Residual gauge symmetries in the front form

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Two lessons from Dirac

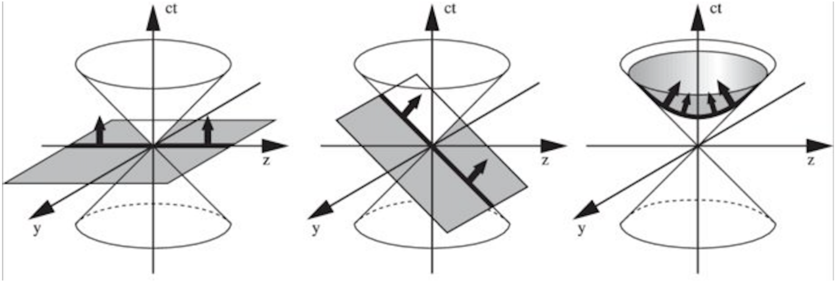
Lesson I: Forms of relativistic dynamics [Dirac 1959]

Three choices of “time” for Hamiltonian dynamics of relativistic systems

Instant form: $t = x^0$

Front form: : $x^+ = (x^0 + x^3)/\sqrt{2}$

Point form: $\tau =$ proper time



Spacelike foliations

Null foliations

Hyperbolic foliations

Two lessons from Dirac

Lesson II: Constrained Hamiltonian systems [Bargmann 1959; Dirac 1959]

Hamiltonian formulation for gauge systems such as electromagnetism, Yang-Mills, gravity, ...

$$\mathcal{S}_H[\phi, \pi_\phi, \lambda_i] = \int dt \int d^3x \left(\pi_\phi \dot{\phi} - \mathcal{H} - \lambda_i \mathcal{G}^i \right)$$

$\mathcal{G}^i \rightarrow$ gauge cosntraints, $\lambda_i \rightarrow$ Largrange multipliers

- algorithm for classifying gauge constraints (primary, first-class, ...)
- symmetries generated by first-class constraints that commute with the Hamiltonian
- precursor to canonical quantization for gauge theories

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The usual route: Instant form + Constrained Hamiltonian systems

BRST quantization, Duality-invariant formulations, *Asymptotic symmetries at spatial infinity*, . . .

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This talk: Front form + Constrained Hamiltonian systems

- Gauge constraint in the front form are often solvable
- Provides a unique framework for studying symmetries of null hypersurfaces

More precisely, the focus of this talk

Gauge theories in **light-cone coordinates** and **light-cone gauge**

(front form) *(constr. Ham. systems)*

Many successes of light-cone physics

- Light-cone formulation of QCD, Discrete light-cone quantization (DLCQ)
- Light-cone gauge quantization of strings
- Proof of UV finiteness of $\mathcal{N} = 4$ SYM
- Links to on-shell methods: spinor helicity formalism, KLT relations, etc.

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A brief outline

- Electromagnetism: Hamiltonian formulation in front form
- Residual or large gauge transformations
- Gravity in the light-cone gauge and BMS symmetry
- Comparison with instant form results

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Warning: Non-covariant and ugly!

Poincarè in the front form

Light-cone coordinates

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}\,, \quad x^- = \frac{x^0 - x^3}{\sqrt{2}}\,, \quad x^i \quad (i = 1, 2)$$

x^+ Light-cone time $\Rightarrow$ $P_+ = i\partial_+ = -P^-$ (Hamiltonian)

Generators of Poincaré algebra

- In the instant form: $(P_\mu, M_{\mu\nu})$

$$[P, P] \sim 0\,, \quad [P, M] \sim P\,, \quad [M, M] \sim M$$

$(P^0, M^{01}, M^{02}, M^{03}) \rightarrow$ four dynamical generators or "Hamiltonians"

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- In the front form

Kinematical $\mathbb{K} = \{P^i, P^+, M^{ij}, M^{+-}\}\,, \quad (i = 1, 2)$

Dynamical $\mathbb{D} = \{P^-, M^{i-} \equiv \underbrace{J^-, \bar{J}^-}_{\text{2 boosts}}\} \rightarrow$ three "Hamiltonians" in the front form

$$[\mathbb{K}, \mathbb{K}] \sim \mathbb{K}\,, \quad [\mathbb{K}, \mathbb{D}] \sim \mathbb{D}\,, \quad [\mathbb{D}, \mathbb{D}] \sim 0$$

Electromagnetism in the front form

- Light-cone gauge

$$A_- = -A^+ = -\frac{A^0 + A^3}{\sqrt{2}} = 0$$

- Maxwell equations: $\partial_\mu F^{\mu\nu} = 0$

a) Constraint

$$(\nu = +): \quad \partial_-^2 A^- + \partial_i \partial_- A^i = 0 \quad \Rightarrow \quad A^- = -\frac{\partial_i A^i}{\partial_-} + \alpha(x^+, x^i)\, x^- + \beta(x^+, x^i)$$

b) Trivial equation

$$(\nu = -): \quad \text{relates } \alpha \text{ and } \beta \quad \Rightarrow \quad \text{only one arbitrary constant}$$

A further choice: set the constants to zero

c) Dynamical equation

$$(\nu = i): \ (2\partial_- \partial_+ - \partial_i \partial^i) A^j = \Box_{lc} A^j = 0 \ \Rightarrow \ \text{two propagating modes of the photon}$$

The “inverse derivative” operator [Mandelstam ’83, Leibbrandt ’83]

$$\partial_- f(x^-) = g(x^-) \quad \Rightarrow \quad f(x^-) = \frac{1}{\partial_-} g(x^-) = -\int \epsilon(x^- - y^-)\, g(y^-)\, dy^- + \text{“} constant \text{''}$$

Electromagnetism in the front form

- Complexify the x^i

$$x = \frac{x^1 + ix^2}{\sqrt{2}}\,, \quad \bar{x} = \frac{x^1 - ix^2}{\sqrt{2}} \quad \partial_i \to (\partial, \bar{\partial})$$

$$A^i \to (A, \bar{A}) : \quad \pm 1 \text{ helicity states of the photon}$$

- Light-cone action for electromagnetism

$$\mathcal{S} = \frac{1}{2}\int d^4x\, \bar{A}\,\Box_{lc}\, A = \int d^4x\;\bar{A}\,(\partial_+\partial_- - \partial\bar{\partial})\,A$$

$\rightarrow$ lc_2 formalism of electromagnetism

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$\rightarrow$ lc_2 formalism of electromagnetism

- Hamiltonian and Poisson brackets (recall: x^+ is time)

$$\pi \;=\; \frac{\delta\mathcal{L}}{\delta(\partial_+ A)} = -\,\partial_-\bar{A}\,, \quad \bar{\pi} \;=\; \frac{\delta\mathcal{L}}{\delta(\partial_+ \bar{A})} = -\,\partial_- A$$

$(\pi, \bar{\pi})$ not independent variables $\Rightarrow$ Half the d.o.f than in the 3+1 formalism

$\rightarrow$ a feature of *all* null-front Hamiltonian systems

- Poisson brackets

$$[A(x), \bar{A}(y)] \;=\; \epsilon(x^- - y^-)\,\delta^{(2)}(x-y)\,, \quad [\,A(x), A(y)\,] \;=\; [\,\bar{A}(x), \bar{A}(y)\,] \;=\; 0\,.$$

Residual gauge transformations

Symmetries in light-cone formulation

- Canonical transformation in the phase space: $(A, \bar{A}) \xrightarrow{\delta_X} (\tilde{A}, \tilde{\bar{A}})$
- *Strict* invariance of action: $\delta_X S[A, \bar{A}] = 0$
- Transformation $=$ Poisson bracket with a generator $G_X[A, \bar{A}]$,

$$\delta_X A = [\, A, G_x \,]_{PB}$$

Is there any residual gauge freedom, $A_\mu(x) \to A_\mu(x) + \partial_\mu \varepsilon(x)$, left?

- All $\varepsilon(x)$ that respects the light-cone gauge choice: $A_- = 0$

$$\partial_- \varepsilon(x) = 0 \quad \Rightarrow \quad \varepsilon = \varepsilon(x^+, x, \bar{x})$$

- But invariance of the light-cone action demands

$$\partial \bar{\partial} \varepsilon(x) = 0 \quad \Rightarrow \quad \varepsilon(x) = f(x) + \bar{f}(\bar{x})$$

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How can we recover all the residual gauge transformations?

Resolution

- Put back the integration constants (zero modes)

$$A^- = -\frac{\partial_I A^I}{\partial_-} + \alpha(x^+, x, \bar{x})x^- + \beta(x^+, x, \bar{x})\,;$$

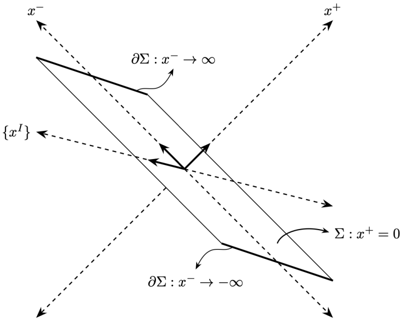
$$\triangle\beta = \partial_+\alpha\,; \quad \triangle = 2\partial\bar{\partial}$$

- Involves relaxing the boundary conditions

$$A^I = \frac{A^I_{(0)}}{x^-} + \frac{A^I_{(1)}}{(x^-)^2} + \dots\,; \quad A^I = (A, \bar{A})$$

$$\downarrow$$

$$A^I = \partial^I\Phi + \frac{A^I_{(0)}}{(x^-)} + \frac{A^I_{(1)}}{(x^-)^2} + \dots$$



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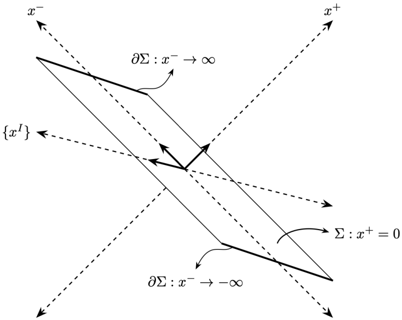

Modified light-cone action

$$\mathcal{S}[A, \bar{A}, \Phi] = \int dx^+ \left\{ \int_\Sigma d^3x\, \bar{A}\,(\partial_+\partial_- - \partial\bar{\partial})\,A - \int_{\partial\Sigma} dx\, d\bar{x}\; \dot{\Phi}\triangle\Phi \right\}$$

Phase space extended to include the boundary d.o.f. $\Phi \to$ a.k.a. lc_4 formalism

[SM, arXiv:2212.10637]

Residual gauge symmetries

- Canonical generator of residual gauge transformations

$$G[\varepsilon] = \int_{\Sigma} dx^- dx d\bar{x}\, \partial_- A_I \partial^I \varepsilon + \int_{\partial\Sigma} dx d\bar{x}\, \triangle \Phi \varepsilon\,,$$

- Light-cone fields transform as

$$\begin{aligned} \{A^I(x)\,, G[\varepsilon]\,\} &= \partial^I \varepsilon(x)\,, \\ \{\Phi(x)\,, G[\varepsilon]\,\} &= \varepsilon(x) \end{aligned}$$

- Complete set of all residual $U(1)$ transformations

  a) Proper GTs: Zero surface charge $\triangle\varepsilon = 0$,

  b) Improper (or large) GTs : Non-vanishing surface charge $\triangle\varepsilon \neq 0$

Residual gauge symmetries

- Canonical generator of residual gauge transformations

$$G[\varepsilon] = \int_{\Sigma} dx^- dx d\bar{x}\, \partial_- A_l \partial^l \varepsilon + \int_{\partial\Sigma} dx d\bar{x}\, \triangle \Phi \varepsilon\,,$$

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Electromagnetism in the front form: Observations

- Putting α, β to zero amounts to residual gauge fixing
- Zero modes are a crucial part of the initial data set
- Going from lc_2 formalism to lc_4 involves improper (or large) gauge transformations

[SM, arXiv:2212.10637]

Residual gauge symmetries $\longleftrightarrow$ Asymptotic symmetries

How are these symmetries related to asymptotic $U(1)$ symmetry in EM?

Consider:

Asymptotic analysis at spatial infinity

Asymptotic analysis of EM [Henneaux-Troessaert '18]

- Hamiltonian action

$$\mathcal{S}[A_i, \pi^i, A_0] \;=\; \int dt \left\{ \int d^3x\, \pi^i \dot{A}_i - \int d^3x \left(\frac{1}{2}\pi^i \pi_i + \frac{1}{4} F^{ij} F_{ij} + A_0 \mathcal{G} \right) + B_\infty \right\}$$

Gauss constraint, $\mathcal{G} \;=\; \partial_i \pi^i \approx 0$

- Fall-off conditions:

$$A_i \;=\; \frac{1}{r}\overline{A}_i + \mathcal{O}(r^{-2})\,, \quad \pi^i \;=\; \frac{1}{r^2}\overline{\pi}^i + \mathcal{O}(r^3)$$

(Gauge-twisted) parity conditions

$$\begin{aligned} \overline{A}_r \;&=\; (\overline{A}_r)^{odd}\,, \qquad & \overline{A}_B \;&=\; (\overline{A}_B)^{even} + \partial_B \Phi\,, \quad \Phi = \text{even} \\ \overline{\pi}^r \;&=\; (\overline{\pi}^r)^{even}\,, \qquad & \overline{\pi}^A \;&=\; (\overline{\pi}^A)^{odd} \end{aligned}$$

Must introduce boundary d.o.f. Ψ through B_∞ to restore invariance under Lorentz boosts

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- Canonical generator for large gauge symmetries

$$G_{\epsilon,\mu}[A_i,\Psi,\pi^i] = \int d^3x\, \epsilon\,\mathcal{G} + \oint d^2x\,(\overline{\epsilon}\,\overline{\pi}^r - \sqrt{\overline{\gamma}}\,\overline{\mu}\overline{A}_r)$$

a) Gauge symmetry: surface charge $= 0 \rightarrow$ Proper
b)True symmetry: surface charge $\neq 0 \rightarrow$ Improper

Does (2+2) equal (3+1)?

(3+1): Asymptotic symmetries at spatial infinity

- Symmetry $\equiv$ invariance of symplectic form or Hamiltonian action
- Boundary value problem on a Cauchy hypersurface
- Spin 1: Must include a surface dof $\overline{\Psi}$ to obtain full U(1) gauge symmetries
  Setting $\overline{\Psi}$ to zero amounts to improper gauge fixing

(2+2): Residual gauge symmetries in light-cone formulation

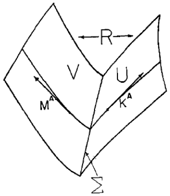
- Symmetry $\equiv$ invariance of light-cone action
- Characteristic initial value problem on a null hypersurface
- Spin 1: Must include the zero mode Φ to obtain all residual gauge symmetries
  Setting Φ to zero amounts to residual gauge fixing

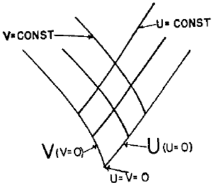
Gravity in the (2+2) formulation

(2+2) or “double-null” formulation of gravity

“On the characteristic initial value problem in gravitational theory” [R. K. Sachs ’62]

“Covariant 2+2 formulation of the initial-value problem in general relativity”
[d’Inverno and Smallwood ’79]
[Gambini-Restuccia, Nagarajan-Goldberg, C. Torre, M. Kaku, S. Hayward...]





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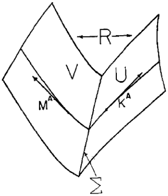

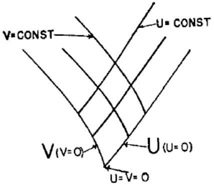

- Spacelike foliation of codim 2 (instead of 1)
- Gravitational d.o.f. identified with the "conformal two-metric"

Our focus

A particular example of 2+2 formulation of gravity: lc_2 gravity [Scherk-Schwarz' 75]

Light-cone gravity à la Scherk-Schwartz

- Light-cone gauge: Set the "minus" components to zero

$$g_{--} = g_{-i} = 0, \quad (i = 1, 2)$$

$10 - 3 = 7$

Parametrization

$$g_{+-} = -e^{\phi}, \quad g_{ij} = e^{\psi}\gamma_{ij}$$

ϕ,ψ,γ_{ij} are real and det $\gamma_{ij} = 1$

Light-cone metric

$$dS^2_{LC} = g_{\mu\nu}dx^{\mu}dx^{\nu} = -2e^{\phi}dx^{+}dx^{+} + g_{++}(dx^{+})^2 + g_{+i}dx^{+}dx^{i} + e^{\psi}\gamma_{ij}dx^{i}dx^{j}$$

given in terms of 7 functions $\{\phi, \psi, \gamma_{ij}, g_{++}, g_{+i}\}$

- "2+2" split of the Einstein field equations $R_{\mu\nu} = 0$ [Sachs, d'Inverno-Smallwood, ...]

  Dynamical equations: $R_{ij} = 0$

  Constraint equations: $R_{--} = R_{-i} = 0$

  Subsidiary equations: $R_{++} = R_{+i} = 0$

  Trivial equations: $R_{+-} = 0$

Gravity in the light-cone gauge

Light-cone metric

$$dS^2_{LC} = g_{\mu\nu} dx^\mu dx^\nu = -2e^{\phi} dx^+ dx^+ + g_{++}(dx^+)^2 + g_{+i} dx^+ dx^i + e^{\psi} \gamma_{ij} dx^i dx^j$$

given in terms of 7 functions $\{\phi, \psi, \gamma_{ij}, g_{++}, g_{+i}\}$

- *Fourth* gauge choice

$$\phi = \frac{\psi}{2}$$

- Constraint equation $R_{--} = 0$ allows us to integrate † out ψ

$$\psi = \frac{1}{4} \frac{1}{{\partial_-}^2} (\partial_- \gamma^{ij} \, \partial_- \gamma^{ij})$$

- Solve rest of the constraints to express the Einstein-Hilbert action as $\mathcal{S}[\gamma_{ij}]$
- Gravitational d.o.f. identified with the "conformal two-metric" γ_{ij}

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† All integration constants set to zero assuming asymptotically flat boundary conditions

Light-cone action for gravity

- Expand Einstein-Hilbert action perturbatively

$$\gamma_{ij} \;=\; (e^{\kappa H})_{ij}\;, \quad H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}; \quad h_{22} \;=\; -h_{11}$$

Complexify

$$h \;=\; \frac{1}{\sqrt{2}}\,(h_{11} + i\,h_{12})\;, \quad \bar{h} \;=\; \frac{1}{\sqrt{2}}\,(h_{11} - i\,h_{12})$$

- Light-cone Lagrangian

$$\mathcal{L} = \frac{1}{2}\bar{h}\,\Box\,h \;+\; 2\kappa\,\bar{h}\,\partial_-{}^2\left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h - h\frac{\bar{\partial}^2}{\partial_-{}^2}h\right) \;+ c.c. \;+\; \text{higher order terms}$$

h and $\bar{h}$ represent gravitons of helicity +2 and -2 respectively

- Poisson brackets

$$[h(x),\bar{h}(y)] \;=\; \epsilon(x^- - y^-)\,\delta^{(2)}(x-y)\,, \quad [\,h(x),h(y)\,] \;=\; [\,\bar{h}(x),\bar{h}(y)\,] \;=\; 0\,.$$

[Scherk-Schwarz' 75, Bengtsson-Cederwall-Lindgren '83]

BMS symmetry from residual gauge invariance

Is there any residual reparameterization freedom $x^\mu \to x^\mu + \xi^\mu$ left?

- Light-cone action for gravity

$$\mathcal{S}[h,\bar{h}] = \int d^4x \left\{ \frac{1}{2}\bar{h}\,\Box\, h + 2\kappa\,\bar{h}\,\partial_-{}^2 \left(\frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h - h\frac{\bar{\partial}^2}{\partial_-{}^2}h \right) + c.c. + \cdots \right\}$$

- Residual reparameterizations

$$\begin{aligned}
\xi^+ &= f = \frac{1}{2}x^+\partial_i Y^i + T(x^k) \\
\xi^i &= -\partial_k f \frac{1}{\partial_-}(g_{-+}g^{ik}) + Y^i(x^k) \\
\xi^- &= -\partial_i Y^i x^- + (\partial_+\xi_i)x^i
\end{aligned}$$

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\xi^i &= -\partial_k f \frac{1}{\partial_-}(g_{-+}g^{ik}) + Y^i(x^k) \\
\xi^- &= -\partial_i Y^i x^- + (\partial_+ \xi_i) x^i
\end{aligned}$$

- The light-cone gravity action invariant

$$\delta_\xi \mathcal{S}[h,\bar{h}] = 0$$

iff $\partial^2 Y = 0 \Rightarrow$ only Lorentz rotations (**no superrotations**)

Only one arbitrary constant: $T(x^i)$

[Ananth, Brink and SM; arXiv:2012.07880 and 2101.00019]

Light-cone representation of the BMS algebra

- Light-cone Poincaré algebra

$$\mathbb{K}: \quad \{P, \bar{P}, P^+, J^{12}, J^+, \bar{J}^+, J^{+-}\}$$
$$\mathbb{D}: \quad \{P^- \equiv H, J^-, \bar{J}^-\}$$

$$[\,\mathbb{K}, \mathbb{K}\,] \;=\; \mathbb{K}\,, \quad [\,\mathbb{K}, \mathbb{D}\,] \;=\; \mathbb{D}\,, \quad [\,\mathbb{D}, \mathbb{D}\,] \;=\; 0\,.$$

- Light-cone BMS algebra

$$\mathbb{K} \;\rightarrow\; \mathbb{K}\,,$$
$$\mathbb{D} \;\rightarrow\; \mathbb{D}(T)\,,$$

$$[\,\mathbb{K}, \mathbb{K}\,] \;=\; \mathbb{K}\,, \quad [\,\mathbb{K}, \mathbb{D}(T)\,] \;=\; \mathbb{D}(T)\,, \quad [\,\mathbb{D}(T), \mathbb{D}(T)\,] \;=\; 0\,.$$

Dynamical part enhanced to infinite-dim supertranslations labeled by a single parameter

- Poincaré part of the BMS

$$\partial^2 T = \bar{\partial}^2 T = 0$$

$\Rightarrow \mathbb{D}(T)$ reduces to $\mathbb{D}: \; \{H, J^-, \bar{J}^-\} \quad \rightarrow$ the three "Hamiltonians" of Dirac

[Ananth, Brink and SM; arXiv:2012.07880 and 2101.00019]

Does (2+2) equal (3+1)?

(3+1): Asymptotic symmetries at spatial infinity

- Symmetry $\equiv$ invariance of symplectic form or Hamiltonian action
- Boundary value problem on a Cauchy hypersurface
- Spin 1: Must include a surface dof $\overline{\Psi}$ to obtain full U(1) gauge symmetries
  Setting $\overline{\Psi}$ to zero amounts to improper gauge fixing
- Spin 2: Supertranslations obtained without any extra surface degrees of freedom

[Henneaux-Troessaert '18]

(2+2): Residual gauge symmetries in light-cone formulation

- Symmetry $\equiv$ invariance of light-cone action
- Characteristic initial value problem on a null hypersurface
- Spin 1: Must include the zero mode α to obtain all residual gauge symmetries
  Setting α to zero amounts to residual gauge fixing
- Spin 2: Supertranslations obtained without reintroducing the zero modes

[Ananth, Brink and SM]

Some concluding remarks...

Connections with amplitudes

- Action in terms of helicity states - closer to on-shell physics
- Various applications- MHV Lagrangians , KLT relations , Double copy methods

[Gorsky-Rosly, Ananth-Theisen, Ananth-Kovacs-Parikh, ...]

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Self-dual, Anti self-dual and all that

- Closely related to Chalmers-Seigel action, double copy construction for SD sectors

  [Campiglia-Nagy '21]
- Double copy for BMS symmetries [work in progress]
- Newmann-Penrose formalism [work in progress], Weyl double copy, ...

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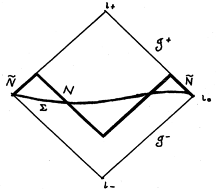
Formal (2+2) Hamiltonian analysis

- Initial (boundary) value problem
- Role of gauge constraints, zero modes, etc. [work in progress]
- Dictionary between residual gauge symmetries in (2+2) with asymptotic symmetries

Stay tuned...

Initial value problem in the front-from

- How does it compare with the initial value problem in the instant form?
- What is the equivalent of Cauchy hypersurfaces in the front form?
- Can we quantize the theory on a single front?



[Nagarajan-Goldberg '85]

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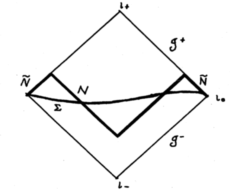

[Nagarajan-Goldberg '85]

Work in progress with

Glenn Barnich, Bruxelles

Simone Speziale, Marseille

Wen-Di Tan, Bruxelles

Thank you!

APPENDIX

Light-cone Hamiltonian for gravity

- Closed form expression

$$S[\gamma_{ij}] \;=\; \frac{1}{2\kappa^2}\int d^4x\, e^{\psi}\left(2\,\partial_+\partial_-\phi \,+\, \partial_+\partial_-\psi - \frac{1}{2}\,\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) - \frac{1}{2}\,e^{\phi-2\psi}\gamma^{ij}\,\frac{1}{\partial_-}R_i\,\frac{1}{\partial_-}R_j$$

$$-e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right)$$

where

$$R_i \;\equiv\; e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\partial_-\psi\right) + \partial_k(e^{\psi}\,\gamma^{jk}\partial_-\gamma_{ij})$$

- Conjugate momenta

$$\pi \;=\; \frac{\delta\mathcal{L}}{\delta(\partial_+ h)} = -\,\partial_-\bar{h}\,, \qquad \bar{\pi} \;=\; \frac{\delta\mathcal{L}}{\delta(\partial_+\bar{h})} = -\,\partial_- h$$

$(\pi, \bar{\pi})$ not independent variables $\Rightarrow$ Half the d.o.f than in the ADM formalism

- Light-cone Hamiltonian for gravity

$$\mathcal{H} \;=\; \partial\bar{h}\,\bar{\partial}h \,+\, 2\,\kappa\,\partial_-^2\,\bar{h}\left(h\,\frac{\bar{\partial}^2}{\partial_-^2}h \,-\, \frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h\right) + c.c. \,+\, \mathcal{O}(\kappa^2)$$

- Poisson brackets

$$[h(x), \bar{h}(y)] \;=\; \epsilon(x^- - y^-)\,\delta^{(2)}(x-y)\,, \quad [\,h(x), h(y)\,] \;=\; [\,\bar{h}(x), \bar{h}(y)\,] \;=\; 0\,.$$

BMS symmetry from residual gauge invariance

Is there any residual reparameterization freedom $x^\mu \to x^\mu + \xi^\mu$ left?

- First gauge condition holds

$$g_{--} = 0 \quad \Rightarrow \quad \partial_- \xi^+ = 0 \quad \Rightarrow \quad \xi^+ = f(x^+, x^j)$$

  Second gauge condition $g_{-i} = 0$ gives

$$\partial_- \xi^j \, g_{ij} \; + \; \partial_i \xi^+ \, g_{+-} = 0$$

  Fourth gauge condition fixes x^+ dependence of $f(x^+, x^j)$

- Residual reparameterizations

$$\begin{aligned} \xi^+ &= f = \frac{1}{2} x^+ \partial_i Y^i + T(x^k) \\ \xi^i &= -\partial_k f \frac{1}{\partial_-}(g_{-+} g^{ik}) + Y^i(x^k) \\ \xi^- &= -\partial_i Y^i x^- + (\partial_+ \xi_i) x^i \end{aligned}$$

- The light-cone gravity action invariant

$$\delta_\xi \mathcal{S}[\, h, \bar{h}\,] = 0$$

  iff $\partial^2 Y = 0 \Rightarrow$ only Lorentz rotations (**no superrotations**)

  Only one arbitrary constant: $T(x^i)$

[Ananth, Brink and SM]

BMS algebra in light-cone gravity

- BMS transformation law (on the initial surface $x^+ = 0$),

$$
\begin{aligned}
\delta_{Y,\overline{Y},T}\, h \quad = \quad & Y(x)\,\bar\partial h + \overline{Y}(\bar x)\,\partial h + (\partial\overline{Y} - \bar\partial Y)\,h + T\,\frac{\partial\bar\partial}{\partial_-}\,h \\
& - 2\,\kappa\,T\,\partial_- \left(h\,\frac{\bar\partial^2}{\partial_-^2} h - \frac{\bar\partial}{\partial_-} h\,\frac{\bar\partial}{\partial_-} h \right) - 2\,\kappa\,T\,\frac{1}{\partial_-}\left(\frac{\partial^2}{\partial_-^2}\bar h\,\partial_-^2\,h \right) \\
& - 2\,\kappa\,T\,\frac{\partial^2}{\partial_-^3}\,(\bar h\,\partial_-^2\,h) + 4\,\kappa\,T\,\frac{\partial}{\partial_-^2}\left(\frac{\partial}{\partial_-}\bar h\,\partial_-^2\,h \right) + \mathcal{O}(\kappa^2)
\end{aligned}
$$

- Symmetry algebra

$$
\Big[\,\delta(Y_1,\overline{Y}_1,T_1)\,,\;\delta(Y_2,\overline{Y}_2,T_2)\Big]\;h \quad = \quad \delta(Y_{12},\overline{Y}_{12},T_{12})\,h\,,
$$

with parameters

$$
\begin{aligned}
Y_{12} \quad & \equiv \quad Y_2\,\bar\partial\,Y_1 \;-\; Y_1\,\bar\partial\,Y_2 \\
\overline{Y}_{12} \quad & \equiv \quad \overline{Y}_2\,\partial\,\overline{Y}_1 \;-\; \overline{Y}_1\,\partial\,\overline{Y}_2 \\
T_{12} \quad & \equiv \quad [Y_2\,\bar\partial\,T_1 \;+\; \overline{Y_2}\,\partial\,T_1 \;+\frac{1}{2}\,T_2(\bar\partial Y_1 \;+\; \partial\overline{Y}_1)] \;-\; (1\leftrightarrow 2)\,.
\end{aligned}
$$

$\rightarrow$ BMS algebra from residual gauge invariance without reintroducing the zero modes