text{on } \mathbb{H}/\Gamma_1(3), \text{ useful numerically}$$

Local $\mathbb{P}^2$ example: the small-volume fate of D2 branes



- **At large volume** D2 brane wraps hyperplane class in $\mathbb{P}^2$
- Below each **wall of marginal stability** it becomes unstable and splits. E.g., for the middle wall it splits as D4 and anti-D4 (with D2 flux) The wall is the locus where Z_{D2}, Z_{D4}, $Z_{\overline{\text{D4}}(-1)}$ have equal phases

How to compute $\Omega_\tau(\gamma)$? Universal wall-crossing

$\Omega_\tau(\gamma)$ is piecewise constant and jumps across **walls of marginal stability** $W(\gamma_1, \gamma_2)$ of $\mathrm{codim}_{\mathbb{R}} = 1$, where $\arg Z(\gamma_1) = \arg Z(\gamma_2)$ and $\gamma = \gamma_1 + \gamma_2$

[Denef Moore '07, Manschot Pioline Sen '11]

Jump governed by **universal wall-crossing formula**

[Kontsevich Soibelman '08, Joyce Song '08]

How to compute $\Omega_\tau(\gamma)$? Universal wall-crossing

$\Omega_\tau(\gamma)$ is piecewise constant and jumps across **walls of marginal stability** $W(\gamma_1, \gamma_2)$ of $\mathrm{codim}_{\mathbb{R}} = 1$, where $\arg Z(\gamma_1) = \arg Z(\gamma_2)$ and $\gamma = \gamma_1 + \gamma_2$

[Denef Moore '07, Manschot Pioline Sen '11]

Jump governed by **universal wall-crossing formula**

[Kontsevich Soibelman '08, Joyce Song '08]

- For 4d $\mathcal{N} = 2$ Seiberg–Witten theories, often finitely many walls
- For IIA on $K\mathbb{P}^2$, dense set of walls but finitely many for a fixed γ
  $\implies$ to depict everything, pick a phase ψ and track only states with

$$Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0, +\infty)$$

How to compute $\Omega_\tau(\gamma)$? Universal wall-crossing

$\Omega_\tau(\gamma)$ is piecewise constant and jumps across **walls of marginal stability** $W(\gamma_1, \gamma_2)$ of $\mathrm{codim}_{\mathbb{R}} = 1$, where $\arg Z(\gamma_1) = \arg Z(\gamma_2)$ and $\gamma = \gamma_1 + \gamma_2$

[Denef Moore '07, Manschot Pioline Sen '11]

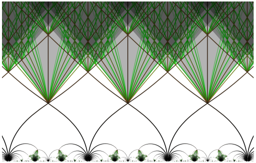
Jump governed by **universal wall-crossing formula**

[Kontsevich Soibelman '08, Joyce Song '08]

- For 4d $\mathcal{N} = 2$ Seiberg–Witten theories, often finitely many walls
- For IIA on $K\mathbb{P}^2$, dense set of walls but finitely many for a fixed γ

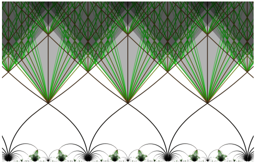
  $\Longrightarrow$ to depict everything, pick a phase ψ and track only states with

$$Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0, +\infty)$$

$\Longrightarrow$ diagrams like 

1. IIA strings and BPS spectrum

2. Local $\mathbb{P}^2$



3. Scattering diagrams ($\psi = 0$)

4. Split attractor flow conjecture and dendroscopy

5. Initial data and orbifold region

6. Scattering diagrams for all values of ψ

Scattering diagram definition

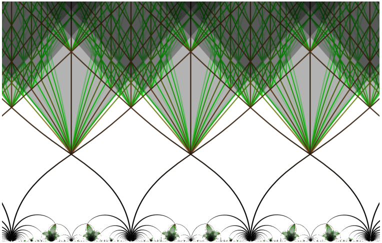
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

Scattering diagram $\mathcal{D}_\psi$:

- codim$_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar{\Omega}_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp\big(\bar{\Omega}_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1}-y)\big)$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\prod^{\curvearrowleft}_{\text{rays around }\tau} \mathcal{U}(\gamma)^{\pm 1} = 1$



Scattering diagram definition

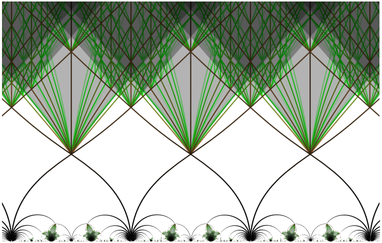
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

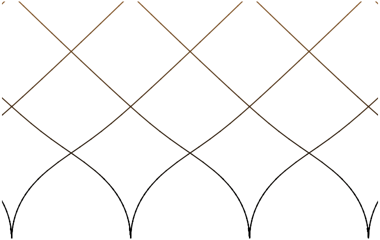
Scattering diagram $\mathcal{D}_\psi$:

- $\mathrm{codim}_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar\Omega_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar\Omega_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1}-y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around }\tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

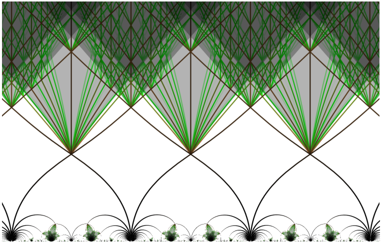
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

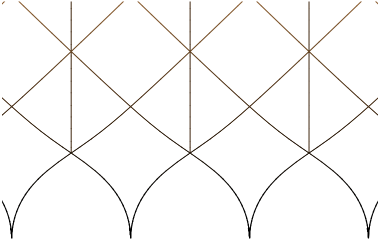
Scattering diagram $\mathcal{D}_\psi$:

- codim$_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0, +\infty),\ \bar{\Omega}_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar{\Omega}_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1} - y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around } \tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

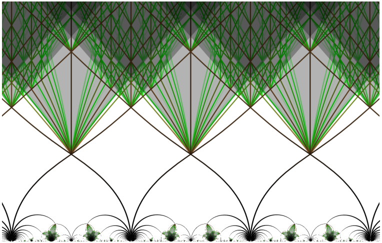
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

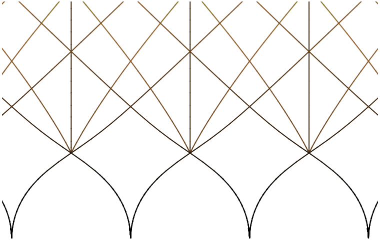
Scattering diagram $\mathcal{D}_\psi$:

- $\mathrm{codim}_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0, +\infty),\ \bar{\Omega}_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar{\Omega}_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1} - y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\, \mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around } \tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

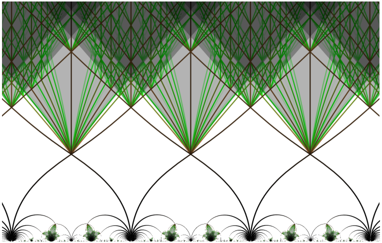
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

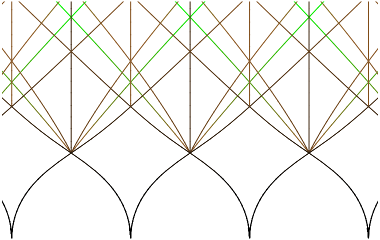
Scattering diagram $\mathcal{D}_\psi$:

- codim$_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0, +\infty),\ \bar{\Omega}_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar{\Omega}_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1} - y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\, \mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around } \tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

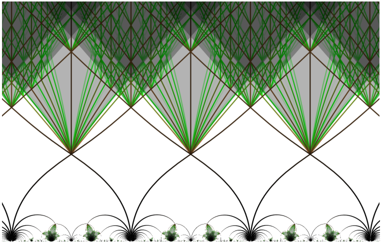
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

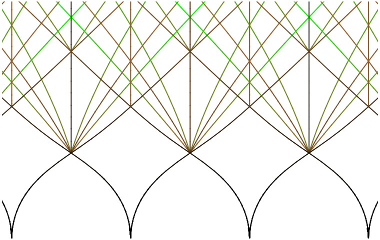
Scattering diagram $\mathcal{D}_\psi$:

- $\operatorname{codim}_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar\Omega_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar\Omega_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1}-y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around }\tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

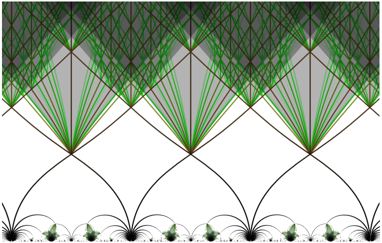
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

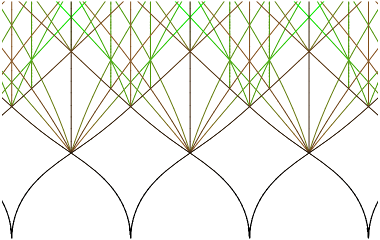
Scattering diagram $\mathcal{D}_\psi$:

- codim$_\mathbb{R}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar\Omega_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar\Omega_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1} - y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around }\tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

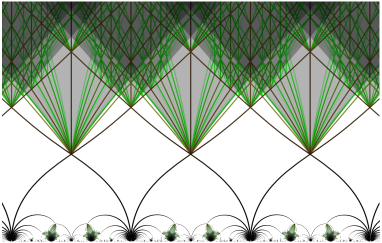
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

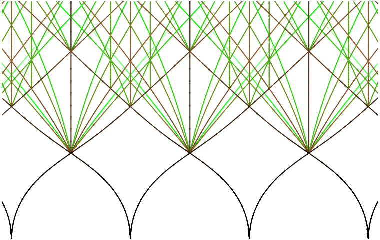
Scattering diagram $\mathcal{D}_\psi$:

- codim$_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar{\Omega}_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar{\Omega}_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1} - y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around }\tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

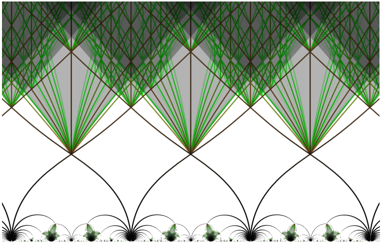
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

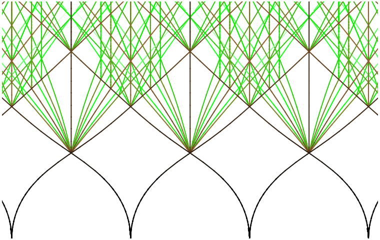
Scattering diagram $\mathcal{D}_\psi$:

- codim$_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar\Omega_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar\Omega_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1}-y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around }\tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

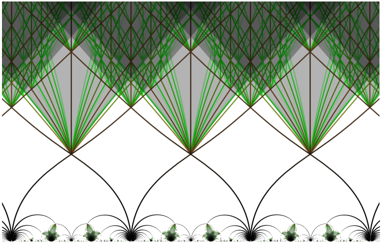
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

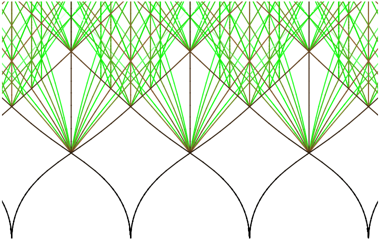
Scattering diagram $\mathcal{D}_\psi$:

- $\mathrm{codim}_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar\Omega_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp\big(\bar\Omega_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1}-y)\big)$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\prod^{\curvearrowleft}_{\text{rays around }\tau} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

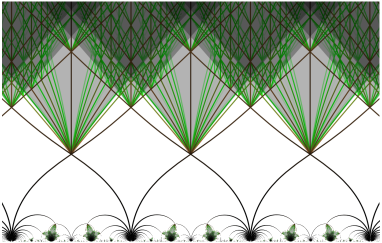
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

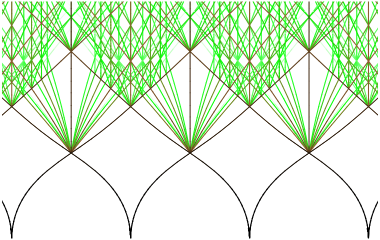
Scattering diagram $\mathcal{D}_\psi$:

- codim$_\mathbb{R}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0, +\infty),\ \bar{\Omega}_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar{\Omega}_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1} - y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\, \mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around }\tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

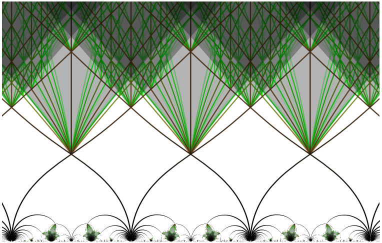
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

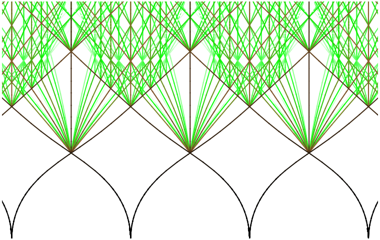
Scattering diagram $\mathcal{D}_\psi$:

- codim$_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar\Omega_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp(\bar\Omega_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1}-y))$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around }\tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

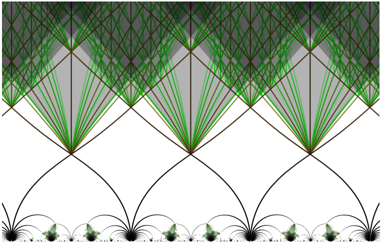
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

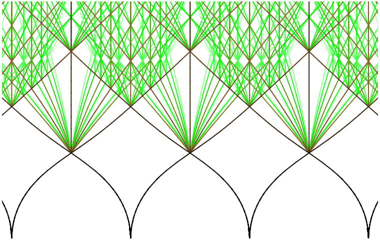
Scattering diagram $\mathcal{D}_\psi$:

- $\mathrm{codim}_{\mathbb{R}}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar{\Omega}_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp\big(\bar{\Omega}_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1} - y)\big)$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\displaystyle\overset{\curvearrowleft}{\prod_{\text{rays around } \tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$





Scattering diagram definition

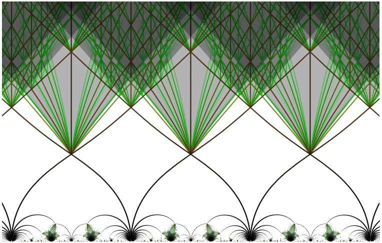
Definition [Kontsevich–Soibelman 2006, Gross–Siebert 2011, Bridgeland 2016]

Scattering diagram $\mathcal{D}_\psi$:

- codim$_\mathbb{R}$-1 rays $\mathcal{R}_\psi(\gamma) = \{\tau \mid Z_\tau(\gamma) \in \mathrm{i}e^{\mathrm{i}\psi}(0,+\infty),\ \bar{\Omega}_\tau(\gamma) \neq 0\}$
- equipped with $\mathcal{U}_\tau(\gamma) = \exp\big(\bar{\Omega}_\tau(\gamma)\mathcal{X}_\gamma/(y^{-1} - y)\big)$

  formal variables $\mathcal{X}_0 = 1$ and $\mathcal{X}_\gamma\,\mathcal{X}_{\gamma'} = (-y)^{\langle\gamma,\gamma'\rangle}\mathcal{X}_{\gamma+\gamma'}$

Wall-crossing $\iff$ **consistency** $\overset{\curvearrowleft}{\prod\limits_{\text{rays around }\tau}} \mathcal{U}(\gamma)^{\pm 1} = 1$



Convergence hard due to dense rays!
Use a cutoff function

$$\varphi_\tau(\gamma) = d - r\,x(\tau)$$

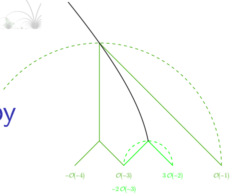
To compute a single $\Omega_\tau(\gamma)$,
simpler approach: attractor flows

1. IIA strings and BPS spectrum

2. Local $\mathbb{P}^2$

3. Scattering diagrams ($\psi = 0$)

4. Split attractor flow conjecture and dendroscopy



5. Initial data and orbifold region

6. Scattering diagrams for all values of ψ

Attractor flow

In principle, to compute one $\Omega_\tau(\gamma)$, move along the moduli space towards simpler points, deal with wall-crossing. **Need to pick path, find walls**

Attractor flow

In principle, to compute one $\Omega_\tau(\gamma)$, move along the moduli space towards simpler points, deal with wall-crossing. **Need to pick path, find walls**

Attractor flow in Kähler moduli space $\mathcal{M}_K$, for a charge γ

[Ferrara–Kallosh–Strominger '95]

$$\frac{\mathrm{d}z^i}{\mathrm{d}\mu} = -g^{i\bar{j}}\partial_{\bar{j}}|Z_z(\gamma)|^2$$

Attractor flow

In principle, to compute one $\Omega_\tau(\gamma)$, move along the moduli space towards simpler points, deal with wall-crossing. **Need to pick path, find walls**

Attractor flow in Kähler moduli space $\mathcal{M}_K$, for a charge γ

[Ferrara–Kallosh–Strominger '95]

$$\frac{\mathrm{d}z^i}{\mathrm{d}\mu} = -g^{i\bar{j}}\partial_{\bar{j}}|\boxed{Z_z(\gamma)}|^2$$

- Kähler potential $K_K = -\log Z_{S^2}[\text{GLSM with target } X]$
- Central charge of D-branes $Z_z(\gamma) = Z_{\text{hemi-}S^2}[\text{GLSM}, \text{brane}]$

Attractor flow

In principle, to compute one $\Omega_\tau(\gamma)$, move along the moduli space towards simpler points, deal with wall-crossing. **Need to pick path, find walls**

Attractor flow in Kähler moduli space $\mathcal{M}_K$, for a charge γ

[Ferrara–Kallosh–Strominger '95]

$$\frac{\mathrm{d}z^i}{\mathrm{d}\mu} = -g^{i\bar{j}}\partial_{\bar{j}}|\boxed{Z_z(\gamma)}|^2$$

- Kähler potential $K_K = -\log Z_{S^2}[\text{GLSM with target } X]$
- Central charge of D-branes $Z_z(\gamma) = Z_{\text{hemi-}S^2}[\text{GLSM}, \text{brane}]$

For non-compact X, $Z_z(\gamma)$ is holomorphic in z so

$$\frac{\mathrm{d}Z_z(\gamma)}{\mathrm{d}\mu} = -g^{i\bar{j}}\partial_i Z_z(\gamma)\partial_{\bar{j}}|Z_z(\gamma)|^2 = -|\partial Z_z(\gamma)|^2 Z_z(\gamma)$$

hence $Z_z(\gamma)$ keeps a **constant phase** and **decreasing norm**

Split attractor flow

Along the attractor flow of $Z(\gamma)$, when crossing wall $W(\gamma_1, \gamma_2)$,

- $\Omega_z(\gamma)$ jumps by a combination of $\Omega_z(k\gamma_1)$ and $\Omega_z(l\gamma_2)$, $k, l \geq 1$
- the **rational DT invariant**

$$\bar{\Omega}_z(\gamma) := \sum_{m|\gamma} \frac{y - y^{-1}}{m(y^m - y^{-m})} \Omega_z(\gamma/m)|_{y \to y^m}$$

jumps by (coef)$\bar{\Omega}_z(\gamma_1)\bar{\Omega}_z(\gamma_2)$, studied in turn using the attractor flow

Split attractor flow

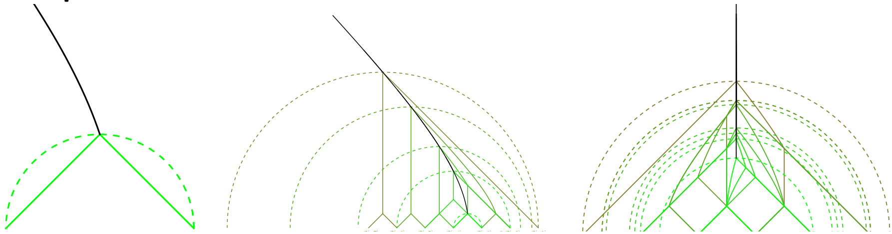
Along the attractor flow of $Z(\gamma)$, when crossing wall $W(\gamma_1, \gamma_2)$,

- $\Omega_z(\gamma)$ jumps by a combination of $\Omega_z(k\gamma_1)$ and $\Omega_z(l\gamma_2)$, $k, l \geq 1$
- the **rational DT invariant**

$$\bar{\Omega}_z(\gamma) := \sum_{m|\gamma} \frac{y - y^{-1}}{m(y^m - y^{-m})} \Omega_z(\gamma/m)|_{y \to y^m}$$

  jumps by (coef)$\bar{\Omega}_z(\gamma_1)\bar{\Omega}_z(\gamma_2)$, studied in turn using the attractor flow

Examples in local $\mathbb{P}^2$:

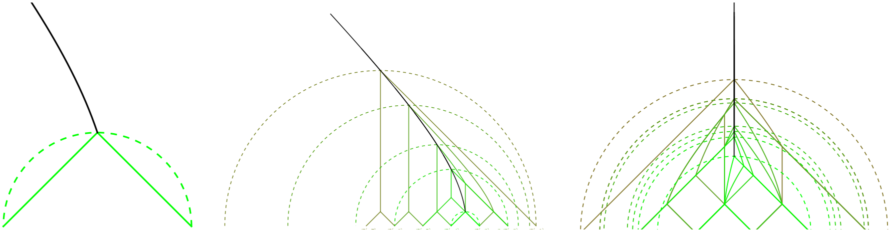


Split attractor flow

Split Attractor Flow Formula

$$\bar{\Omega}_z(\gamma) = \sum_{\text{tree that splits } \gamma=\gamma_1+\cdots+\gamma_n} (\text{combinatorics}) \prod_{i=1}^{n} \bar{\Omega}_{z_i}(\gamma_i)$$

Sum over attractor flow trees rooted at z and ending at attractor points z_i where $Z_{z_i}(\gamma_i) = 0$.



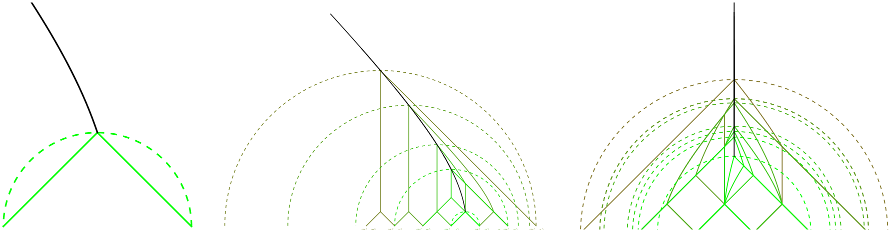
Split attractor flow

Split Attractor Flow Formula

$$\bar{\Omega}_z(\gamma) = \sum_{\text{tree that splits } \gamma=\gamma_1+\cdots+\gamma_n} (\text{combinatorics}) \prod_{i=1}^{n} \bar{\Omega}_{z_i}(\gamma_i)$$

Sum over attractor flow trees rooted at z and ending at attractor points z_i where $Z_{z_i}(\gamma_i) = 0$. **Conjecture: the sum is finite**

we proved it for local $\mathbb{P}^2$



Split attractor flow

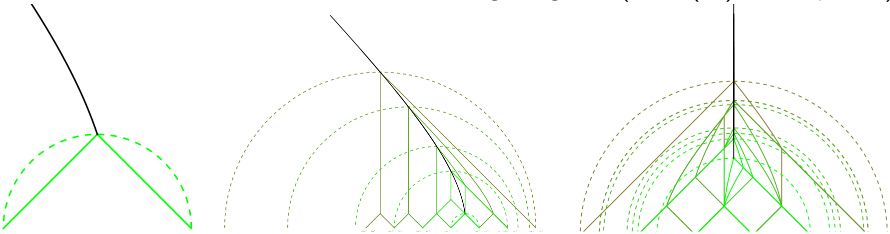
Split Attractor Flow Formula

$$\bar{\Omega}_z(\gamma) = \sum_{\text{tree that splits } \gamma=\gamma_1+\cdots+\gamma_n} (\text{combinatorics}) \prod_{i=1}^{n} \bar{\Omega}_{z_i}(\gamma_i)$$

Sum over attractor flow trees rooted at z and ending at attractor points z_i where $Z_{z_i}(\gamma_i) = 0$. **Conjecture: the sum is finite**

we proved it for local $\mathbb{P}^2$

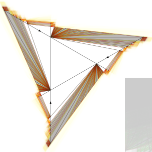
Remark: attractor flow trees $\subset$ scattering diagram (all $Z(\dots)$ same phase)



1. IIA strings and BPS spectrum

2. Local $\mathbb{P}^2$
3. Scattering diagrams ($\psi = 0$)

4. Split attractor flow conjecture and dendroscopy

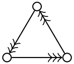


5. **Initial data and orbifold region**

6. Scattering diagrams for all values of ψ

Quiver description near the orbifold point

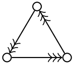
$K\mathbb{P}^2$ is a resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$

Near $\tau = \tau_o$, quiver description  with superpotential

- $K\mathbb{P}^2 \to$ Higgs branch
- Kähler parameters of $K\mathbb{P}^2 \to$ Fayet–Iliopoulos parameters θ
- $\Omega_\tau(\gamma) = \#\{\text{BPS states of charge } \gamma = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3\}$
  $=$ index of world-volume theory of n_i fractional D0 branes
  (counts stable representations)

Quiver description near the orbifold point

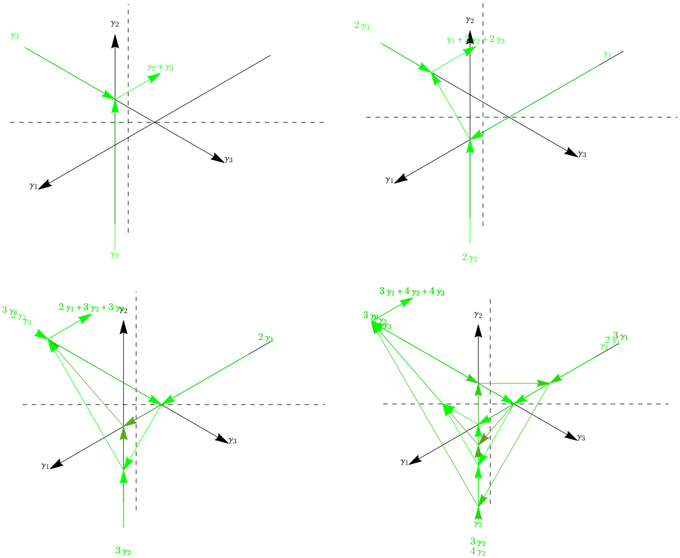
$K\mathbb{P}^2$ is a resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$

Near $\tau = \tau_o$, quiver description  with superpotential

- $K\mathbb{P}^2 \to$ Higgs branch
- Kähler parameters of $K\mathbb{P}^2 \to$ Fayet–Iliopoulos parameters θ
- $\Omega_\tau(\gamma) = \#\{\text{BPS states of charge } \gamma = n_1\gamma_1 + n_2\gamma_2 + n_3\gamma_3\}$
  $=$ index of world-volume theory of n_i fractional D0 branes
  (counts stable representations)

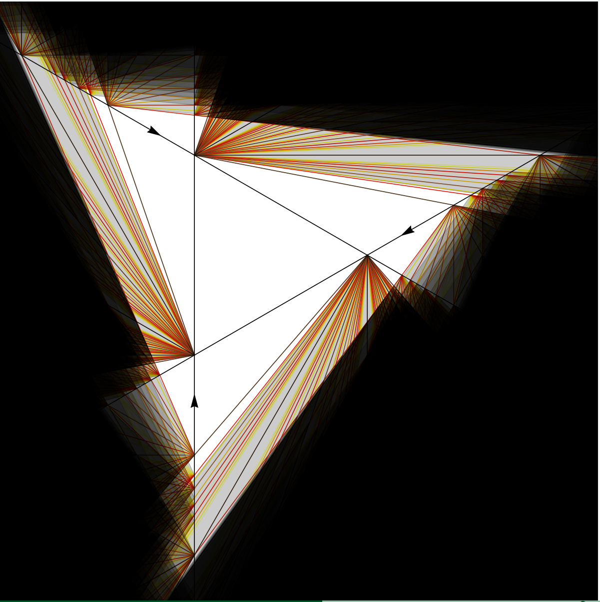
Quiver representations have been very extensively studied

- Theorem: consistent scattering diagram [Bridgeland]
- Formula of $\Omega_\theta(\gamma)$ in terms of Ω_* complicated:
  - flow tree formula [Alexandrov–Pioline, Argüz–Bousseau]
  - operadic approach [Mozgovoy]
- Theorem: split attractor flow conjecture true [Bousseau-Argüz]



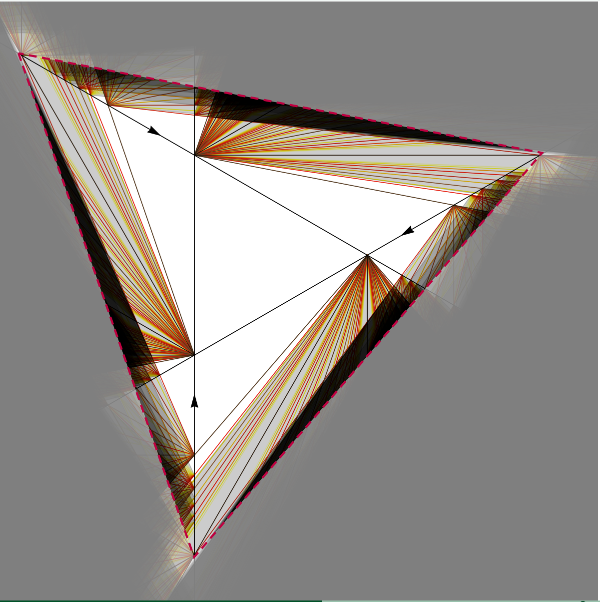
Scattering sequences for $\gamma = (n-1, n, n)$ corresponding to the Hilbert scheme of n points on $\mathbb{P}^2$, with $n = 1, 2, 3, 4$

Quiver for local $\mathbb{P}^2$: scattering diagram



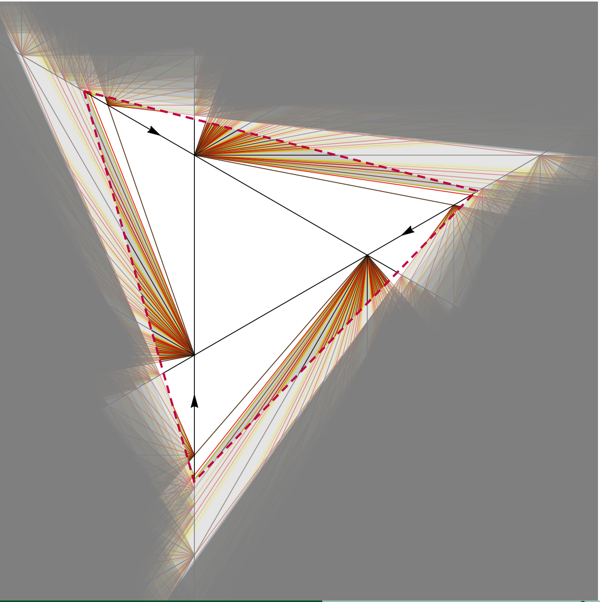
Side-note on first scattering: $\mathcal{R}(1,0,0)$ with $\mathcal{R}(0,1,0)$ gives $\mathcal{R}(n_1,n_2,0)$ with $\Omega(n_1,n_2,0)$ counting representations of dimension $(n_1,n_2,0)$, hence representations of the Kronecker quiver $\circ \!\!\xrightarrow{\;\;\ggg\;\;}\!\! \circ$

Quiver for local $\mathbb{P}^2$: scattering diagram



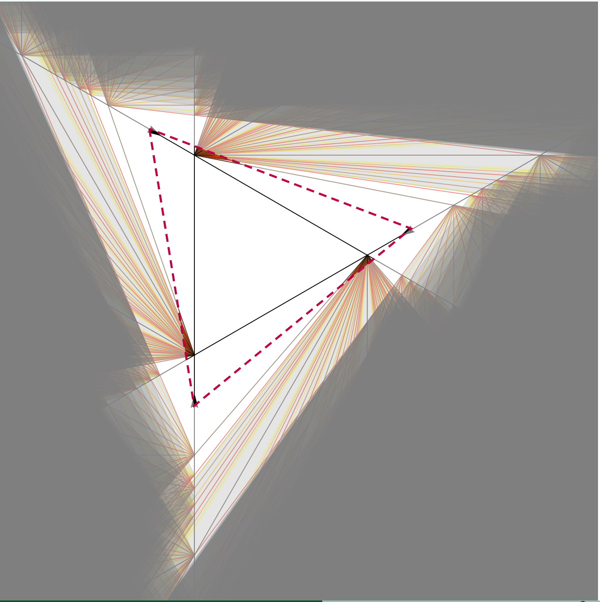
Region relevant for the exact diagram
with ψ large

Quiver for local $\mathbb{P}^2$: scattering diagram



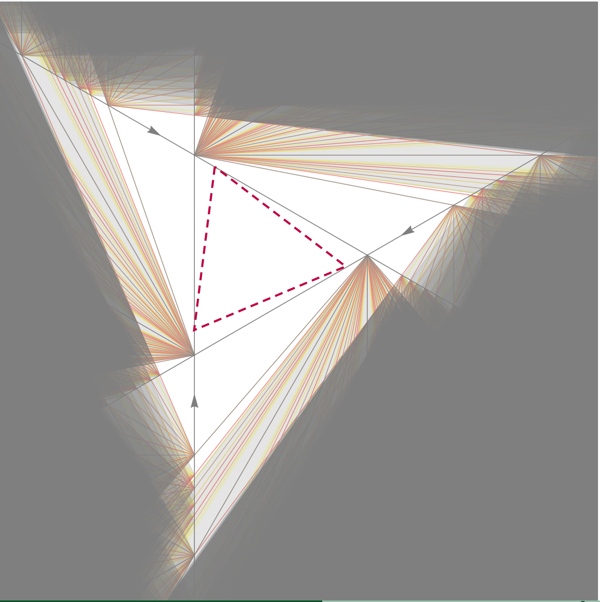
Region relevant for the exact diagram
with ψ medium

Quiver for local $\mathbb{P}^2$: scattering diagram

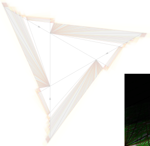


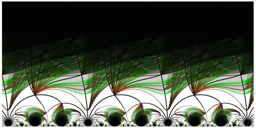
Region relevant for the exact diagram
with ψ small

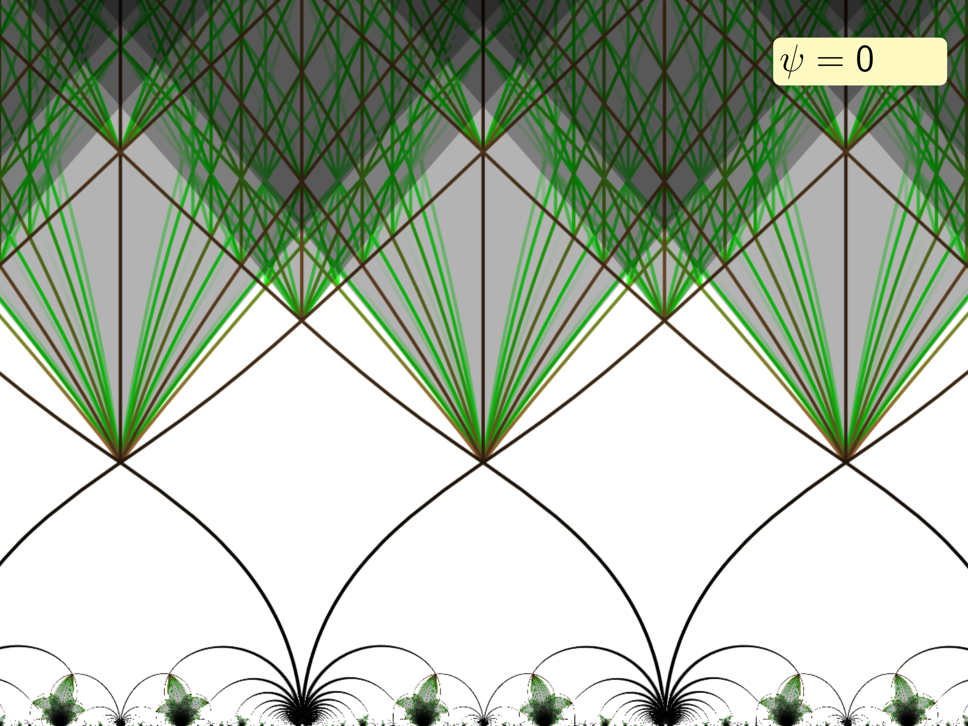
Quiver for local $\mathbb{P}^2$: scattering diagram

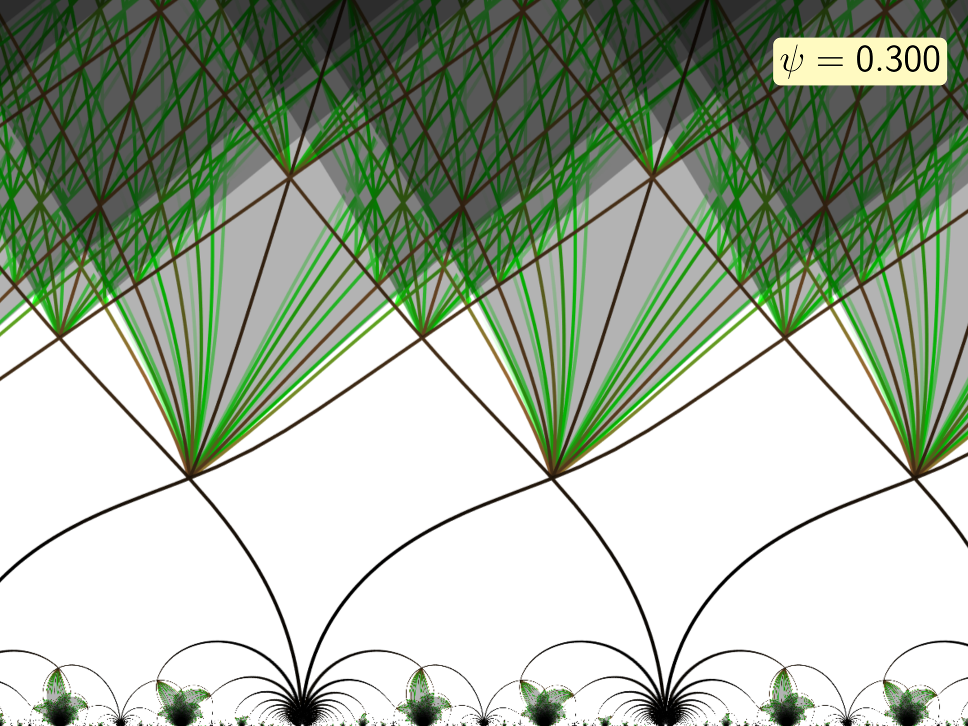


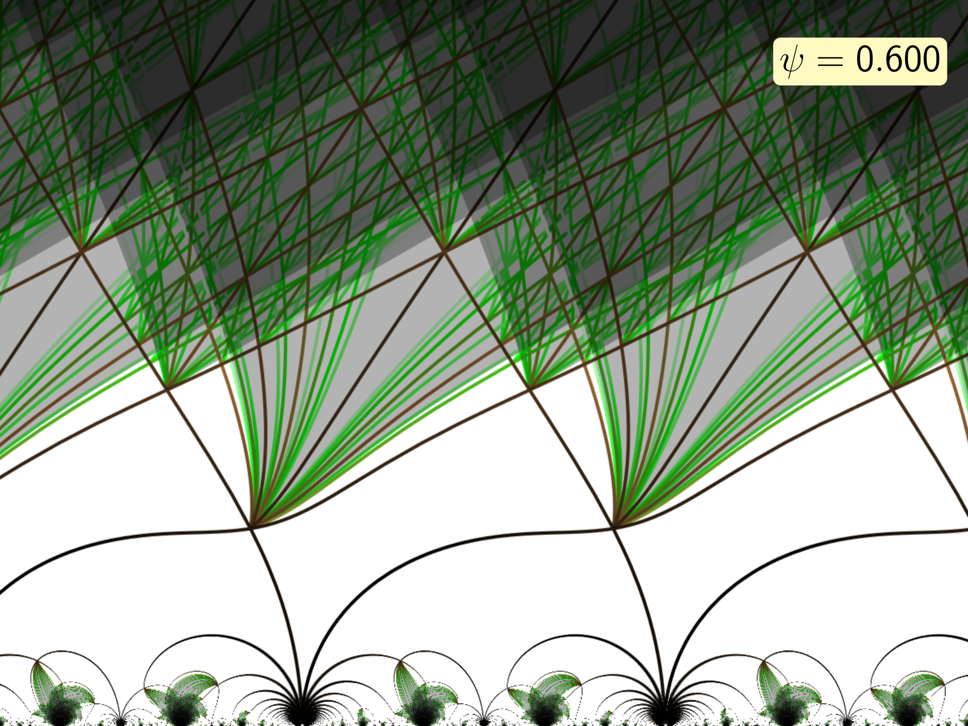
Region relevant for the exact diagram
with ψ tiny

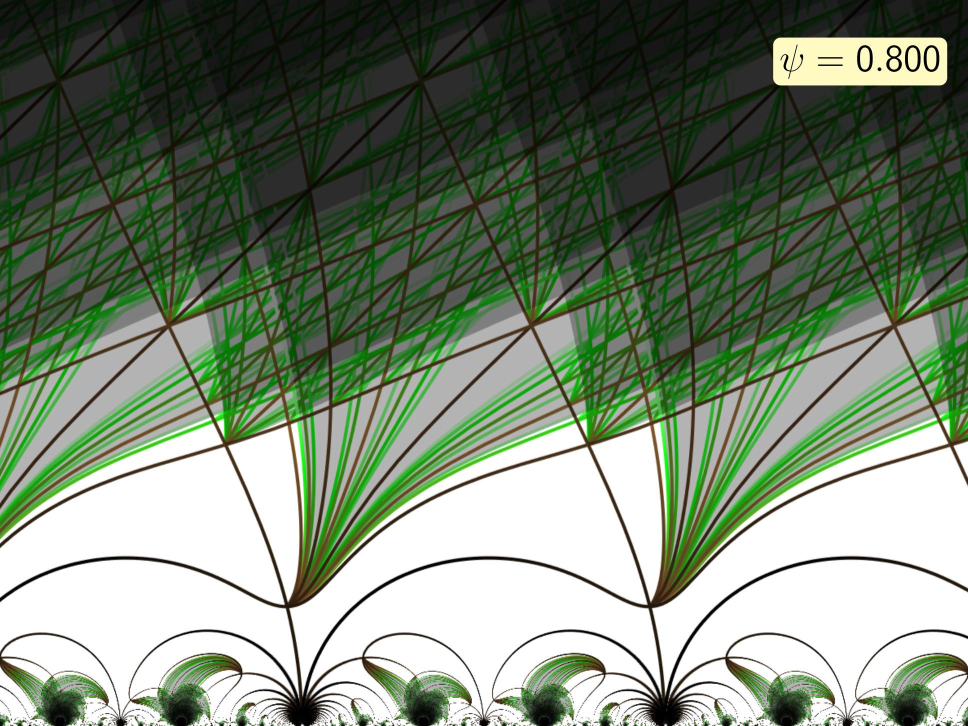


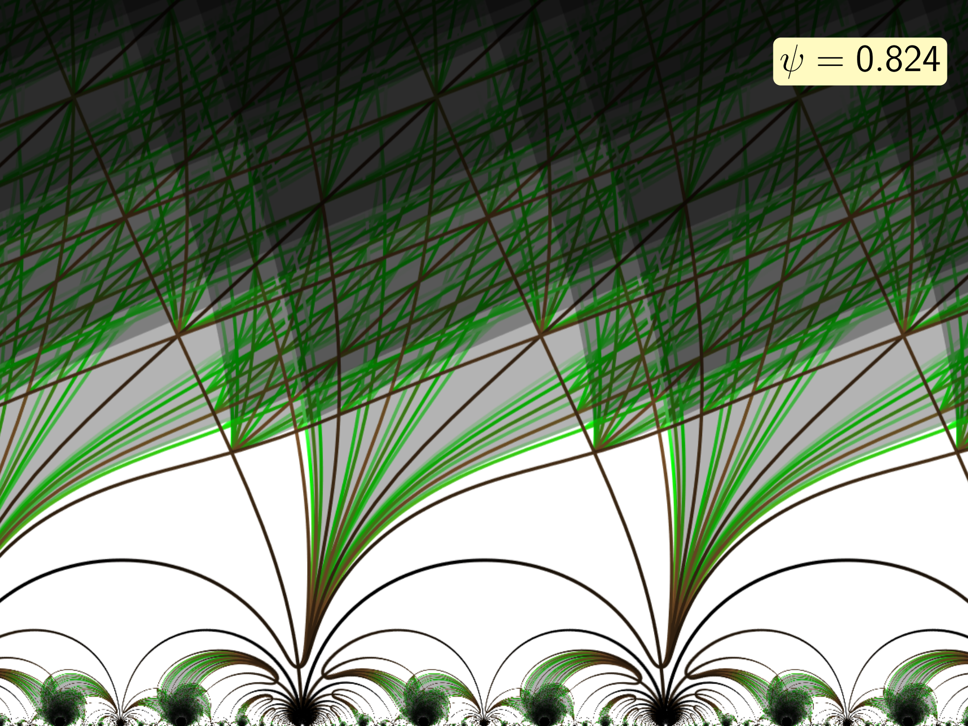


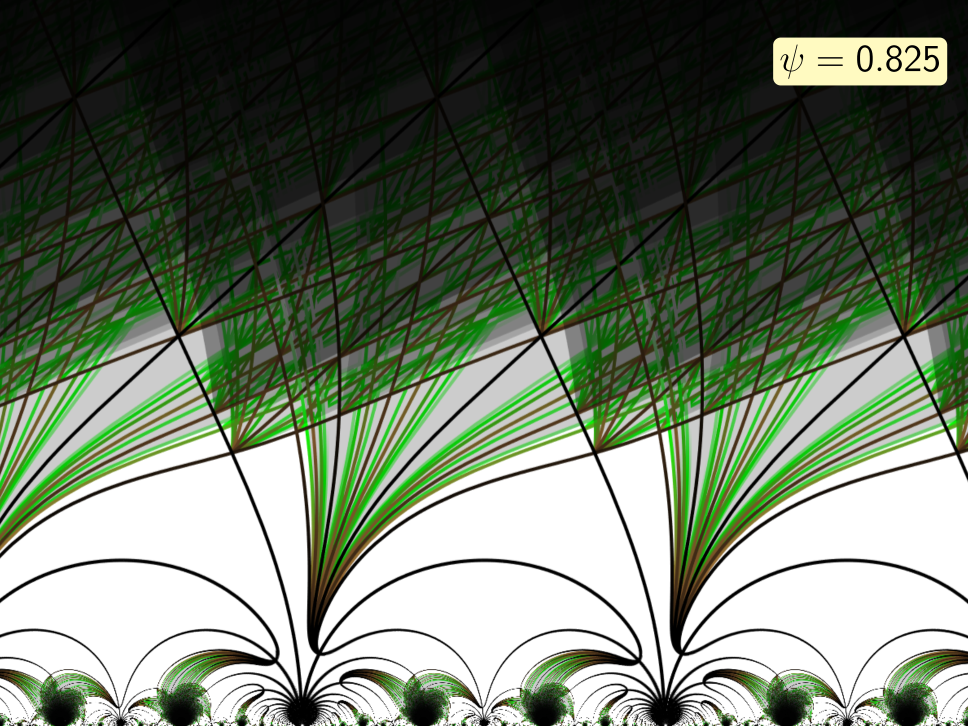

$\psi = 0$

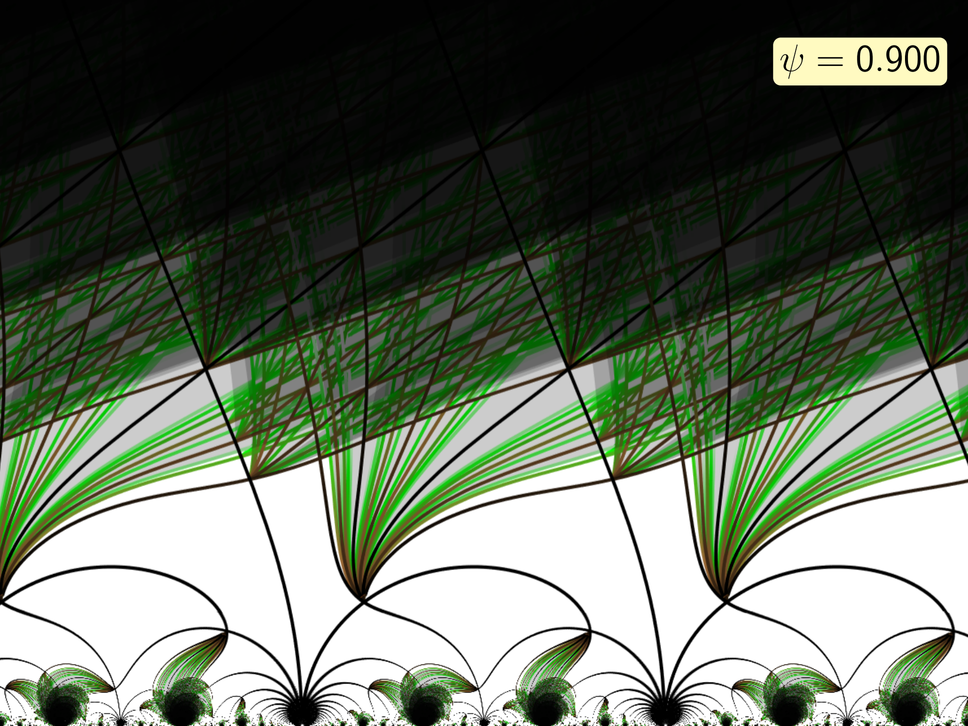

$\psi = 0.300$

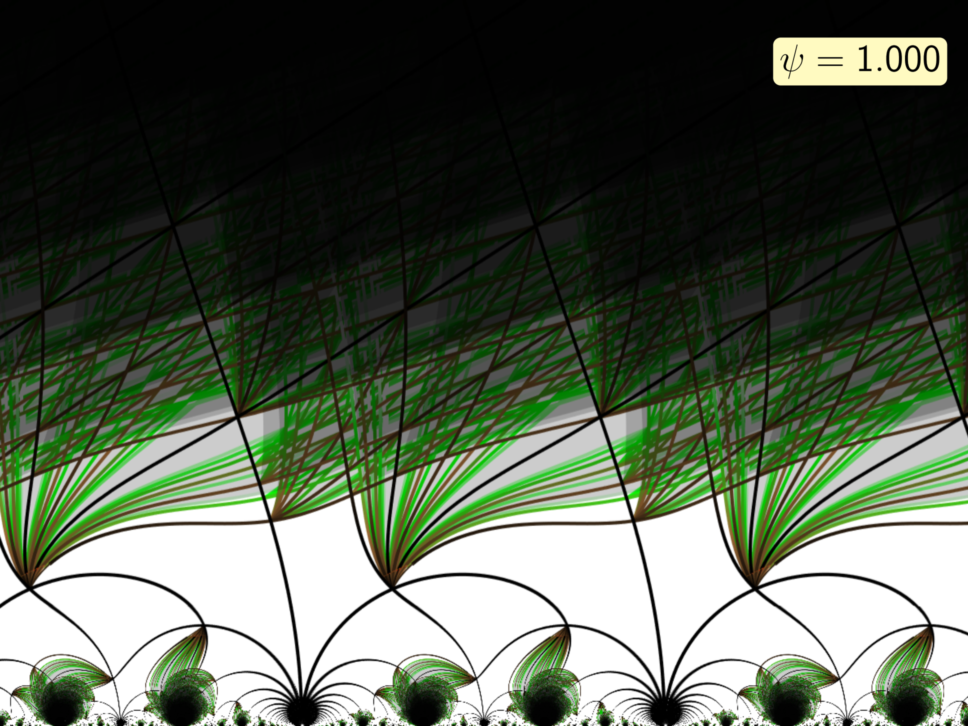

$\psi = 0.600$

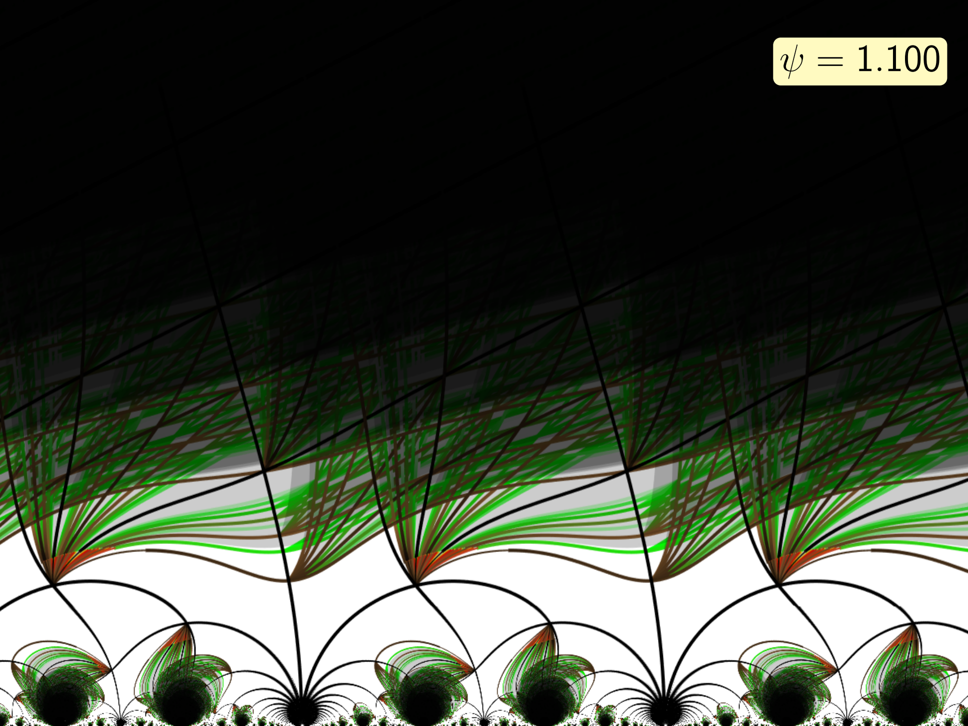

$\psi = 0.800$

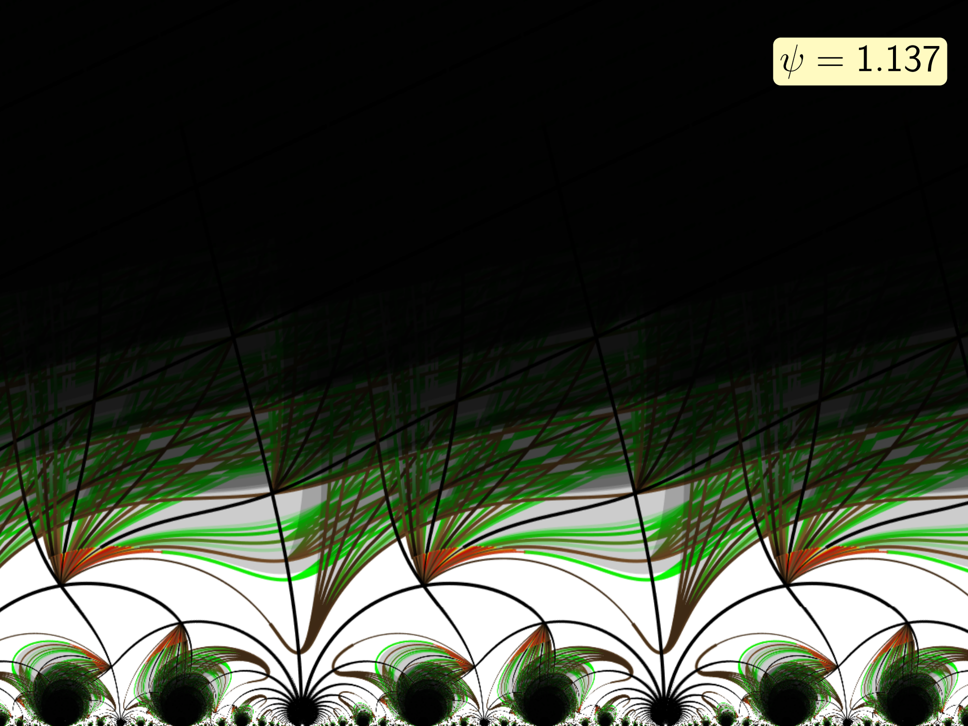

$\psi = 0.824$

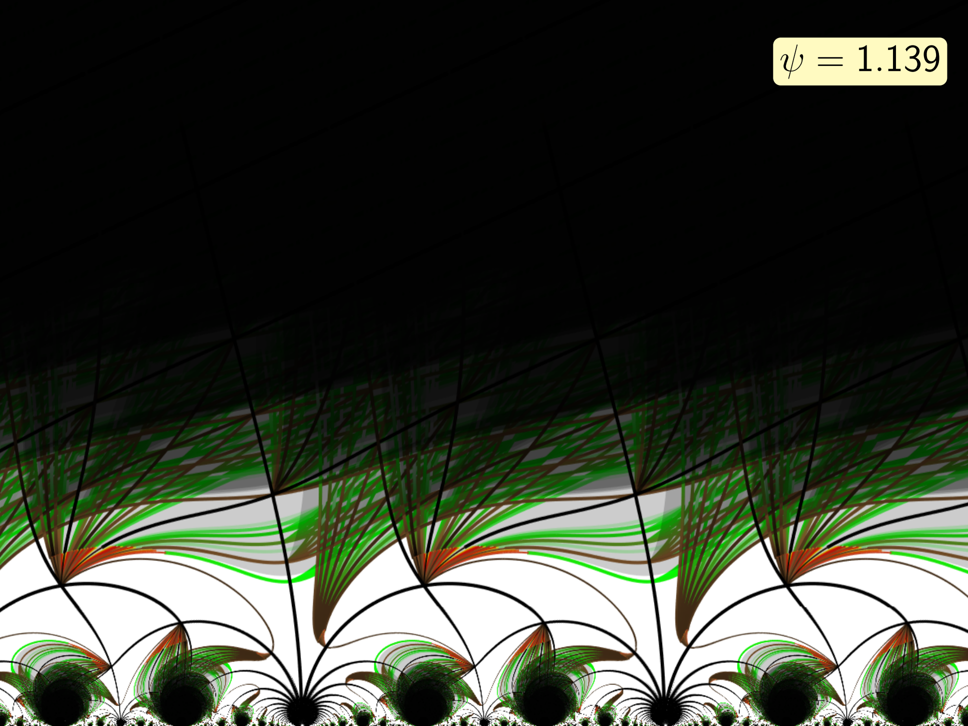

$\psi = 0.825$

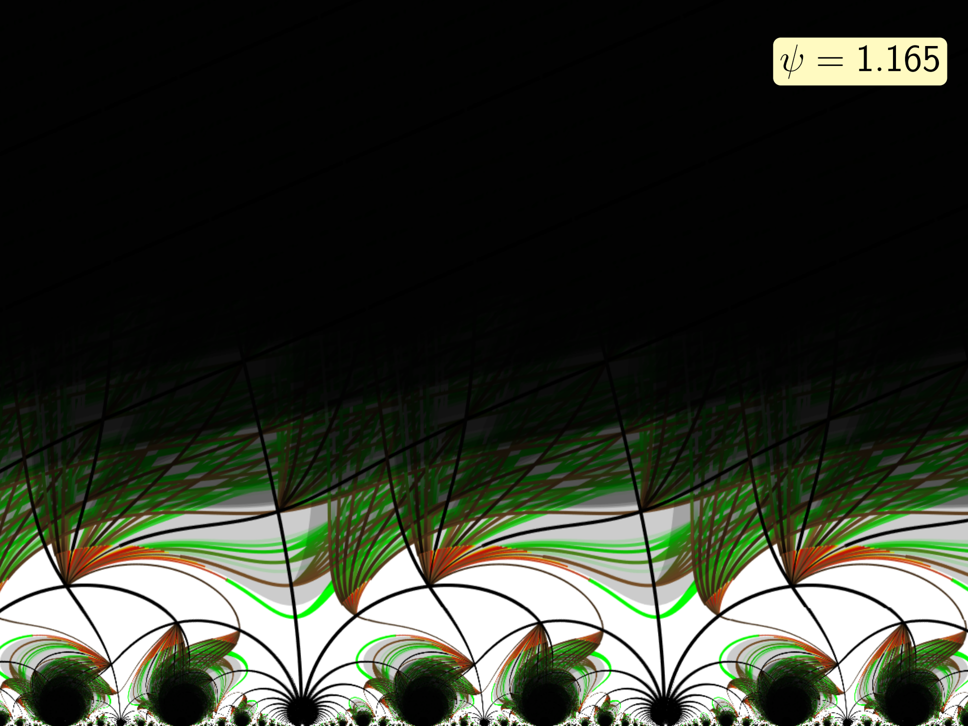

$\psi = 0.900$

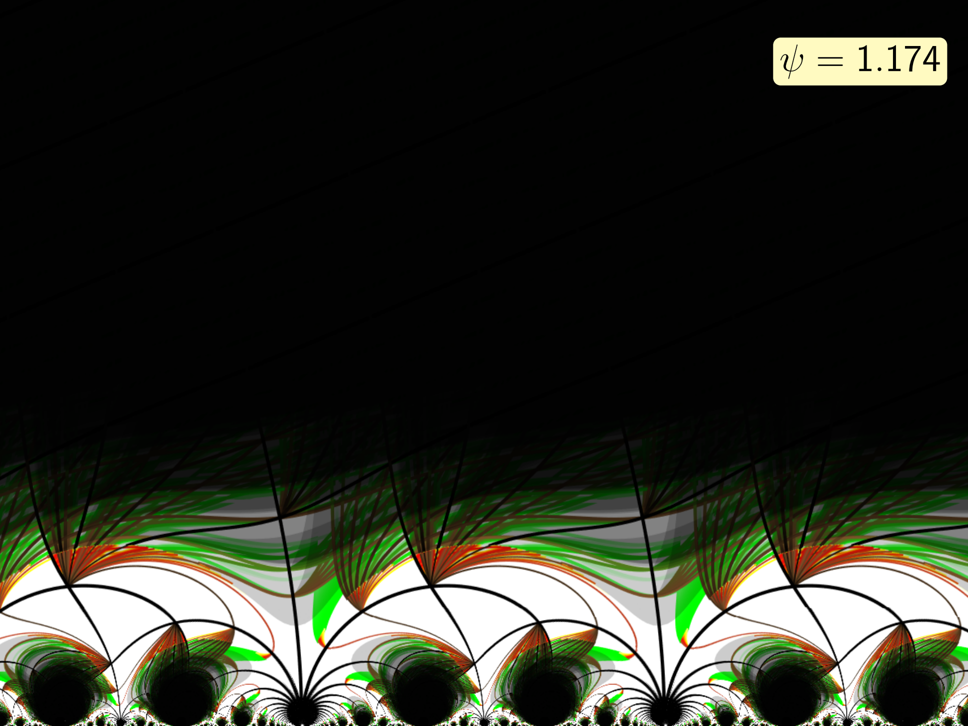

$\psi = 1.000$

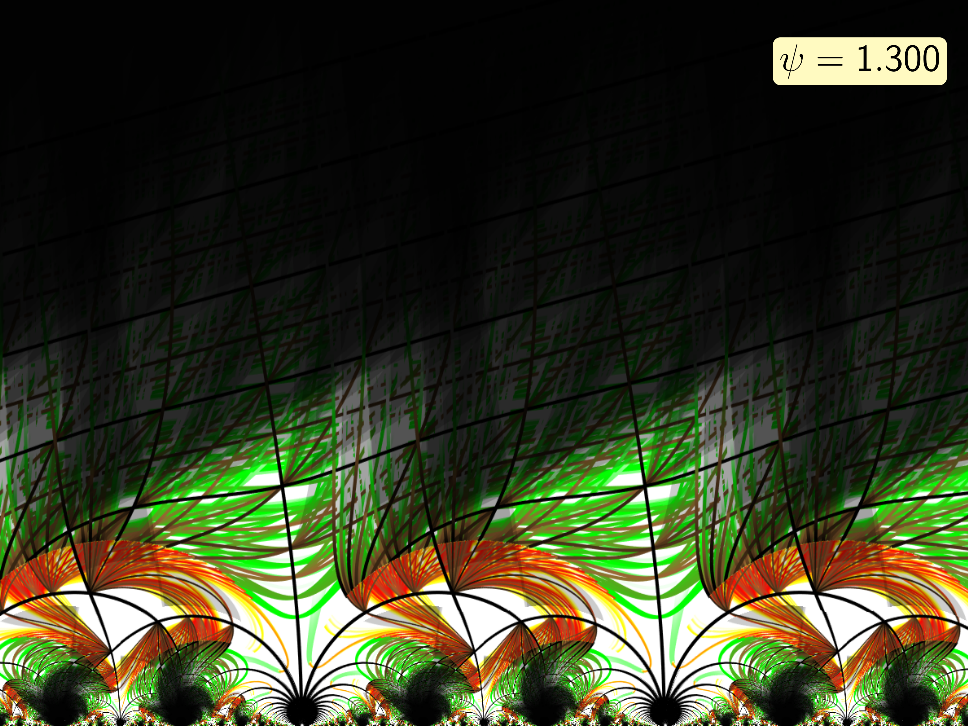

$\psi = 1.100$

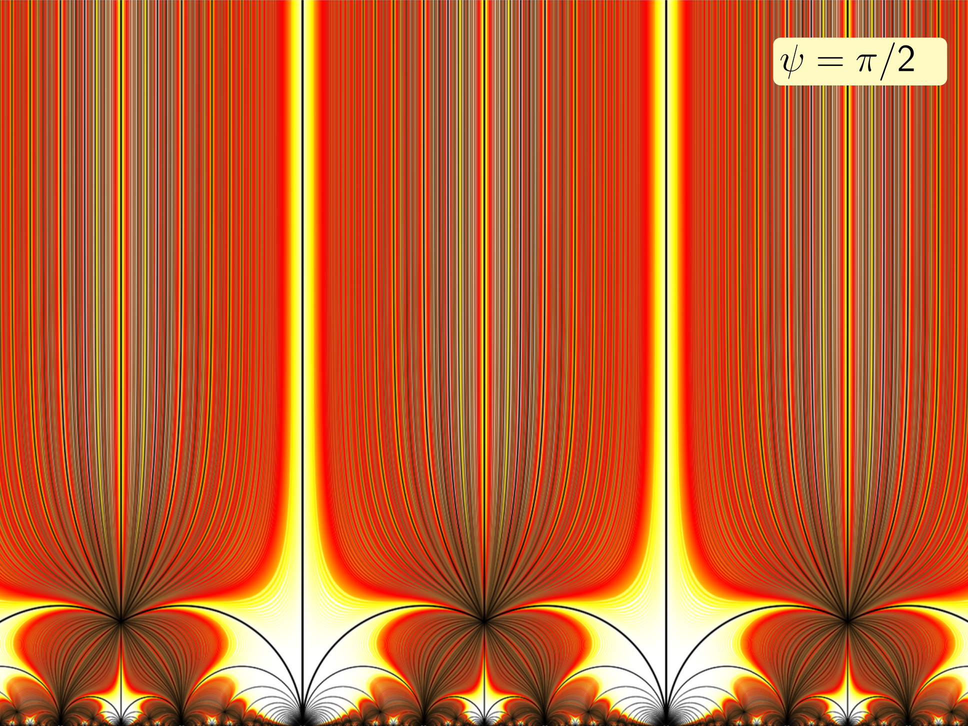

$\psi = 1.137$


$\psi = 1.139$


$\psi = 1.165$


$\psi = 1.174$


$\psi = 1.300$


$\psi = \pi/2$

Future directions

- Dendroscopy: moduli space, not just $\Omega_\tau(\gamma)$
  (attractor trees$\leftrightarrow$strata in $\mathcal{M}_\tau(\gamma)$)
- Polynomial-time algorithm to compute $\Omega_\tau(\gamma)$ using memoization
  (exponentially-many trees with common subtrees)
- $X =$ local del Pezzo surface
  ($\dim_{\mathbb{R}} = 1$ attractor flows) $\subset$ ($\mathrm{codim}_{\mathbb{R}} = 1$ scattering diagrams)
- Add surface operators, relate to (exponential) spectral networks

Definition and integrality of refined DT invariants
Relation with topological string partition function
Modularity properties of generating series arXiv:2301.08066

Future directions

- Dendroscopy: moduli space, not just $\Omega_\tau(\gamma)$
  (attractor trees$\leftrightarrow$strata in $\mathcal{M}_\tau(\gamma)$)
- Polynomial-time algorithm to compute $\Omega_\tau(\gamma)$ using memoization
  (exponentially-many trees with common subtrees)
- $X =$ local del Pezzo surface
  ($\dim_{\mathbb{R}} = 1$ attractor flows) $\subset$ ($\mathrm{codim}_{\mathbb{R}} = 1$ scattering diagrams)
- Add surface operators, relate to (exponential) spectral networks

Thorsten
Schimannek
.jpg

Definition and integrality of refined DT invariants
Relation with topological string partition function
Modularity properties of generating series arXiv:2301.08066

Let us thank the conference organizers!

M-theory on a local Calabi–Yau 3-fold

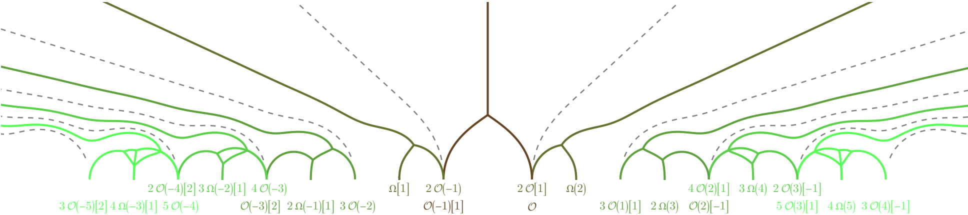
Take $X = KS$ canonical bundle over a Fano 2-fold S

S	n_{params}	5d theory	Lagrangian description
dP_8	9	E_8 SCFT	UV limit of $SU(2)$ $N_f = 7$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
dP_2	3	E_2 SCFT	UV limit of $SU(2)$ $N_f = 1$
$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$	2	E_1 SCFT	UV limit of pure $SU(2)$
$\mathbb{F}_1 = \mathrm{dP}_1$	2	$\widetilde{E}_1$ SCFT	UV limit of pure $SU(2)_\pi$
$\mathbb{P}^2$	1	E_0 SCFT	

Dendroscopy for local $\mathbb{P}^2$ as we vary ψ beyond ± 0.824

D2 brane $\gamma = [0,1,1)$: Chern vector of $\mathcal{O}_C$ of curve $C \subset \mathbb{P}^2$ (hyperplane class)

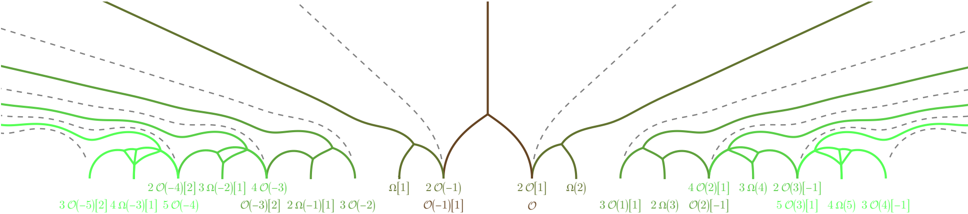
$\Omega_{\tau_2 \gg 1}(\gamma) = K_3(1,2)K_3(1,3)^{n-1} = y^2 + 1 + 1/y^2$



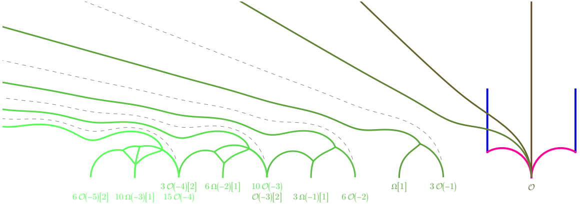
Dendroscopy for local $\mathbb{P}^2$ as we vary ψ beyond ± 0.824

D2 brane $\gamma = [0, 1, 1)$: Chern vector of $\mathcal{O}_C$ of curve $C \subset \mathbb{P}^2$ (hyperplane class)

$\Omega_{\tau_2 \gg 1}(\gamma) = K_3(1,2) K_3(1,3)^{n-1} = y^2 + 1 + 1/y^2$



D4 brane $\gamma = [1, 0, 1)$: Chern vector of $\mathcal{O}$; $\Omega_{\tau_2 \gg 1} = K_3(1,3) \ldots K_{3n}(1, 3n) = 1$



Split attractor flows for arbitrary $|\psi| < \pi/2$

Split Attractor Flow Conjecture

The BPS index $\Omega_\tau(\gamma)$ for a given charge γ and moduli τ is a finite sum, over attractor flow trees rooted at τ and ending at "attractor points", of combinations of "attractor indices" $\Omega_*(\gamma_i)$ that count BPS states at these points.

Proof for local $\mathbb{P}^2$: analyze leaves, define monotonic φ, reconstruct trees.

Split attractor flows for arbitrary $|\psi| < \pi/2$

Split Attractor Flow Conjecture

The BPS index $\Omega_\tau(\gamma)$ for a given charge γ and moduli τ is a finite sum, over attractor flow trees rooted at τ and ending at "attractor points", of combinations of "attractor indices" $\Omega_*(\gamma_i)$ that count BPS states at these points.

Proof for local $\mathbb{P}^2$: analyze leaves, define monotonic φ, reconstruct trees.

1a. By compactness, any split attractor flow must eventually end
1b. By $\tau \to \mathrm{i}\infty$ asymptotics it cannot end at large volume
1c. Ending at a conifold point is constrained using $(3,3,3)$ and $(3,3,6)$ quivers that cover the conifold neighborhood

Split attractor flows for arbitrary $|\psi| < \pi/2$

Split Attractor Flow Conjecture

The BPS index $\Omega_\tau(\gamma)$ for a given charge γ and moduli τ is a finite sum, over attractor flow trees rooted at τ and ending at "attractor points", of combinations of "attractor indices" $\Omega_*(\gamma_i)$ that count BPS states at these points.

Proof for local $\mathbb{P}^2$: analyze leaves, define monotonic φ, reconstruct trees.

1a. By compactness, any split attractor flow must eventually end
1b. By $\tau \to \mathrm{i}\infty$ asymptotics it cannot end at large volume
1c. Ending at a conifold point is constrained using $(3,3,3)$ and $(3,3,6)$ quivers that cover the conifold neighborhood

2a. Piecewise potential φ: $\begin{cases} c(n_1+n_2+n_3) & \text{in orbifold region} \\ 2(d - r\lfloor x - \mathcal{V}_\psi \rfloor) & \text{in large volume region} \end{cases}$
2b. Check φ decreases along flow; $\varphi(\text{leaves}) \geq C > 0$

Split attractor flows for arbitrary $|\psi| < \pi/2$

Split Attractor Flow Conjecture

The BPS index $\Omega_\tau(\gamma)$ for a given charge γ and moduli τ is a finite sum, over attractor flow trees rooted at τ and ending at "attractor points", of combinations of "attractor indices" $\Omega_*(\gamma_i)$ that count BPS states at these points.

Proof for local $\mathbb{P}^2$: analyze leaves, define monotonic φ, reconstruct trees.

1a. By compactness, any split attractor flow must eventually end
1b. By $\tau \to \mathrm{i}\infty$ asymptotics it cannot end at large volume
1c. Ending at a conifold point is constrained using $(3, 3, 3)$ and $(3, 3, 6)$ quivers that cover the conifold neighborhood

2a. Piecewise potential φ: $\begin{cases} c(n_1 + n_2 + n_3) & \text{in orbifold region} \\ 2(d - r\lfloor x - \mathcal{V}_\psi \rfloor) & \text{in large volume region} \end{cases}$
2b. Check φ decreases along flow; $\varphi(\text{leaves}) \geq C > 0$

3a. Geometry allows only some constituents to contribute to $\Omega_\tau(\gamma)$
3b. Finite number of trees with given constituents

$\square$