Ambient-space variational calculus for gauge fields on constant-curvature spacetimes

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Plan

1. Higher-spin (HS) fields
   - The HS zoo: species, origins and natural habitats
   - Metric-like formulation in flat space: constraints, equations of motion, gauge symmetries
2. Anti-de Sitter (AdS) space embedded in flat ambient space
   - Ambient-space description of the intrinsic AdS geometry
   - Radial reduction of ambient-space fields
3. String-like BRST formulation in flat space
   - Tensionless limit of the bosonic string
   - "Triplet" formulation of HS fields
4. BRST-invariant inner product and codimension-one Lagrangians

Some higher-spin basics

HS basics

HS fields naturally arise as generalisations of the lower-spin particles of the Standard Model:

- spin-0 (scalar) particles (*e.g.*, the Brout-Englert-Higgs boson)
- spin-$\frac{1}{2}$ particles (*e.g.*, electron, neutrino),
- spin-1 particles (*e.g.*, photon, $W^{\pm}$, Z-bosons),

as well as of hypothetical particles arising in (super-)gravity theories:

- spin-$\frac{3}{2}$ gravitino,
- spin-2 graviton.

In particle experiments, short-living massive higher-spin particles are observed as *baryon resonances*.

HS basics

In the [hep-th] world, HS fields are encountered:

- in the spectrum of *string theory*.
  For example, massless higher-spin fields arise in the tensionless limit of the open bosonic string [Bengtsson, Ouvry-Stern'86].
- in the *AdS/CFT correspondence*.
  In particular, conserved currents of the $O(N)$-vector model in a d-dimensional conformally flat space are dual to massless higher-spin particles in the $(d+1)$-dimensional AdS spacetime [Klebanov-Polyakov'02].

HS basics

Metric-like formulation for a symmetric spin-s field in the *Minkowski spacetime* (equations of motion):

- a symmetric rank-s tensor $\phi_{\mu_1\ldots\mu_s}(x)$ subject to the off-shell constraints (*transversality* and *tracelessness*)

$$\partial^\nu \phi_{\nu\mu_1\ldots\mu_{s-1}}(x) = 0\,, \quad \phi^\nu{}_{\nu\mu_1\ldots\mu_{s-2}}(x) = 0\,.$$

- On-shell condition - the Klein-Gordon equation:

$$(\Box + m^2)\,\phi_{\mu_1\ldots\mu_s}(x) = 0\,.$$

- In the massless limit $m^2 \to 0$ some of the degrees of freedom decouple, which signals the emergence of the gauge symmetry

$$\phi_{\mu_1\ldots\mu_s} \sim \phi_{\mu_1\ldots\mu_s} + \partial_{(\mu_1}\varepsilon_{\mu_2\ldots\mu_s)}\,,$$

  with $\partial^\nu\varepsilon_{\nu\mu_2\ldots\mu_{s-1}} = 0$, $\varepsilon^\nu{}_{\nu\mu_3\ldots\mu_{s-1}} = 0$ and $\Box\varepsilon_{\mu_1\ldots\mu_{s-1}} = 0$ (recall Maxwell's electrodynamics in the *Lorentz gauge*).

HS basics

Metric-like formulation for a symmetric spin-s field in the *Minkowski spacetime* (Lagrangian formulation) `[Singh-Hagen'74,Fronsdal'78]`:

- a multiplet of traceless symmetric tensors

$$\{\phi_{\mu(s)}(x), \phi_{\mu(s-2)}(x), \phi_{\mu(s-3)}(x), \ldots, \phi\}$$

- In the massless limit $m^2 \to 0$ some of the degrees of freedom decouple, leaving one with a pair of traceless fields

$$\{\phi_{\mu(s)}(x), \phi_{\mu(s-2)}(x)\} \quad \Leftrightarrow \quad \bar{\phi}^{\lambda\nu}{}_{\lambda\nu\,\mu(s)}(x) = 0\,.$$

Emergence of the gauge symmetry

$$\phi_{\mu(s)}(x) \sim \phi_{\mu(s)}(x) + \partial_\mu \varepsilon_{\mu(s-1)}(x)\,.$$

with $\varepsilon^{\nu}{}_{\nu\mu(s-2)}(x) = 0$.

Ambient-space description of AdS fields

Ambient space

The $(d+1)$-dimensional AdS spacetime AdS_{d+1} can be realised as a quadric in the flat *ambient space* $\mathbb{R}^{2,d}$ parametrised by the Cartesian coordinates X^A $(A = 0', 0, 1, \dots, d)$:

$$-r^2 := -(X^{0'})^2 - (X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 = -\ell^2 \,.$$

The region $r > 0$ is $\mathbb{R}^{2,d}_+ \cong AdS_{d+1} \times \mathbb{R}_+$.

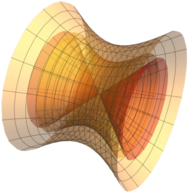


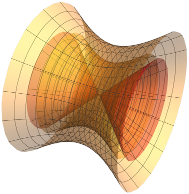
Figure 1: Foliation of the open region $r > 0$ in $\mathbb{R}^{2,d}$ by AdS_{d+1} (the light cone $r = 0$ pictured in red)

Ambient space

Intrinsic AdS geometry is induced from the ambient-space. For example, the volume form on AdS_{d+1}:

- Radial vector field $X \cdot \partial = r\partial_r$ is orthogonal to AdS_{d+1}.
- Adapted coordinates (r, x^μ) $(\mu = 0, 1, \ldots, d)$.
- From the volume form $\mathcal{V} = \mathrm{d}^{d+2}X \in \Omega^{d+2}(\mathbb{R}^{2,d})$ one constructs

$$\mathcal{V}_X = i_X \mathcal{V} = r^{d+2}\sqrt{-g}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^d \in \Omega^{d+1}(\mathbb{R}^{2,d})\,.$$



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Radially-constrained fields in the ambient space

Fields $\Phi(X)$ in $\mathbb{R}^{2,d}_{+}$ can be used to parametrise fields $\phi(x)$ in AdS_{d+1}.

- Fixing radial behaviour of $\Phi(X)$ by imposing the *radial constraint*:

$$\big(X^A\frac{\partial}{\partial X^A}+\Delta\big)\,\Phi(X)=0\,,\quad \Delta\in\mathbb{R}\,.$$

  By virtue of $X^A\frac{\partial}{\partial X^A}=r\frac{\partial}{\partial r}$, one can resolve the above constraint

$$\Phi(X)=r^{-\Delta}\,\phi(x)\,.$$

- The ambient d'Alembertian decomposes into its radial and "angular" parts. For fields subject to the radial constraint:

$$\Box\Phi(X)=r^{-\Delta}\,\big(\nabla^2+\frac{\Delta(d-\Delta)}{r^2}\big)\phi(x)$$

  Radial dimensional reduction: *massless ambient field + radial constraint = AdS field with mass $m(\Delta)$.*

Radially-constrained fields in the ambient space

A simple example: spherical harmonics $Y_{\ell,m}(\theta,\varphi)$.

- Eigenfunctions of the Laplace operator on S^2,

$$\Delta_{S^2} Y_{\ell,m} = -\ell(\ell+1) Y_{\ell,m}\,.$$

- The "ambient space" $\mathbb{R}^3\backslash\{0\} \cong S^2 \times \mathbb{R}_+$ with the radial variable $r^2 = x^2 + y^2 + z^2$. Polynomial functions $\Phi(x,y,z)$ subject to the radial constraint
- Harmonic polynomials subject to the radial constraint:

$$\Delta_{\mathbb{R}^3}\Phi = 0\,,\;\; r\partial_r \Phi = \ell\,\Phi \quad \Leftrightarrow \quad \Phi = \sum_{m=-\ell}^{\ell} c_m\, r^\ell Y_{\ell,m}(\theta,\varphi)$$

- Correspondence:

$$S^2 \to AdS_{d+1}\,,\;\; \mathbb{R}^3\backslash\{0\} \to \mathbb{R}^{2,d}_+\,,\;\; SO(3) \to SO(2,d)\,.$$

String-like BRST description of HS fields in the flat ambient space

BRST formulation of strings

String theories in flat space admit a BRST formulation
`[Kato-Ogawa'83, Witten'85, Ohta'86]`

- String oscillators α_m^μ with $m \in \mathbb{Z}$ and the conjugation rule $(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu$

$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}\,, \quad \alpha_m^\mu\,|0\rangle = 0 \ \text{ for } \ m \geqslant 1\,,$$

  the ghosts c_m and anti-ghosts b_m with $m \in \mathbb{Z}$

$$\{c_m, b_n\} = \delta_{m+n,0}\,,$$

$$b_0\,|0\rangle = 0\,, \ \ b_m\,|0\rangle = 0\,, \ \ c_m\,|0\rangle = 0 \ \text{ for } \ m \geqslant 1\,.$$

- The string field $|\phi\rangle$ is spanned over the basis generated from $|0\rangle$, with coefficients $\phi_{\mu_1\dots\mu_k}(x)$.
- The nilpotent BRST operator Q built from α_m^μ, c_m, b_m singles out physical states, with $\text{gh}(|\phi\rangle) = 0$, $\text{gh}(|\epsilon\rangle) = -1$:

$$Q\,|\phi\rangle = 0\,, \quad |\phi\rangle \sim |\phi\rangle + Q\,|\epsilon\rangle\ .$$

String-like BRST description of ambient HS fields

Symmetric HS fields *in flat spacetimes* via the tensionless limit of an open bosonic string [Bengtsson'86, Sagnotti-Tsulaia'03] and the related BRST formulation [Barnich-Grigoriev-Semikhatov-Tipunin'04]:

- We consider BRST description of strings on $\mathbb{R}^{2,d}$, and concentrate on the ghost-degree-0 "triplets"

$$
\begin{aligned}
|\Phi_s\rangle = \quad & \tfrac{1}{s!}\, B_{A_1\ldots A_s}(X)\, \alpha_{-1}^{A_1}\ldots\alpha_{-1}^{A_s}\,|0\rangle \\
& + \tfrac{1}{(s-1)!}\, C_{A_1\ldots A_{s-1}}(X)\, \alpha_{-1}^{A_1}\ldots\alpha_{-1}^{A_{s-1}}\, c_0 b_{-1}\,|0\rangle \\
& + \tfrac{1}{(s-2)!}\, D_{A_1\ldots A_{s-2}}(X)\, \alpha_{-1}^{A_1}\ldots\alpha_{-1}^{A_{s-2}}\, c_{-1} b_{-1}\,|0\rangle
\end{aligned}
$$

- Upon appropriate rescaling of the oscillators one can take the tensionless limit $\alpha' \to 0$, which leads to the following nilpotent BRST operator on the space of "triplets":

$$
Q_0 = c_0 \Box + c_{-1}\, \alpha_{+1}^{A} \frac{\partial}{\partial X^A} - c_{+1} \alpha_{-1}^{A} \frac{\partial}{\partial X^A} + \text{cub. term}\,.
$$

String-like BRST description of ambient HS fields

The string-like BRST description of HS fields in *flat space* admits a natural Lagrangian formulation: for the "triplet" $|\Phi_s\rangle$ and the BRST operator

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one has the quadratic Lagrangian

$$L_s = \langle \Phi_s | \, Q_0 \, | \Phi_s \rangle \ .$$

Question: Describe radial reduction of the string-like BRST Lagrangian formulation in the ambient space.

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Question: Describe radial reduction of the string-like BRST Lagrangian formulation in the ambient space.

Motivation$_1$: Avoid working with the intrinsic AdS geometry (see `[Sagnotti-Tsulaia’03, Buchbinder-Krykhtin-Lavrov’06]`). Ambient flat space is technically simpler than AdS.
Motivation$_2$: Constructing yet unknown AdS Lagrangians in the metric-like formulation for some “exotic” HS fields (*e.g.*, partially massless fields).

Dimensional radial reduction: possible issues

Hidden traps:

- *Mismatch of dimensions*: AdS_{d+1} is a codimension-1 submanifold in $\mathbb{R}^{2,d}_{+}$. Suitable Lagrangians are $(d+1)$-forms on the $(d+2)$-dimensional ambient space:

  $$\mathbb{L}[\Phi_s] = L[\Phi_s]\,\mathcal{V}_X \in \Omega^{d+1}(\mathbb{R}^{2,d}_{+})\,, \quad \mathbb{L}[\Phi_s] \sim \mathbb{L}[\Phi_s] + \mathrm{d}H\,,$$

  where $H \in \Omega^{d}(\mathbb{R}^{2,d}_{+})$.

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  where $H \in \Omega^{d}(\mathbb{R}^{2,d}_{+})$.

- What is the *variational principle* in codimension-1?

  For example, an analog of the *fundamental lemma of calculus of variations* for constrained fields:

$$\int \delta\Phi\, F[\Phi]\,\mathcal{V}_X = 0 \quad \Rightarrow \quad F[\Phi] = ?$$

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  where $H \in \Omega^d(\mathbb{R}^{2,d}_+)$.
- What is the *variational principle* in codimension-1?
- in order to obtain the correct equations of motion $Q_0\Phi_s = 0$ from the Lagrangian

$$\mathbb{L}[\Phi_s] = \langle \Phi_s, Q_0\Phi_s \rangle\ \mathcal{V}_X\,,$$

  one needs $Q_0 = Q_0^\dagger$ with respect to the *inner product* $\langle\cdot,\cdot\rangle\,\mathcal{V}_X$. This is *not* the case when radial constraint is imposed on Φ_s.

Dimensional radial reduction: possible issues

Example: consider $\psi(t, y)$ subject to the constraint $(\partial_y + w)\psi(t, y) = 0$, *i.e.* $\psi(t, y) = h(t)e^{-wy}$. Take the inner product

$$\big(\psi, \chi\big) = \psi(t, y)\, \chi(t, y)\, \mathrm{d}t$$

(note that usually one has $\mathrm{d}t\, \mathrm{d}y$).

- Then since ∂_y acts algebraically, one has $\partial_y^\dagger = \partial_y = -w$ (instead of the usual conjugation rule for partial derivatives $\partial_y^\dagger = -\partial_y$). Hence, if an operator is self-adjoint in the usual case, and contains first-order derivatives ∂_y, it is *not* self-adjoint with respect to $\big(\cdot, \cdot\big)$.

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- To cure the problem, one deforms the inner product:

$$\big(\psi,\chi\big)_w = \big(e^{2w\,y}\psi,\chi\big)\,,$$

such that $\partial_y^\dagger = -\partial_y$.

Dimensional radial reduction: the deformed inner product

There exists a unique $\Omega^{d+1}(\mathbb{R}^{2,d})$-valued pairing of "triplets"

$$\big(\Psi_s, \Phi_s\big)_K = \langle K\Psi_s, \Phi_s\rangle\, \mathcal{V}_X\,,$$

such that $Q_0^\dagger = Q_0$.

- Radial decomposition of ambient oscillators α^A_{-1}:

  $$\rho_{-1} = r^{-1}\, X_A \alpha^A_{-1} \ \ \textit{(the radial oscillator)}, \quad \alpha^\mu_{-1}\,.$$

  Radial decomposition of fields (with respect to the degree in the radial oscillator):

  $$\Phi_s = \bigoplus_{m=0}^{s} \Phi_s^{(m)}\,.$$

- The deformation operator

  $$K = r^{-(d-2\Delta)} U \mathfrak{D} U^{-1}\,, \ \text{ where } \ U = 1 + \frac{2}{r}\, c_0\, (b_{-1}\rho_{+1} + b_{+1}\rho_{-1})$$

  And $\mathfrak{D}$ is diagonal on the radial decomposition.

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- *Radial decomposition* of fields (with respect to the degree in the radial oscillator):

$$\Phi_s = \bigoplus_{m=0}^{s} \Phi_s^{(m)}\,.$$

- The deformation operator $K = r^{-(d-2\Delta)}\, U\mathfrak{D}U^{-1}$, with $\mathfrak{D}\Phi_s^{(m)} = \nu_\Delta^{(s|n)}\, \Phi_s^{(n)}$ and

$$\nu_\Delta^{(s|n)} = \frac{\big[\frac{d}{2}+s-2\big]_n}{\big[\Delta+s-2\big]_n}\; {}_2F_1\left(\begin{matrix}-n,\ \frac{d}{2}-\Delta\\ \frac{d}{2}+s-1-n\end{matrix}\,;\,-1\right)\,,$$

where $[x]_n = x(x-1)\dots(x-n+1)$ denotes the falling Pochhammer symbol.

Summary

1. A uniform Lagrangian description of free symmetric HS fields on AdS_{d+1}, massless and (partially) massless, is available via the radial dimensional reduction of the string-like BRST description of massless symmetric HS fields on the *flat* ambient space $\mathbb{R}^{2,d}$:
   - Lagrangians are $(d+1)$-forms $\mathbb{L}[\Phi] = L[\Phi]\,\mathcal{V}_X$, which belong to a particular subset in $\Omega^{d+1}(\mathbb{R}^{2,d}_{+})$.
   - Pullback of $\mathbb{L}[\Phi]$ to AdS_{d+1} in the case of partially massless fields leads to yet unknown Lagrangians in the metric-like formulation.
2. An adapted ambient (codimension-one) variational calculus is developed in terms of jet bundles.
   - The formalism is suitable for *generalised* ambient spaces $\mathcal{A}$ with a volume form $\mathcal{V}$, which are trivial line bundles over a codimension-one manifold $\Sigma \hookrightarrow \mathcal{A}$. In particular, neither *metric* nor *flatness* are assumed for $\mathcal{A}$.
   - Variational derivative is given by a simple explicit formula: one can vary ambient Lagrangians *like if* ambient fields were *unconstrained*.

Thank you for your attention.