The 4d Chern-Simons Perspective of Integrability in Gauge Theories

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May 24, 2022

French Strings Meeting, LAPTh, Annecy

Based on arXiv:2110.15112, arxiv:2211.00049
and ongoing works with

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Background

Main motivation

Reach an elementary understanding of why integrable spin chain structures show up in various supersymmetric QFTs.

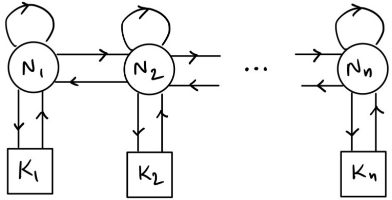
Some examples:

- Bethe/Gauge correspondence (Certain gauge theory vacua $\sim$ Bethe states of spin chains) [Nekrasov-Shatashvili '09, Nekrasov '18]
- Yangians from quantized monopole operators in 3d $\mathcal{N} = 4$.[Bullimore-Dimofte-Gaiotto '15, Braverman-Finkelberg-Nakajima '16]
- ◦ R-matrix from Janus interfaces.[Dedushenko-Nekrasov '21]
- ◦ Ω-deformed 4d $\mathcal{N} = 2$ theories with/without surface defects [Jeon-Lee-Nekrasov '21, '23, Nekrasov-Tsymbaliuk '21], [Seiberg-Witten '94, Nekrasov-Witten '10, Chen-Dorey-Hollowood-Lee '11].
- ◦ Gaudin model and chiral $\beta\gamma$-theory with critical Affince Kac-Moody current algebra [Feigin-Frenkel-Reshetikhin '94].
- ◦ ...

Background

2d Bethe/Gauge Correspondence for *compact* $\mathfrak{gl}_n$ Spin Chain

Vacua of massive 2d $\mathcal{N} = (4,4)$ A_n-quiver theory = Bethe eigenstates of $\mathfrak{gl}_n$ Spin Chains. [Nekrasov-Shatashvili '09]

$Q(N_i) :=$ 

$$\Downarrow$$

$$\bigoplus_{N_i} H^{\bullet}_{T_F}(\mathcal{M}_{\text{Higgs}}(Q(N_i)))$$

$$\cong$$

Hilbert space of a compact $\mathfrak{gl}_n$ spin chain. Representations determined by twisted masses.

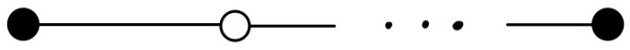
In math: Bethe eigenstates $\sim$ Stable basis [Maulik-Okounkov '12, Aganagic-Okounkov '16, '17, Bullimore-Kim-Lukowski '17, Dedushenko-Nekrasov '21]

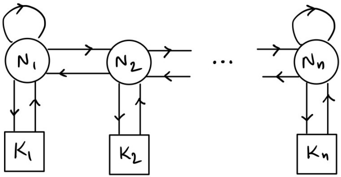
Background

Bethe/Gauge Correspondence for *compact* $\mathfrak{gl}_{m|n}$ Spin Chain

Vacua of massive 2d $\mathcal{N} = (2, 2)$ $A_{m|n}$-quiver theory = Bethe eigenstates of $\mathfrak{gl}_{m|n}$ Spin Chains. [Nekrasov '18]

$A_{m|n}$ Dynkin diagram:

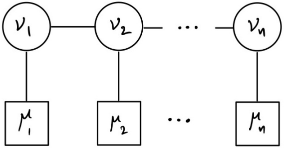




Background

3d Monopoles and *noncompact* $\mathfrak{gl}_n$ Spin Chain

Monopole operators in 3d $\mathcal{N} = 4$ A_n quiver gauge theories generate a Yangian algebra $Y(\mathfrak{gl}_n)$.[1] [Bullimore-Dimofte-Gaiotto '15, Braverman-Finkelberg-Nakajima '16] The algebra acts on verma modules spanned by vortex configurations. [Bullimore-Dimofte-Gaiotto-Hilburn-Kim '18]



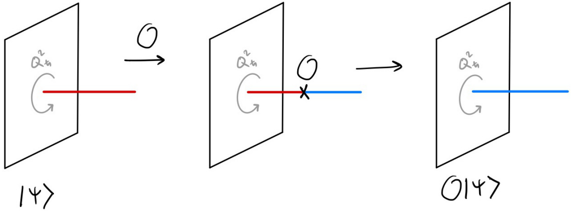
$$Y(\mathfrak{gl}_n) \curvearrowright \mathcal{H} := \bigoplus_{\mathfrak{m}} H^\bullet_{T_{\text{vortex}}}(\mathcal{M}_{\text{vortex}}(\mathfrak{m}))$$

$\mathcal{H}$ can be interpreted as the Hilbert space of a *noncompact* integrable $\mathfrak{gl}_n$ spin chain.

[1] A shifted truncated version to be exact.

Background

Cartoon of the action of a monopole operator on a vortex:



Vortices in the 3d theory define states at transverse boundaries.
Monopole operators act by changing the vortices.

Vortices = Bethe states.

Background

These two bosonic (A_n) correspondences are talking about closely related spin chains.

The gauge theories however are not obviously related.

A generalization of the 3d monopole algebra to include the $Y(\mathfrak{gl}_{m|n})$ case is not obvious.

Look for a unifying explanation that also naturally includes the super case.

Our Approach

Schema

Integrable spin chain

↓ Brane construction

String theory

↓ String dualities

Different SQFTs

Brane Construction of Spin Chains

A central tool: 4d Chern-Simons (CS) theory

A 4d gauge theory that is topological in two directions and holomorphic in the remaining directions, has the action:

$$S_{4\mathrm{dCS}} := \frac{1}{\hbar} \int_{\mathbb{R}^2 \times \mathbb{C}} \mathrm{d}z \, \mathrm{tr} \left(A \mathrm{d} A + \frac{2}{3} A^3 \right) .$$

The gauge field is complex:

$$A \in \Omega^1(\mathbb{R}^2 \times \mathbb{C}) \otimes \mathfrak{gl}_n .$$

The gauge group is GL_n.

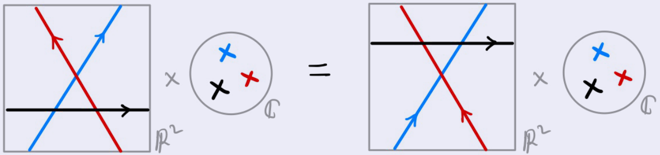
Brane Construction of Spin Chains

Two key results:

Yang-Baxter from line operators

Line operators in 4d CS theory satisfy Yang-Baxter equations.

[Costello-Witten-Yamazaki '17]



Brane construction of 4d GL_n Chern-Simons

4d GL_n CS is the the world-volume theory of a stack of n D5 branes in Ω-background. [Costello-Yagi '18]

Brane Construction of Spin Chains

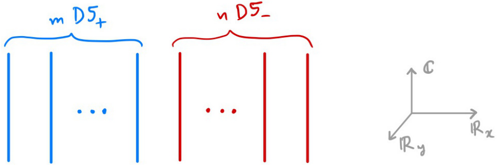
The previous result generalizes.

Brane construction of 4d $GL_{m|n}$ Chern-Simons

A stack of m D5 branes and a *rotated* stack of n D5 branes reduce, in an Ω-background, to 4d CS with $GL_{m|n}$ gauge group.

D5 world-volumes: $$\overbrace{\underbrace{\mathbb{R}^2\times\mathbb{C}}_{n\ \mathrm{D5}_-}\times\mathbb{R}^2_{+\hbar}}^{m\ \mathrm{D5}_+}\times\underbrace{\mathbb{R}^2_{-\hbar}}_{n\ \mathrm{D5}_-}\times\mathbb{R}_x\times\mathbb{R}_y$$

Brane picture:



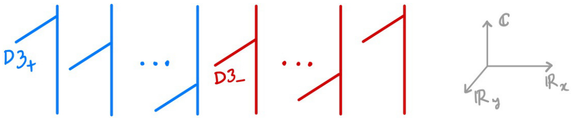
Brane Construction of Spin Chains

Line Operators in 4d CS

D3 branes ending on the D5 branes create 3d BPS defects in 6d $\mathcal{N} = (1,1)$ SYM $\xrightarrow{\Omega\text{-deformation}}$ line operators in the 4d CS theory.

D3 world-volumes: $\overbrace{\underbrace{\mathbb{R}}_{\text{D3}_-}}^{\text{D3}_+} \times \mathbb{R} \times \mathbb{C} \times \overbrace{\mathbb{R}^2_{+\hbar}}^{\text{D3}_+} \times \underbrace{\mathbb{R}^2_{-\hbar}}_{\text{D3}_-} \times \mathbb{R}_x \times \overbrace{\underbrace{\mathbb{R}_y}_{\text{D3}_-}}^{\text{D3}_+}$

Brane picture for a single line operator:



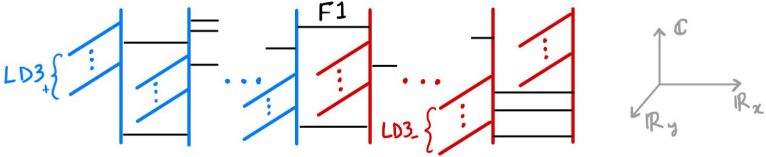
Representations labeling line operators are determined by locations of the D3 branes in the $\mathbb{C}$ direction.

Brane Construction of Spin Chains

Ω-deformation essentially means looking at a subsector protected by a supercharge $Q_\hbar$ that squares to a space-time rotation:

$$Q_\hbar^2 = \hbar J_{\mathrm{U}(1)} \,.$$

Atop the the D3-D5 configuration just mentioned, there's a tower of F1 string configurations that are supersymmetric, and provides the states in the spin chain Hilbert space.



Brane construction for a spin chain with L sites.

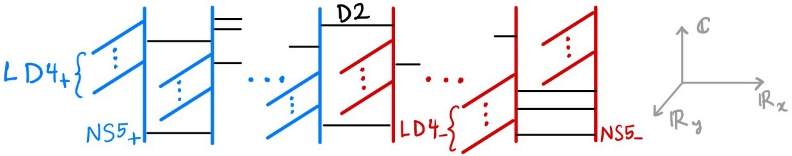
We Have The Spin Chain, What's Next?

Course of action

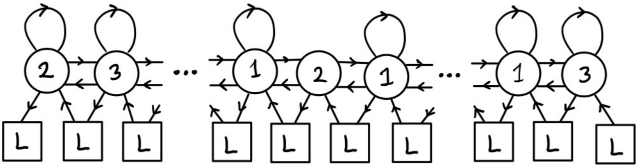
Map the supersymmetric F1-D3-D5 configurations to other supersymmetric configurations by applying string dualities and see what interpretations these new configurations have in terms of various D-brane world-volume theories.

2d Bethe/Gauge Correspondence for Superspin Chains

Application of an S and a T duality transforms the spin chain setup into a D2-D4-NS5 setup:



The world-volume theory of the D2 branes is a 2d $\mathcal{N} = (2,2)$ quiver gauge theory. [Hanany-Hori]



Quiver $Q_L(2,3,\cdots,1,2,1,\cdots,1,3)$ from the above brane diagram.

2d Bethe/Gauge Correspondence for Superspin Chains

Bethe/Gauge corr. for *noncompact* $\mathrm{GL}_{m|n}$ spin chains

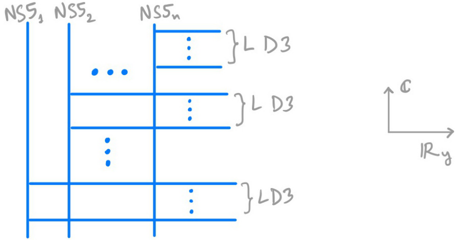
Supersymmetric F1s $\xrightarrow{T\circ S}$ Supersymmetric D2s = Vacua of the 2d theory (the D2 world-volume theory).

- Vacua of the massive 2d theory correspond to (flavor) equivariant cohomology of the Higgs branch.
- Full spectrum of line operators in the 4d CS is generated by all possible supersymmetric F1 configurations.
- Dually, the full spectrum corresponds to the vacua of all gauge theories found by varying ranks.

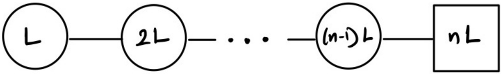
$$\mathrm{GL}_{m|n} \text{ spin chain spectrum} = \bigoplus_{\text{ranks}} H^{\bullet}_{T_F}(\mathcal{M}_{\text{Higgs}}(Q(\text{ranks})))$$

3d Monopoles

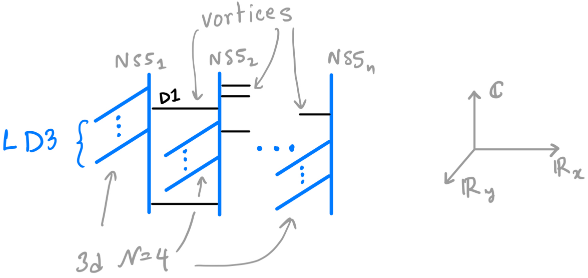
Apply S-duality to the F1-D3-D5 spin chain configuration turning it into a D1-D3-NS5 configuration. (For simplicity) look at the $D3_+$ branes:



This leads to 3d $\mathcal{N} = 4$ quiver gauge theories. [Hanany-Witten '96]



3d Monopoles



Monopole Operators and Yangians

Supersymmetric F1s $\xrightarrow{S}$ Supersymmetric D1s = Vortices.
Monopole operators create and destroy vortices.

$$\Downarrow$$

Monopole operators generate an integrable spin chain spectrum, i.e., a representation of the Yangian.

Vacuum Branches and Spin Chains

Any Ω-deformed Hanany-Witten type D3-NS5 configuration = S dual to a line operator in 4d CS. The Line operator is given by:

$$S_{\text{4dCS}} + \frac{1}{\hbar}\int_{\mathbb{R}} p^i \wedge dq_i \,,$$

where

$$(p^i, q_i) : \mathbb{R} \to \mathcal{M}_{\text{Higgs}}(\mathcal{T})$$

are the local momentaum/position coordinates on the Higgs branch of a 3d $\mathcal{N} = 4$ theory.

Geometric Characterization of Integrable Spins

Phase spaces of line operators in 4d CS
$\sim$ Phase space of spins in integrable spin chains
$\sim$ Cherkis bow varieties (a generalization of quiver varieties)

A Couple of Exemplary Vacuum Branches

T-operator in $\mathfrak{gl}_n$ spin chain

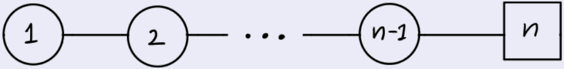


Figure 1: $\mathcal{M}_{\text{Higgs}} = T^*\text{Fl}_n$, where Fl_n is the complete Flag variety with framing n. Recall the classical result that geometric quantization of these flag varieties provide the usual representations of $\mathfrak{gl}_n$ – what we want for T-operators/Wilson lines.

Q-operator in $\mathfrak{gl}_n$ spin chain

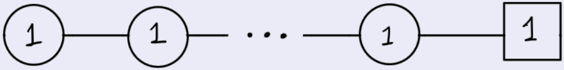


Figure 2: $\mathcal{M}_{\text{Higgs}} = T^*\mathbb{C}^{n-1}$. Its quantization therefore produces $n-1$ decoupled oscillators, what we want for Q-operators/'t Hooft lines [Bazhanov et al. '11, Costello-Gaiotto-Yagi '21].

Summary of Results and Outlooks

- 2d Bethe/gauge correspondence for non-compact A-type superspin chains.
- Proposal for new quiver varieties to study geometric representation theory of Yangians of A-type Lie superalgebras.
- Relating Bethe/gauge correspondence to quantization of monopole operators (supercase ongoing).
- A geometric classification of representations in integrable spin chains in terms of Cherkis bow varieites (supercase ongoing).
- ○ Chiral defects in 4d CS leads to spin chain - Gaudin correspondence (ongoing).
- ○ Computing elliptic R-matrices for $\mathfrak{gl}_{m|n}$-spin chains from Janus interfaces in 3d $\mathcal{N} = 2$ theories (ongoing).
- ○ Making various stacks of D-branes infinitely heavy leads to holographic setup and we can look for look for integrable degrees of freedom in twisted gravitational theories.

Thank You!