Branes and vertex operators in eight dimensions

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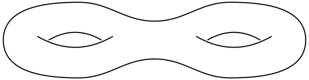
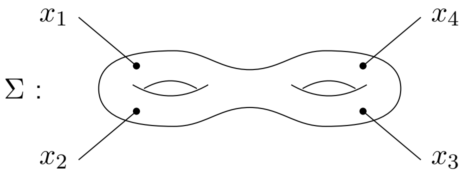
Based on [arXiv:2212.03870] and work in progress

Vertex operator in string theory

Σ :



Vertex operator in string theory



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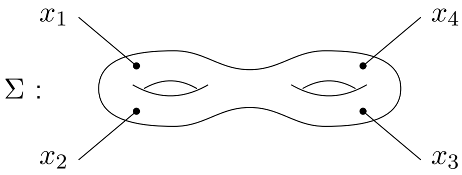
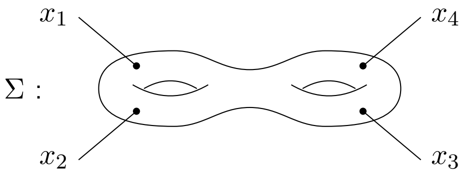


$$= \langle V_{\alpha_1}(x_1) V_{\alpha_2}(x_2) V_{\alpha_3}(x_3) V_{\alpha_4}(x_4) \rangle_\Sigma$$

Vertex operator in string theory



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- Vertex operator creates **string interaction**, **boundary**, **D-branes (?)**
- Goal : vertex operator = D-brane creation operator

D0-D4-D8 system on $\mathbb{C}^4$

Main formula

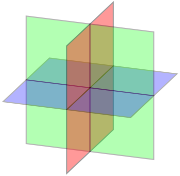
kD0-nD4-n'D4'-mD8 system (gauge origami [Nekrasov]) partition function is given by

$$Z_k = \frac{1}{k!} \oint d^k\phi \, \langle \prod_{I=1}^{k} \mathsf{A}^{-1}(\phi_I) \cdot \mathsf{X}_{12}^{[n]} \, \mathsf{X}_{34}^{[n']} \, \mathsf{Z}^{[m]} \rangle$$

where $\quad \mathsf{X}_{12}^{[n]} = : \prod_{\alpha=1}^{n} \mathsf{X}_{12}(a_\alpha) :$ with Coulomb moduli $(a_\alpha)_{\alpha=1,\ldots,n}$, etc.

- A : D0, $\quad \mathsf{X}_{12}$: D4 on $\mathbb{C}_1 \times \mathbb{C}_2$, $\quad \mathsf{X}_{34}$: D4' on $\mathbb{C}_3 \times \mathbb{C}_4$, $\quad \mathsf{Z}$: D8 on $\mathbb{C}^4$

[K, 2212.03870]



D0-D4 system on $\mathbb{C}_1 \times \mathbb{C}_2$

- 4d $\mathcal{N} = 2^*$ U(n) theory on $\mathbb{C}_1 \times \mathbb{C}_2$ (5d $\mathcal{N} = 1^*$ theory on $\mathbb{C}_1 \times \mathbb{C}_2 \times \mathbb{S}^1$)

$$Z_k = \frac{1}{k!} \oint \mathrm{d}^k \phi \, \langle \prod_{I=1}^{k} \mathsf{A}^{-1}(\phi_I) \cdot \mathsf{X}_{12}^{[n]} \rangle$$

$$= \frac{1}{k!} \oint \mathrm{d}^k \phi \prod_{\substack{I=1,\ldots,k \\ \alpha=1,\ldots,n}} \frac{[\phi_I - a_\alpha - \epsilon_3][\phi_I - a_\alpha - \epsilon_4]}{[\phi_I - a_\alpha][\phi_I - a_\alpha + \epsilon_{12}]} \prod_{I<J}^{k} \frac{[\phi_{IJ} \pm \epsilon_{12,23,31}]}{[\phi_{IJ} \pm \epsilon_{1,2,3,4}]} [\phi_{IJ}]^2$$

  - Coulomb moduli : $(a_\alpha)_{\alpha=1,\ldots,n}$, $[x] = 2\sinh(x/2)$, $\phi_{IJ} = \phi_I - \phi_J$
  - $\epsilon_{ij} = \epsilon_i + \epsilon_j$, $\epsilon_3 = -m_{\mathsf{adj}}$, $\epsilon_4 = m_{\mathsf{adj}} - \epsilon_{12}$ ⟶ CY4 condition $\sum_{i=1}^{4} \epsilon_i = 0$

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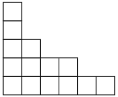
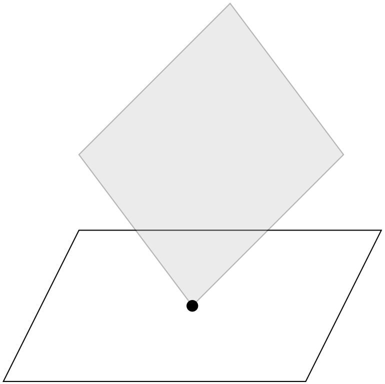
- **LMNS formula**: (JK) residue calculus ⟶ **Nekrasov function**: combinatorics

$$Z_k = \sum_{\lambda \in \{\text{partitions}\}} Z_\lambda \, , \qquad \lambda =$$



D0-D4 system on $\mathbb{C}_1 \times \mathbb{C}_2$

- D4': defect brane (point; co-dim 4) → chiral ring generator $(\mathcal{O}_n)_{n \in \mathbb{N}}$
- D8 : flavor brane



D4' on $\mathbb{C}_3 \times \mathbb{C}_4$

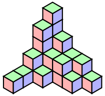
D4 on $\mathbb{C}_1 \times \mathbb{C}_2$

D0-D8(-$\overline{\text{D8}}$) system on $\mathbb{C}^4$

- Magnificent four on $\mathbb{C}^4$ [Nekrasov] [Nekrasov–Piazzalunga]

$$Z_k = \frac{1}{k!} \oint \mathrm{d}^k\phi \, \langle \prod_{I=1}^{k} \mathsf{A}^{-1}(\phi_I) \cdot \mathsf{Z}^{[m|m]} \rangle$$

$$= \frac{1}{k!} \oint \mathrm{d}^k\phi \prod_{\substack{I=1,\ldots,k \\ \alpha=1,\ldots,m}} \frac{[\phi_I - b_\alpha]}{[\phi_I - a_\alpha]} \prod_{I<J}^{k} \frac{[\phi_{IJ} \pm \epsilon_{12,23,31}]}{[\phi_{IJ} \pm \epsilon_{1,2,3,4}]} [\phi_{IJ}]^2 = \sum_{\varpi \in \{\textbf{4d} \text{ partitions}\}} Z_\varpi$$

- Donaldson–Thomas theory of CY4 [Borisov–Joyce] [Cao–Kool] [Oh–Thomas]...
  - CY4 condition : $\sum_{i=1}^{4} \epsilon_i = 0$
  - DT theory of CY3 : $Z_k = \sum_{\pi \in \{\textbf{3d} \text{ partitions}\}} Z_\pi$, $\pi =$ 

Free field realization

- D0 vertex operator : $\mathsf{A}(x) = : \exp\left(\sum_{n \in \mathbb{Z}^\times} a_n x^{-n}\right) :$ (see also [K–Pestun])

where $[a_n, a_m] = -\frac{1}{n}(1 - q_1^n)(1 - q_2^n)(1 - q_3^n)(1 - q_4^n)\, \delta_{n+m,0}\,, \qquad q_i = e^{\epsilon_i}$

$$\longrightarrow \quad \frac{\mathsf{A}(x)\mathsf{A}(x')}{:\mathsf{A}(x)\mathsf{A}(x'):} = \frac{[z - z' \pm \epsilon_{12,23,31}]}{[z - z' \pm \epsilon_{1,2,3,4}]}[z - z']^2\,, \qquad x = e^z$$

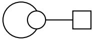
- Orbifold geometry : $\mathbb{C}^4 \longrightarrow \mathbb{C}^4/\Gamma$ with $\Gamma \subset \mathrm{SU}(4)$

Orbifold geometry : $\mathbb{C}^4/\Gamma$ with $\Gamma \subset \mathrm{SU}(4)$

- e.g., $\dfrac{\mathbb{C}_1 \times \mathbb{C}_2}{\Gamma_{12}} \times \dfrac{\mathbb{C}_3 \times \mathbb{C}_4}{\Gamma_{34}}$ where $\Gamma_{12,34} = ADE$-type $\subset \mathrm{SU}(2)$

$$\mathsf{A}_i^j(x) = :\exp\left(\sum_{n\in\mathbb{Z}^\times} a_{i,n}^j x^{-n}\right): \quad \text{where} \quad [a_{i,n}^j, a_{i',m}^{j'}] = -\frac{1}{n} c_{\widehat{\Gamma}_{12},jj'}^{[n]} \, c_{\widehat{\Gamma}_{34},ii'}^{[n]} \, \delta_{n+m,0}$$

- McKay–Nakajima correspondence : $\Gamma = ADE \subset \mathrm{SU}(2) \longrightarrow \widehat{ADE}$ quiver
  - c_Γ : q-Cartan matrix of quiver Γ (geometry $\to$ algebra)
  - $\Gamma = 1 \ (= \mathbb{Z}_1) \longrightarrow \widehat{\Gamma} = \widehat{A}_0$ (Jordan; ADHM quiver)



$$c_{\widehat{A}_0} = 1 + q_1 q_2 - q_1 - q_2 = (1-q_1)(1-q_2)$$

Summary

- Vertex operator creates **D-branes (!)**

$$Z_k = \frac{1}{k!} \oint \mathrm{d}^k\phi \, \langle \prod_{I=1}^{k} \mathsf{A}^{-1}(\phi_I) \cdot \mathsf{X}_{12}^{[n]} \, \mathsf{X}_{34}^{[n']} \, \mathsf{Z}^{[m]} \rangle$$

- Geometry to Algebra : McKay–Nakajima correspondence, q-Cartan matrix

- Future directions :
  - D0-D2-D4-D6-D8 system
  - Underlying (infinite dim) algebraic structure : W-algebra, toroidal algebra...
  - D-instanton?

Thank you for your attention!