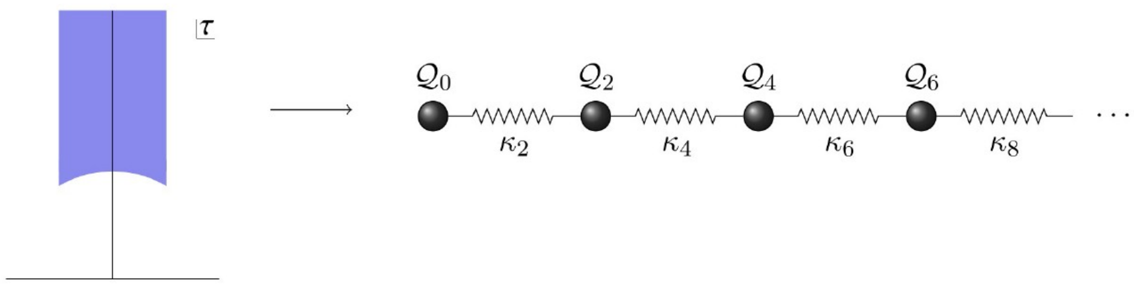
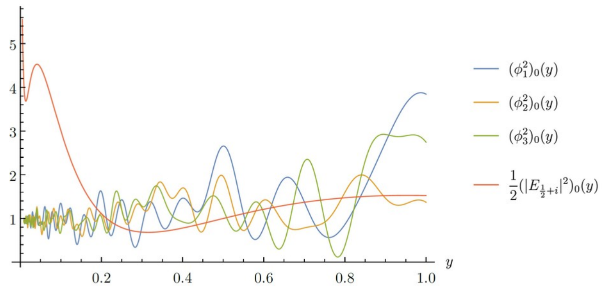
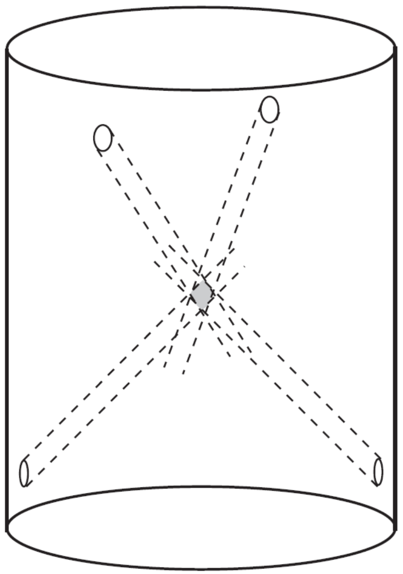




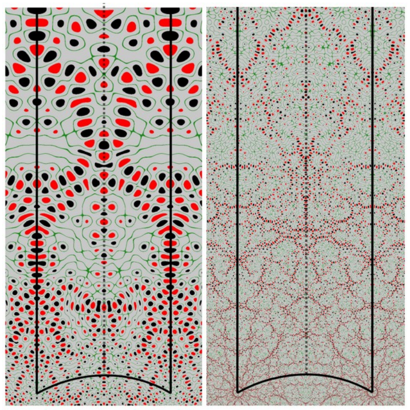
Bootstrapping String Theory

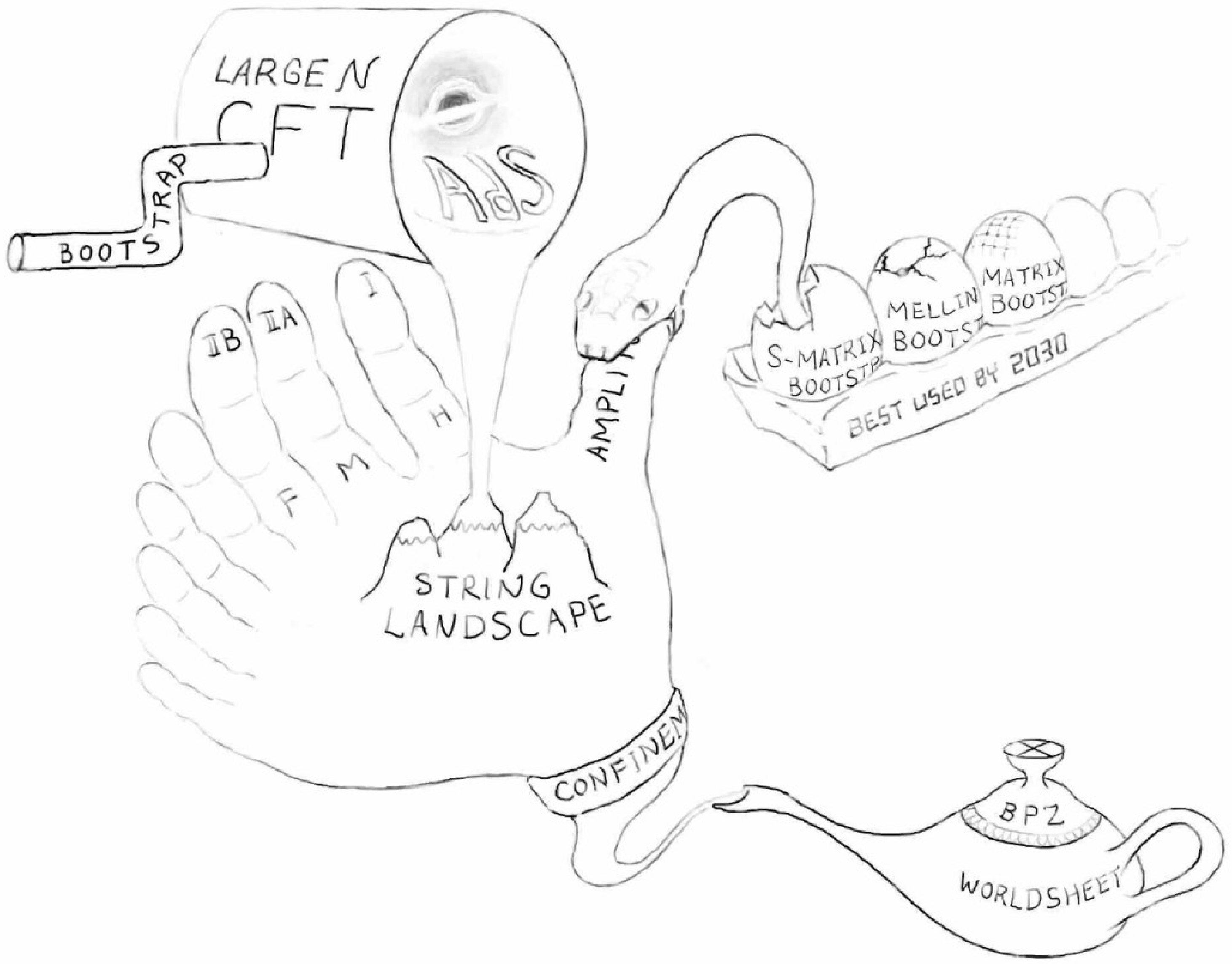


Eric Perlmutter, IPhT Saclay

French Strings Meeting, Annecy

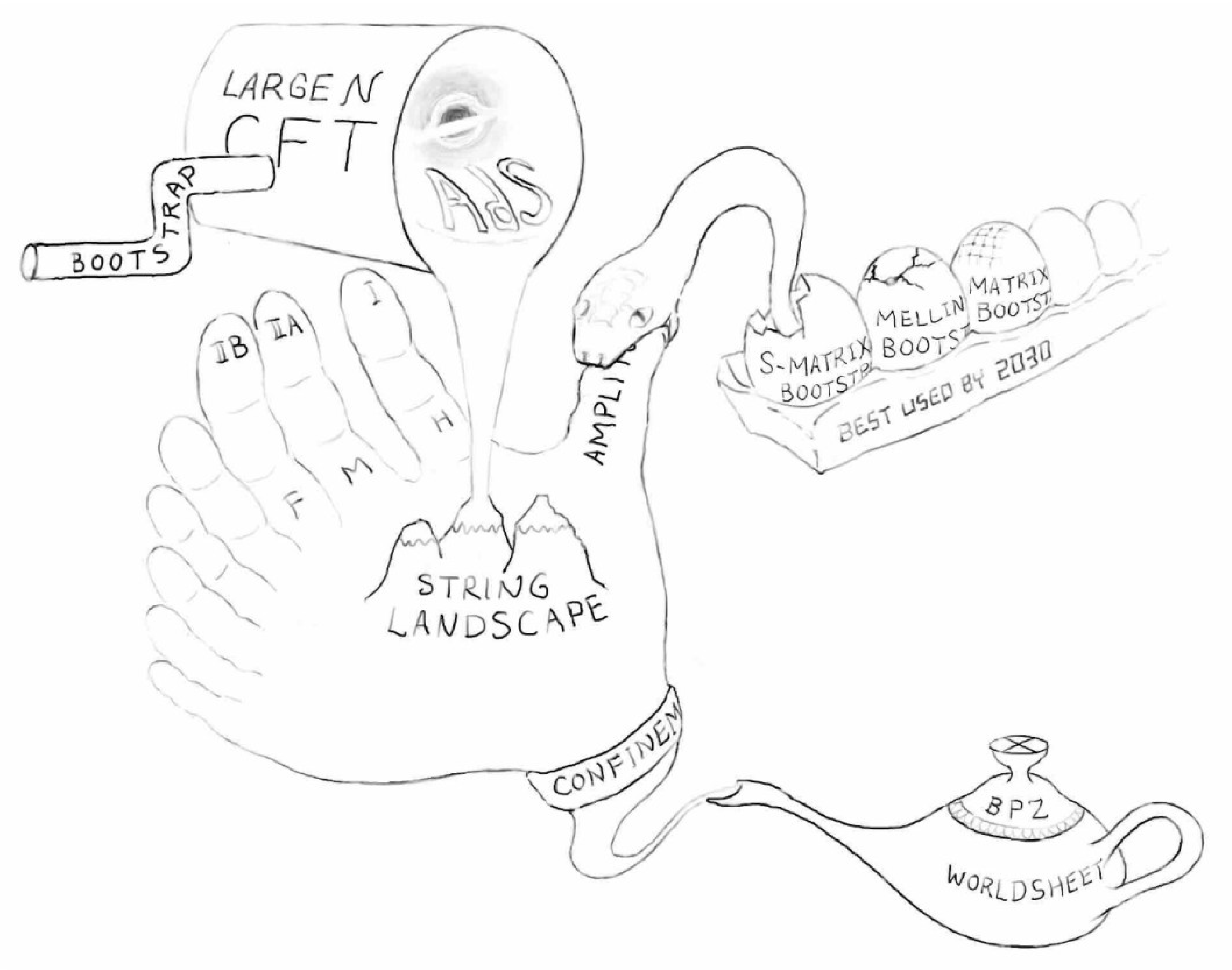
May 23, 2023




LARGE N
CFT
AdS
BOOTS
TRAP
IIB
IIA
I
H
M
F
STRING
LANDSCAPE
AMPLIT
S-MATRIX
BOOTSTR
MELLIN
BOOTST
MATRIX
BOOTST
BEST USED BY 2030
CONFINEM
BPZ
WORLDSHEET

[Yin]

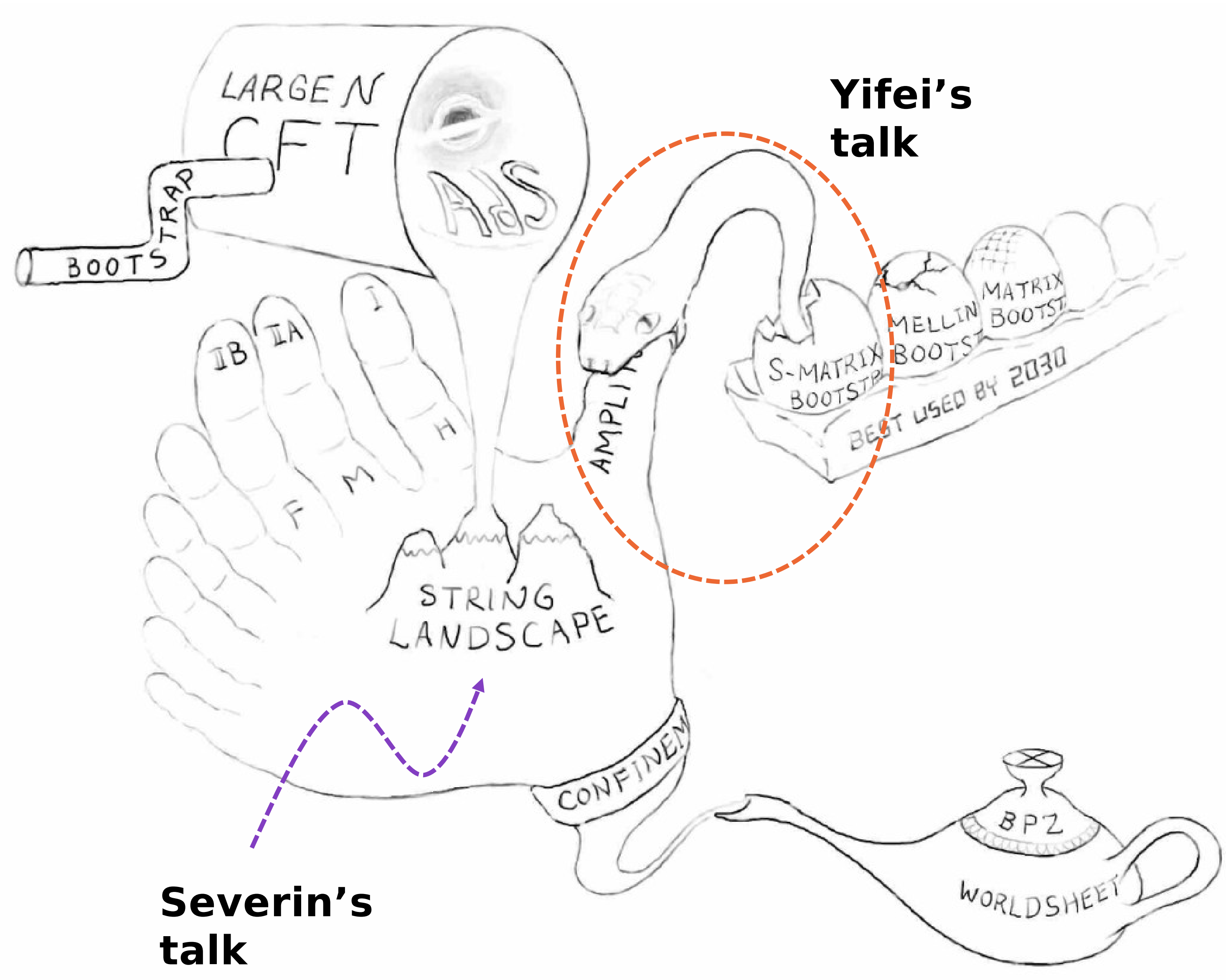
My talk



LARGE N
CFT
AdS
BOOTSTRAP
IB
IIA
I
H
M
F
STRING LANDSCAPE
AMPLIT
S-MATRIX BOOTSTR
MELLIN BOOTST
MATRIX BOOTST
BEST USED BY 2030
CONFINEM
BPZ
WORLDSHEET

[Yin]

My talk



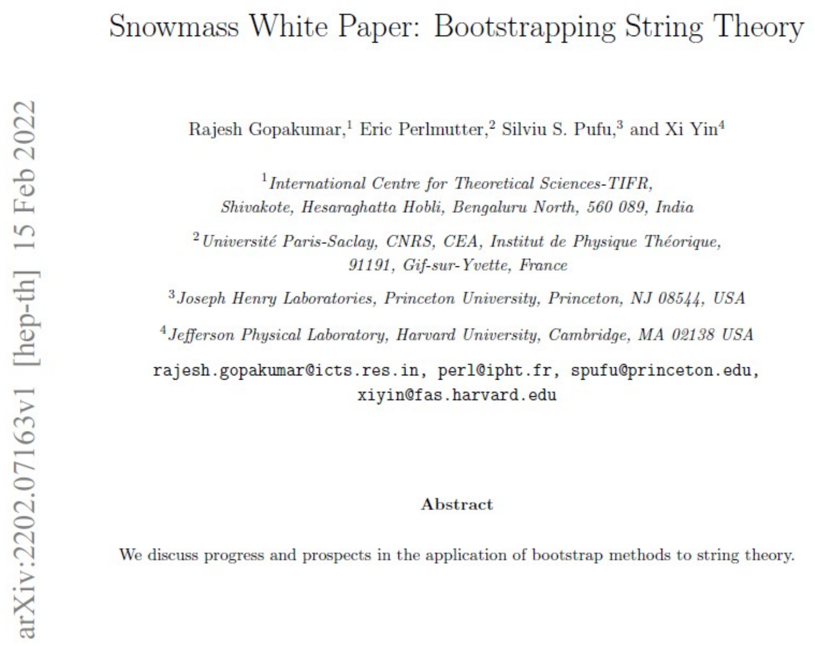
Yifei's talk

Severin's talk

[Yin]

A White Paper Wish List:

- Obtain the terms in the type II string theory and M-theory effective actions.
- Write down the Virasoro-Shapiro amplitude for $AdS_5 \times \mathbf{S}^5$.
- Definitively rule in/out scale-separated AdS vacua using large N conformal bootstrap.
- Bootstrap the worldsheet description of weakly-coupled super-Yang-Mills theory



Snowmass White Paper: Bootstrapping String Theory

Rajesh Gopakumar,[1] Eric Perlmutter,[2] Silviu S. Pufu,[3] and Xi Yin[4]

[1] *International Centre for Theoretical Sciences-TIFR, Shivakote, Hesaraghatta Hobli, Bengaluru North, 560 089, India*

[2] *Université Paris-Saclay, CNRS, CEA, Institut de Physique Théorique, 91191, Gif-sur-Yvette, France*

[3] *Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA*

[4] *Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138 USA*

rajesh.gopakumar@icts.res.in, perl@ipht.fr, spufu@princeton.edu, xiyin@fas.harvard.edu

Abstract

We discuss progress and prospects in the application of bootstrap methods to string theory.

This talk has three themes:

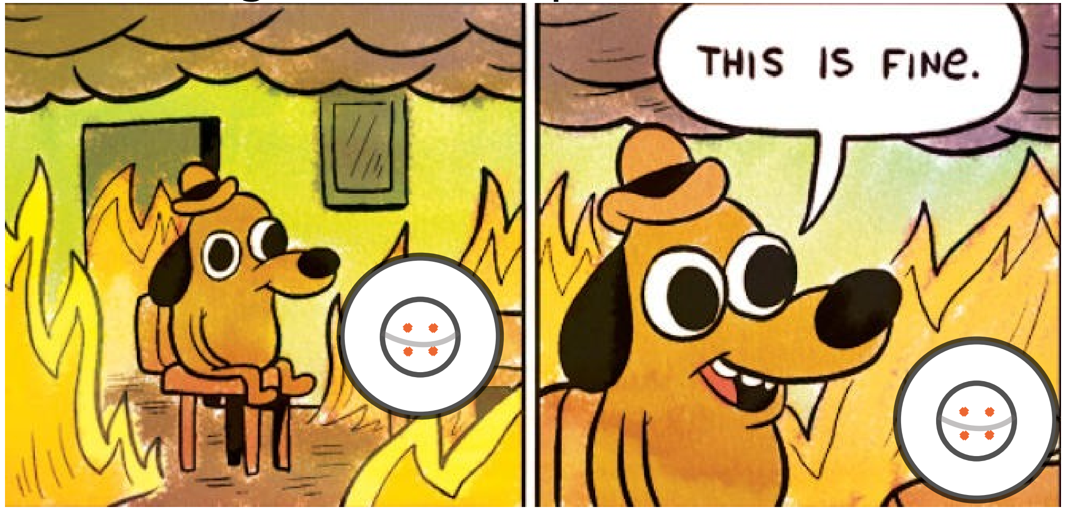
String amplitudes at finite α'

Exact results for $\mathcal{N} = 4$ super Yang-Mills correlators

Modular symmetries in conformal field theory

We all want to solve Planar $\mathcal{N}=4$ super Yang-Mills = Classical string theory in AdS_5 x **S^5**

Even “just” the tree-level four-graviton amplitude at finite string length would be fine



This is dual to the stress-tensor four-point function in the planar theory.

But this is too hard on the string side so far.

- Worldsheet: RR flux
- Spacetime: action unknown
  - In flat space, unknown beyond .
  - In AdS, even more complicated.

Looking for a handle…

The large N conformal bootstrap can determine string effective actions:

1 Bootstrap CFT correlator

2 Interpret as AdS amplitude (+ flat space limit if desired)

But this is too hard on the string side so far.

- Worldsheet: RR flux
- Spacetime: action unknown
  - In flat space, unknown beyond .
  - In AdS, even more complicated.

Looking for a handle…

The large N conformal bootstrap can determine string effective actions:

1. Bootstrap CFT correlator
2. Interpret as AdS amplitude (+ flat space limit if desired)

"I know. What's new?"

Most work in this vein has been in a low-energy expansion. We want to do better.

Localization leads to exact results for SUSY quantities.

Sometimes, these quantities are independent of the coupling (partition functions, SUSY indices)

But not always.

e.g. 4d $\mathscr{N}$=2 SQCD extremal 2-pt functions.

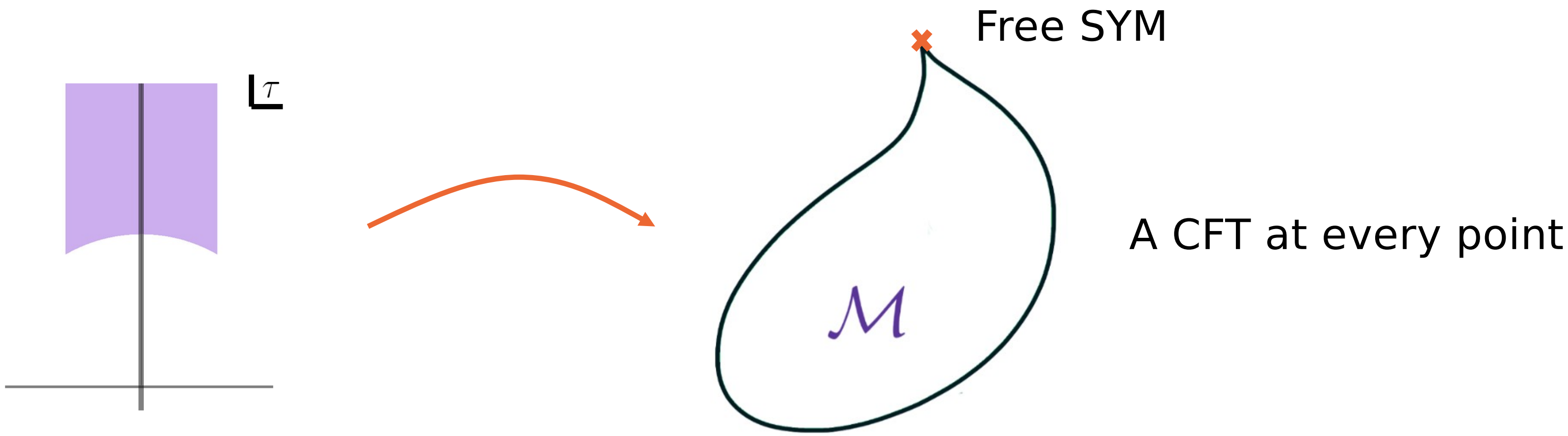
What about in $\mathscr{N}$=4 SYM?

Consider = 4 super Yang-Mills, with gauge group G.

It contains a complexified gauge coupling on which generic observables depend.

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$$

Being exactly marginal, parameterizes a “conformal manifold” preserving = 4 SUSY:



All single-trace operators are ½-BPS or unprotected.

Unprotected correlators are too hard.

2-, 3-point functions of ½-BPS operators are independent of coupling.

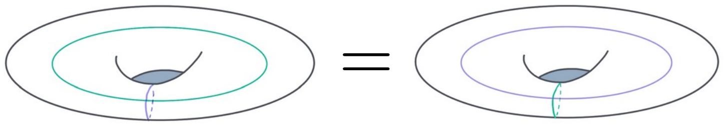
4-point functions depend on cross-ratios and receive unprotected contributions...

Remarkably, there is a class of **integrated 4**-point functions which can be computed exactly from localization, even at finite N.

$$\int d\mu(z,\bar{z})\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle$$

Combined with large N bootstrap methods, these teach us about string effective actions.

They are also amazing observables in their own right.



= 4 SYM enjoys **S-duality**.

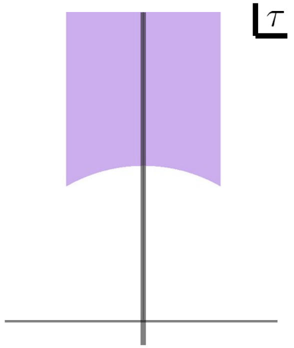
For simply-laced G, this is a self-duality under transformations of (up to global identifications).

$$\mathbb{O}(\gamma\tau) = \mathbb{O}(\tau)\,, \quad \gamma \in SL(2,\mathbb{Z})$$

e.g $\Delta_i, \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \mathcal{O}_\ell \rangle$

.

(Non-local observables invariant under congruence subgroups.)



S-duality of = 4 SYM is beyond a reasonable doubt…

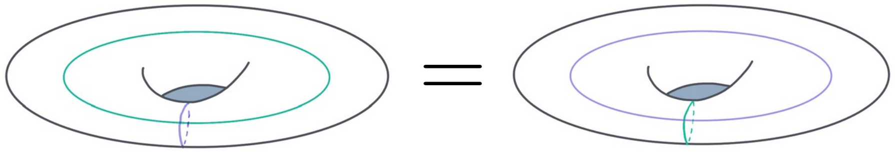
- Field theory (D-instantons, 1/N, bound states, partition functions, …)
- Holography + string theory (D-instantons, graviton scattering, …)

But its abstract consequences for CFT observables have not been fully understood.

[Montonen, Olive; Olive, Witten; Osborn; Argyres, Kapustin, Seiberg; Vafa, Witten; Sen; Gomis, Okuda; Dorey, Hollowood, Khoze, Mattis, Vandoren; Green, Gutperle; Banks, Green; Green, Miller, Vanhove; D’Hoker et al; Beem, Rastelli, Sen, van Rees; Chester, Green, Pufu, Wang, Wen; …]

This is just one instantiation of invariance:

In conformal field theory, modularity is everywhere.



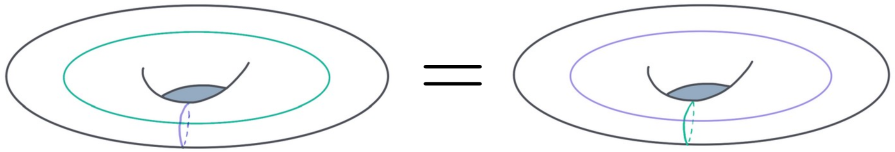
- Spacetime symmetry of 2d CFT
- Electric-magnetic duality symmetry of Maxwell theory
- Generalized modularity of counting functions in superconformal field theories
- …

Everybody knows that we should first process symmetries, then compute.

Why can’t we do this for modular invariance?

This is just one instantiation of invariance:

In conformal field theory, modularity is everywhere.

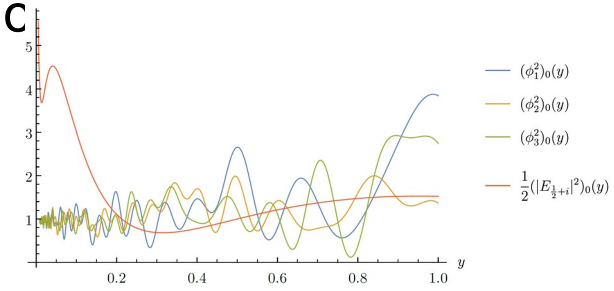


- Spacetime symmetry of 2d CFT
- Electric-magnetic duality symmetry of Maxwell theory
- Generalized modularity of counting functions in superconformal field theories
- …

Everybody knows that we should first process symmetries, then cc

Why can’t we do this for modular invariance?

(We can.)



The rest of the talk is a story about these intertwined concepts,

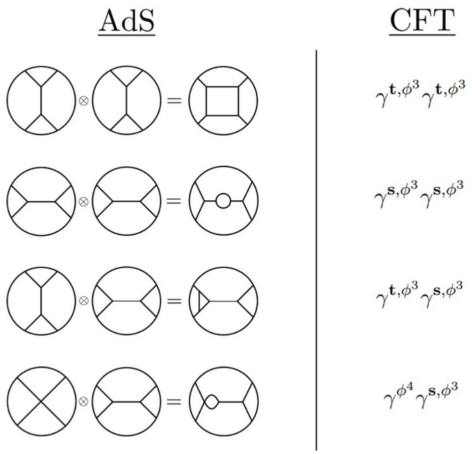
and how they are leading us toward exact (quantum) string theory observables.

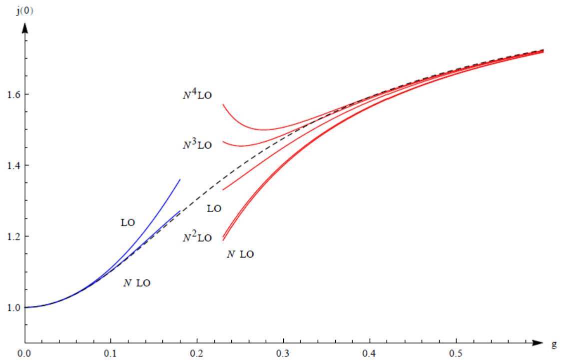
Outline:

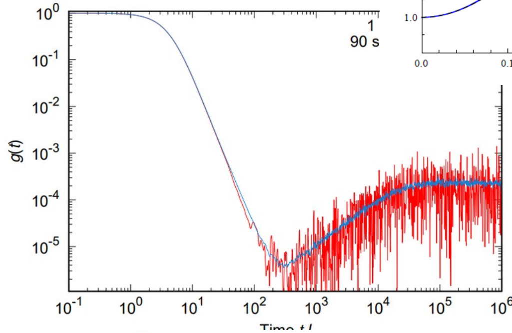
1) String tree amplitudes…
2) … from N=4 SYM correlators
   - Unintegrated
   - Integrated
3) spectral theory
   - Application to Integrated correlators & semiclassical string theory

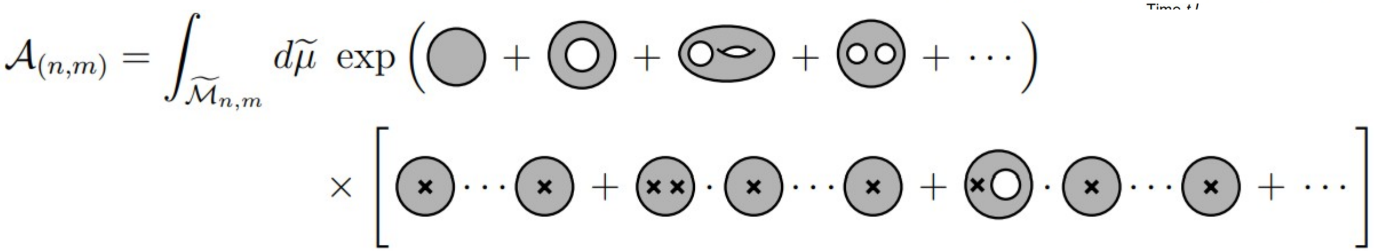
Some relevant things that I won't/can't talk about:

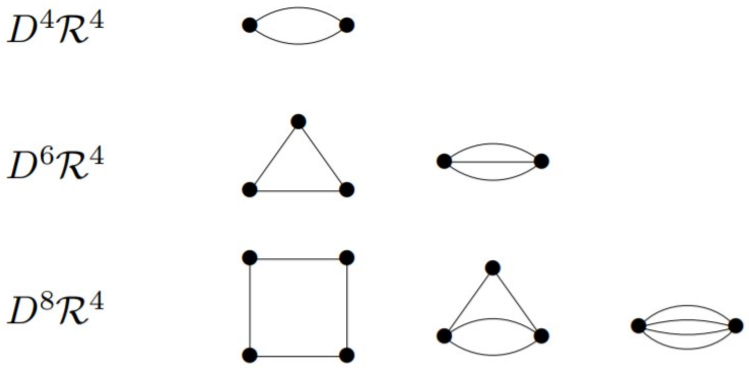
- Modular structure of string worldsheet integrands
- AdS Unitarity Method/Loop amplitudes
- Integrability
- String Field Theory methods
- Ensemble averages and gravity











String tree amplitudes and holography

Quartic supergraviton scattering amplitudes in (say) type IIB string theory in flat space take the form

$$\mathcal{A}_{\mathbb{M}_{10}} \approx \frac{1}{stu} + \alpha'^3 \sum_{m,n=0}^{\infty} \alpha'^{2m+3n} \sigma_2^m \sigma_3^n f_{mn}(\tau)$$

$$\sigma_2 := s^2 + t^2 + u^2$$
$$\sigma_3 := stu$$
$$s + t + u = 0$$
$$\tau := \chi + ie^{-\phi}$$

There is a corresponding effective action:

$$\mathcal{L}_{\mathbb{M}_{10}} = R + f_{00}(\tau)\,\alpha'^3 R^4 + f_{10}(\tau)\,\alpha'^5 D^4 R^4 + f_{01}(\tau)\,\alpha'^6 D^6 R^4 + \ldots$$

In type IIB string theory, these are -invariant functions, order-by-order. (In type IIA, only dilaton)

Higher orders are not fully known.

String tree amplitudes and holography

Here are the ones that we know (cf. explicit worldsheet/S-duality/on-shell methods):

$$f_{R^4}(\tau) \propto E_{\frac{3}{2}}(\tau)\,, \quad f_{D^4R^4}(\tau) \propto E_{\frac{5}{2}}(\tau)\,, \quad f_{D^6R^4}(\tau) \propto F_{3;\frac{3}{2},\frac{3}{2}}(\tau)$$

You're probably familiar with non-holomorphic Eisenstein series, eigenfunctions of hyperbolic Laplacian:

$$(\Delta_\tau - s(1-s))E_s(\tau) = 0$$

Note the Maass cusp forms, also eigenfunctions (more later)

$$\left(\Delta_\tau - \left(\frac{1}{4} + t_n^2\right)\right)\phi_n(\tau) = 0$$

Less familiar are the "generalized Eisensteins"

$$(\Delta_\tau - s(s+1))F_{s;s_1,s_2}(\tau) = -E_{s_1}(\tau)E_{s_2}(\tau)$$

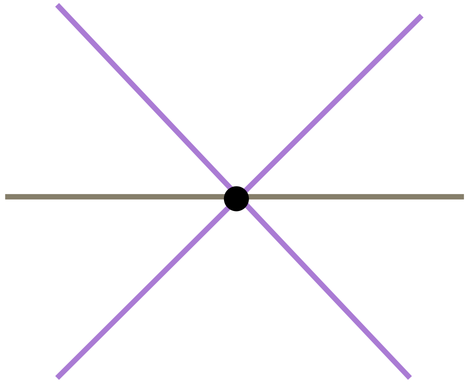
Higher-order terms, and beyond, are unknown in general (modulo low-loop perturbative data)

[Green, Gutperle; Green, Gutperle, Kwon; D'Hoker, Green, Vanhove; Green, Russo, Vanhove; Green, Miller, Vanhove; Wang, Yin; ...]

[Kleinschmidt, Dorigoni, Schlotterer; Klinger-Logan; Klinger-Logan, Miller, Radchenko; Chester, Green, Pufu, Wang, Wen]

String tree amplitudes and holography

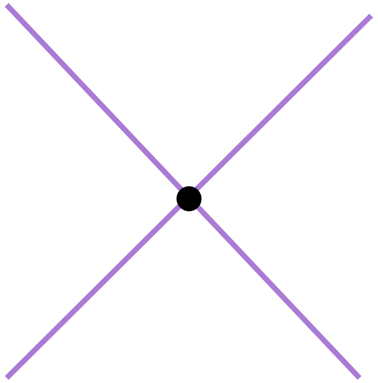
What about AdS? It's easy to understand why the AdS string effective action is more complicated...



$$\ell_s\,,\quad \ell_p\,,\quad L_{\rm AdS}$$

$$\mathcal{L}_{\mathbb{M}_{10}} \supset \alpha'^5 R^6 \qquad \Rightarrow \qquad \mathcal{L}_{{\rm AdS}_5} \supset \alpha'^5 R^4$$

(Put two legs on S^5)

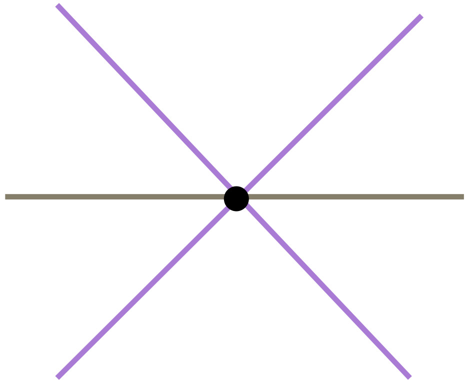


$$\mathcal{A}_{\mathbb{M}_{10}} \approx \frac{1}{stu} + \alpha'^3 f_{R^4}(\tau) + \alpha'^5 \sigma_2 f_{D^4R^4}(\tau) + \ldots \qquad \mathcal{A}_{{\rm AdS}_5} \approx \frac{1}{stu} + \alpha'^3 f_{R^4}(\tau) + \alpha'^5 \left(\sigma_2 f_{D^4R^4}(\tau) + \frac{1}{L_{\rm AdS}^2} f_{R^6}(\tau)\right) + \ldots$$

Finite-size correction

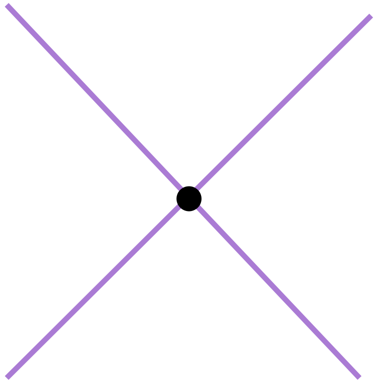
String tree amplitudes and holography

What about AdS? It's easy to understand why the AdS string effective action is more complicated...



$$\ell_s\,, \quad \ell_p\,, \quad L_{\mathrm{AdS}}$$

$$\mathcal{L}_{\mathbb{M}_{10}} \supset \alpha'^5 R^6 \qquad \Rightarrow \qquad \mathcal{L}_{\mathrm{AdS}_5} \supset \alpha'^5 R^4$$



At fixed order in α', amplitudes are *not* homogeneous in momenta

$$\mathcal{A}_{\mathrm{AdS}_5} \approx \mathcal{A}_{\mathrm{sugra}} + \alpha'^3 \sum_{m,n=0}^{\infty} \alpha'^{2m+3n} \sum_{j=0}^{m} \sum_{k=0}^{n} \sigma_2^{m-j} \sigma_3^{n-k} f_{mn}^{(jk)}(\tau)$$

At fixed order in momenta, coefficients have an infinite expansion in

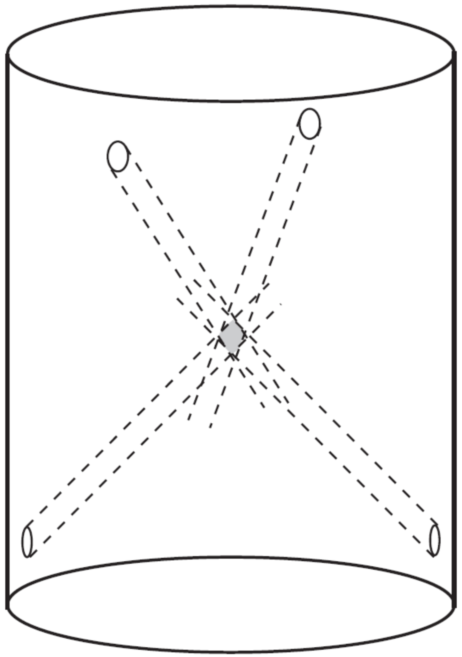
$$\mathcal{A}_{\mathrm{AdS}_5} \approx \mathcal{A}_{\mathrm{sugra}} + \alpha'^3 \sum_{m,n=0}^{\infty} \sigma_2^m \sigma_3^n \sum_{j=0}^{\infty} \alpha'^{2m+3n+j} f_{mn}^{(j)}(\tau)$$

String tree amplitudes and holography

… but the AdS effective action is of clear interest:

1) It encodes CFT correlators

2) It can be derived from CFT correlators



[Polchinski; Penedones] [Maldacena, Simmons-Duffin, Zhiboedov]

[Komatsu, Paulos, van Rees, Zhao]

One can also take the flat space limit.

This isolates the leading term in momenta, order-by-order in α'.

$$\mathcal{A}_{\mathrm{AdS}_5} \approx \frac{1}{stu} + \alpha'^3 f_{R^4}(\tau) + \alpha'^5 \left(\sigma_2 f_{D^4R^4}(\tau) + \frac{1}{L^2_{\mathrm{AdS}}} f_{R^6}(\tau) \right) + \ldots$$

String tree amplitudes and holography

Implementing this program,

$$f_{R^4}(\tau) \propto E_{\frac{3}{2}}(\tau)\,, \quad f_{D^4R^4}(\tau) \propto E_{\frac{5}{2}}(\tau)\,, \quad f_{D^6R^4}(\tau) \propto F_{3;\frac{3}{2},\frac{3}{2}}(\tau)$$

can be recovered from certain computations of $\mathcal{N}$=4 SYM four-point functions.

The holographic results also determine an infinite set of SUSY-protected terms at higher-orders.

We now turn to how this works.

(N.B: Analogous results exist in M-theory.)

$\mathcal{N}$=4 SYM correlators

Define ½-BPS superconformal primaries

$$\mathcal{O}_p \sim \mathrm{tr}(\phi^{(I_1} \cdots \phi^{I_p)}) \in [0\,p\,0] \text{ irrep of } \mathfrak{su}(4)_R \qquad (\Delta_p, J_p) = (p, 0)$$

Holographically dual to scalar KK modes on S^5 (modulo multi-trace admixture details)

At p>2, there is degeneracy (multi-traces), e.g.

$$\mathcal{O}_p := [\mathcal{O}_2]^{p/2}\,, \quad p \in 2\mathbb{Z}_+$$

Dual to ½-BPS gas of gravitons

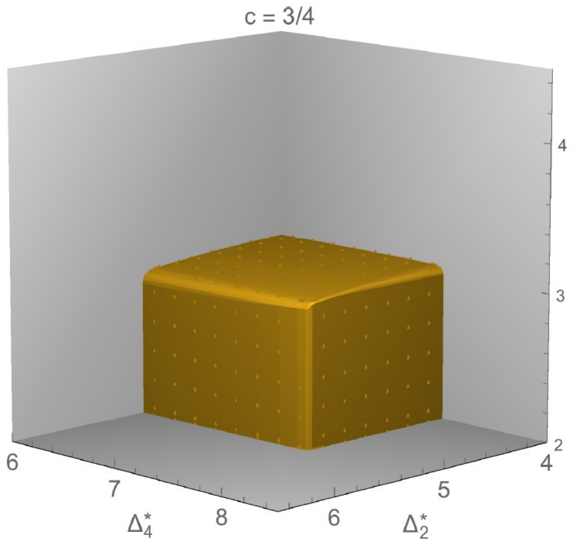
Consider four-point functions: schematically,

$$\langle 22pp \rangle = \texttt{(free part)} + \mathcal{I}\,\mathcal{H}_p^{(N)}(u, v; \tau)$$

$\mathcal{I}$ = SC Ward identity factor

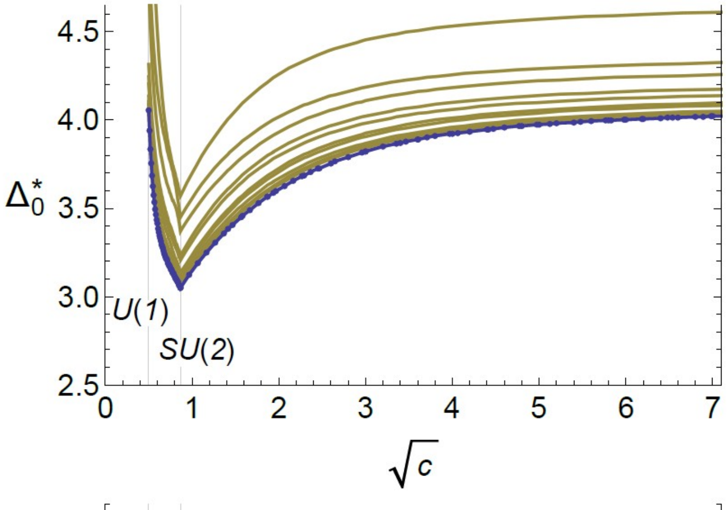
Specialize (for now) to simplest case of p=2 □ Four-point supergraviton amplitude in AdS_5

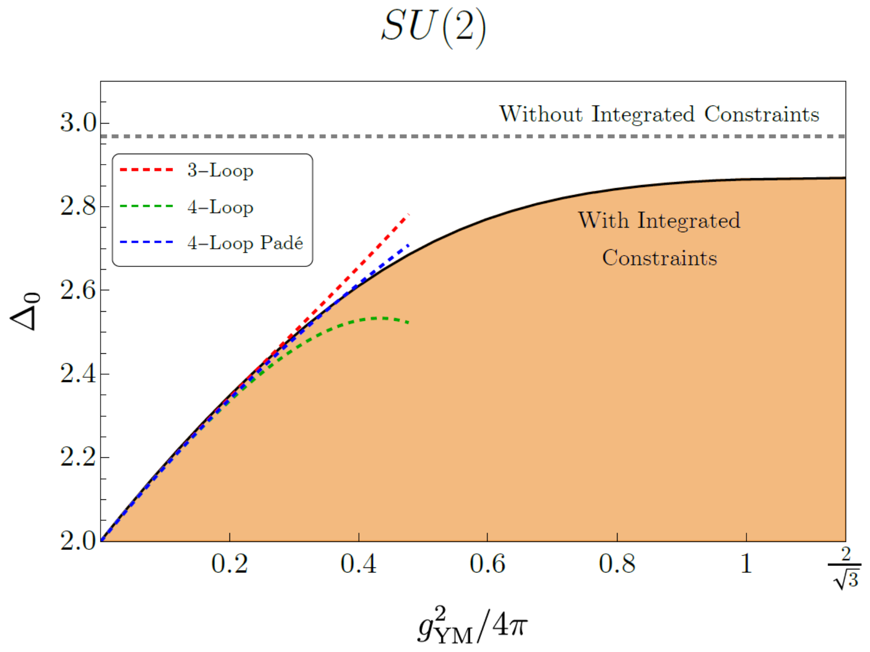
$\mathcal{N}$=4 SYM correlators: Finite N



Some is known about *<22pp>* at finite N

- Perturbation theory
  - Two loops (generic p) [D'Alessandro, Genovese]
  - Three loops (p=2) [Drummond, Duhr, Eden, Heslop, Pennington, Smirnov; Eden, Heslop, Korchemsky, Sokatchev; Fleury, Pereira]
- Numerical superconformal bootstrap [Beem, Rastelli, van Rees; Bissi, Manenti, Vichi; Alday, Chester; Chester, Dempsey, Pufu]
  - Mainly p = 2
  - Some p = 2, 3 (mixed correlator system)





$\mathcal{N}$=4 SYM correlators: Large N

At large N, the usual ‘t Hooft limit is well-studied: $N \to \infty\,, \quad \lambda := g_{\rm YM}^2 N$ fixed

- Perturbation theory in (Feynman diagrams, symbology, etc), (holography)
- Integrability
- Planar numerical conformal bootstrap [Caron-Huot, Coronado, Trinh, Zahraee] $0 = W^{\rm protected} + \sum_{(\Delta,J)\ \rm long} \lambda_{\Delta,J}^2 W[\Delta, J]$

Less familiar is the “very strongly coupled” limit: $N \to \infty\,, \quad g_{\rm YM}^2$ fixed

This preserves SL(2,ℤ) properties of SYM observables. [Azeyanagi, Hanada, Honda, Matsuo, Shiba]

In particular, *<22pp>* is SL(2,ℤ)-invariant.

$$\mathcal{H}_p(u,v;\tau) = \mathcal{H}_p(u,v;\gamma\tau)\,, \quad \gamma \in SL(2,\mathbb{Z})$$

Semiclassical AdS_5 x S^5 string amplitudes

In the ‘t Hooft limit, bulk string theory is weakly coupled.

The tree-level four-graviton string amplitude is the “AdS_5 x **S^5** -Virasoro-Shapiro” amplitude

$$\mathcal{A}_{\mathrm{AdS}_5} \approx \mathcal{A}_{\mathrm{sugra}} + \alpha'^3 \sum_{m,n=0}^{\infty} \sigma_2^m \sigma_3^n \sum_{j=0}^{\infty} \alpha'^{2m+3n+j} c_{mn}^{(j)}$$

$$\sim f_{mn}^{(j)}(\tau)\big|_{\mathrm{tree}}$$

Various partial results here [Alday, Hansen, Silva]

Recent bootstrap + Integrability approach:

$$c_{mn}^{(1)}$$

(Dispersive sum rules) + *(Integrability data)* + *(svMZV assumption)* All

$$\sum_{\delta=1}^{\infty} \sum_{n=0}^{\delta-1} \frac{1}{\delta^4} \mathcal{D}_n(\delta) \frac{y+2}{1-x-y} \binom{z+\delta-\frac{n}{2}-1}{\delta-n-1}^2$$

Nice structural generalization of VS amplitude…

… with complicated details

$$\begin{aligned}\mathcal{D}_n(\delta) = {} & h_n(\delta) - g_n(\delta)\left(3 + 2x\partial_x + 3y\partial_y\right) + \delta_{n,0}\Big(-\left(\tfrac{1}{2}x\partial_x + \tfrac{3}{4}y\partial_y + \tfrac{27}{4}\right)(z+\delta)\partial_z z\partial_z \\ & - 6(x\partial_x)^2 y\partial_y - 9x\partial_x(y\partial_y)^2 - 16x\partial_x y\partial_y - \tfrac{4}{3}(x\partial_x)^3 - \tfrac{16}{3}(x\partial_x)^2 - \tfrac{1}{3}x\partial_x - \tfrac{9}{2}(y\partial_y)^3 \\ & - 12(y\partial_y)^2 - \tfrac{1}{2}y\partial_y - \tfrac{277}{32} + \left(6x\partial_x y\partial_y + 2(x\partial_x)^2 + \tfrac{26}{3}x\partial_x + \tfrac{9}{2}(y\partial_y)^2 + 13y\partial_y - \tfrac{21}{8}\right) z\partial_z \\ & + \left(3x\partial_x y\partial_y + (x\partial_x)^2 + \tfrac{55}{12}x\partial_x + \tfrac{9}{4}(y\partial_y)^2 + \tfrac{55}{8}y\partial_y + \tfrac{33}{16}\right)\delta\partial_z\Big)\,. \qquad (5.16)\end{aligned}$$

(Instead expand in momenta?)

Semiclassical AdS_5 x S^5 string amplitudes

We henceforth focus on the very strongly coupled limit, where SL(2,ℤ) is manifest.

The <2222> correlator has been computed to several orders in 1/N. In Mellin space, $s+t+u=4$

$$c=\frac{N^2-1}{4}$$

$$\widetilde{\mathcal{M}}_2(s,t;\tau)\Big|_{\text{tree}} \approx \frac{1}{c}\frac{8}{(s-2)(t-2)(u-2)} + \frac{1}{c^{7/4}}\frac{15\zeta(3)}{2\sqrt{2\pi^3}}E_{\frac{3}{2}}(\tau) + \frac{1}{c^{9/4}}\frac{315\zeta(5)}{64\sqrt{2\pi^5}}E_{\frac{5}{2}}(\tau)(\sigma_2-3) + \frac{1}{c^{5/2}}\frac{315\zeta(3)^2}{16\pi^3}F_{3;\frac{3}{2},\frac{3}{2}}(\tau)\left(\sigma_3-\frac{1}{4}\sigma_2-4\right)+\dots$$

Supergravity

Orange = Finite-size corrections

The supergravity term has been bootstrapped from CFT: Ward IDs + holographic consistency conditions

Indeed, <p_1 p_2 p_3 p_4> supergravity correlators follow from a single “master” formula: [Rastelli, Zhou]

$$\mathcal{H}^{\text{sugra}}_{p_1p_2p_3p_4}(u,v) = \mathcal{D}_{p_1p_2p_3p_4}\mathcal{H}^{\text{sugra}}_{2222}(u,v)$$

“Hidden 10d conformal sym” [Caron-Huot, Trinh]

Corrections to supergravity have been reproduced from CFT by different, more involved means….

Semiclassical AdS_5 x S^5 string amplitudes

Consider the first correction ():

$$\widetilde{\mathcal{M}}_2(s,t;\tau)\Big|_{\text{tree}} \approx \frac{1}{c}\frac{8}{(s-2)(t-2)(u-2)} + \frac{1}{c^{7/4}}\frac{15\zeta(3)}{2\sqrt{2\pi^3}}E_{\frac{3}{2}}(\tau) + \ldots$$

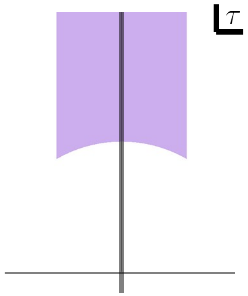
- Constructed in late 90's via instantons + string duality [Gutperle, Green]
- Required by on-shell superamplitude identities (Laplace eq with) [Wang, Yin]
- What about S-matrix bootstrap approach:

In the space of consistent S-matrices, Where is String Theory?

[Guerrieri, Penedones, Vieira]

$$E_{\frac{3}{2}}(\tau) = y^{3/2} + \frac{\pi^2}{3\zeta(3)}y^{-1/2} + \frac{8\pi\sqrt{y}}{\zeta(3)}\sum_{k=1}^{\infty}\cos(2\pi k x)\frac{\sigma_2(k)}{k}K_1(2k\pi y)$$

with



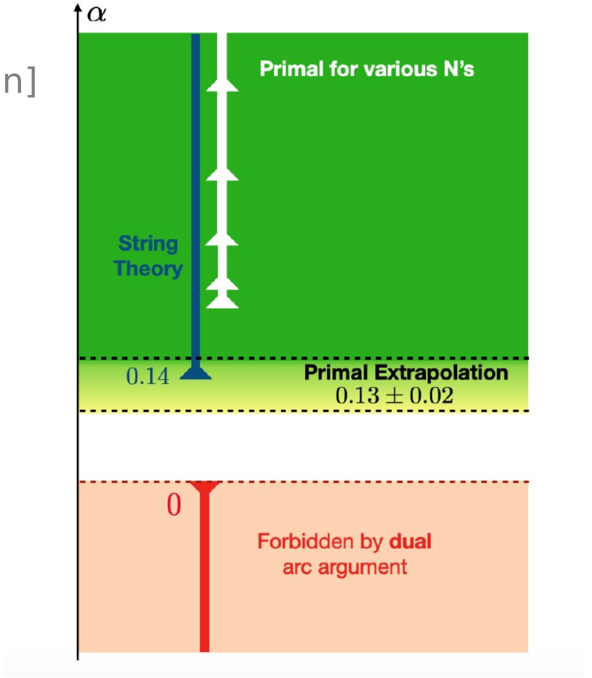


FIG. 3. String Theory covers all or almost all the allowed quantum gravity theory space.

Semiclassical AdS_5 x S^5 string amplitudes

Consider the first correction ():

$$\widetilde{\mathcal{M}}_2(s,t;\tau)\Big|_{\text{tree}} \approx \frac{1}{c}\frac{8}{(s-2)(t-2)(u-2)} + \frac{1}{c^{7/4}}\frac{15\zeta(3)}{2\sqrt{2\pi^3}}E_{\frac{3}{2}}(\tau) + \dots$$

- Constructed in late 90's via instantons + string duality [Gutperle, Green]
- Required by on-shell superamplitude identities (Laplace eq with) [Wang, Yin]
- What about S-matrix bootstrap approach:

In the space of consistent S-matrices, Where is String Theory?

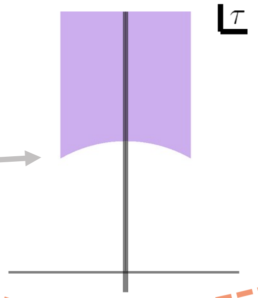
[Guerrieri, Penedones, Vieira]

$$E_{\frac{3}{2}}(\tau) = y^{3/2} + \frac{\pi^2}{3\zeta(3)}y^{-1/2} + \frac{8\pi\sqrt{y}}{\zeta(3)}\sum_{k=1}^{\infty}\cos(2\pi k x)\frac{\sigma_2(k)}{k}K_1(2k\pi y)$$

with

Maximized at cusp,

Minimized at corner, with near-extremal value!



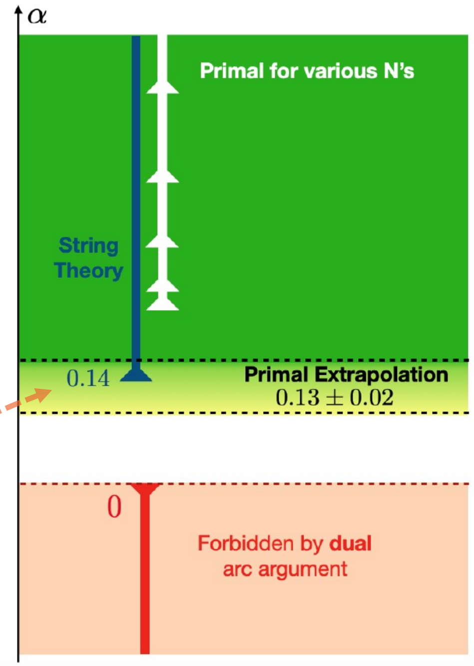


FIG. 3. String Theory covers all or almost all the allowed quantum gravity theory space.

Semiclassical $AdS_5 \times S^5$ string amplitudes from CFT

$$\widetilde{\mathcal{M}}_2(s,t;\tau)\Big|_{\text{tree}} \approx \frac{1}{c}\frac{8}{(s-2)(t-2)(u-2)} + \frac{1}{c^{7/4}}\frac{15\zeta(3)}{2\sqrt{2\pi^3}}E_{\frac{3}{2}}(\tau) + \frac{1}{c^{9/4}}\frac{315\zeta(5)}{64\sqrt{2\pi^5}}E_{\frac{5}{2}}(\tau)(\sigma_2-3) + \frac{1}{c^{5/2}}\frac{315\zeta(3)^2}{16\pi^3}F_{3;\frac{3}{2},\frac{3}{2}}(\tau)\left(\sigma_3-\frac{1}{4}\sigma_2-4\right)+\ldots$$

Let us now delve into how the corrections were computed from the CFT.

As a segue, recall the earlier result of [Chester-Dempsey-Pufu]

Q: How did they track position along ?

A: By using certain exact quantities – *integrated correlators* – obtained from localization.

Remarkably, *<22pp>* correlators may be found exactly, for all and (!)...

IF we integrate against a specific spacetime measure.

$\mathcal{N}$=4 SYM Integrated Correlators

Here is the basic relation:

Mass-deformed free energy on S^4 with source

$$\int d\mu(u,v)\,\mathcal{H}_p^{(N)}(u,v;\tau) = \text{“}\,\partial_{\tau_p}^2\,\partial_m^2 \log Z_{S^4}(\tau,\tau_p;m)\big|_{m=\tau_p=0}\,\text{”}$$

Specific measure

Brings down two ’s

Brings down two ’s

[Binder, Chester, Pufu, Wang; Chester, Pufu]

$\mathcal{N}$=4 SYM Integrated Correlators

Here is the basic relation:

Mass-deformed free energy on S^4 with source

$$\int d\mu(u,v)\,\mathcal{H}_p^{(N)}(u,v;\tau) = \text{``}\,\partial_{\tau_p}^2\partial_m^2 \log Z_{S^4}(\tau,\tau_p;m)\big|_{m=\tau_p=0}\,\text{''}$$

Specific measure

Brings down two 's

Brings down two 's

$$\mathcal{G}_p^{(N)}(\tau) := -\frac{2}{\pi}\int_0^\infty dr \int_0^\pi d\theta\, \frac{r^3\sin^2\theta}{u^2}\,\mathcal{H}_p^{(N)}(u,r^2;\tau)\,, \qquad u = 1 + r^2 - 2r\cos\theta$$

This is tremendously powerful.

In the planar limit, e.g., $\mathcal{G}_p^{(\mathfrak{g}=0)}(\lambda) = \int_0^\infty d\omega\, \frac{\omega}{\sinh^2\omega}\left(J_1\left(\frac{\sqrt{\lambda}}{\pi}\omega\right)^2 - J_p\left(\frac{\sqrt{\lambda}}{\pi}\omega\right)^2\right)$

Other derivatives are possible: $\int d\mu'(u,v)\,\mathcal{H}_2^{(N)}(u,v;\tau) = \text{``}\partial_m^4 \log Z_{S^4}(\tau,\tau_p;m)\big|_{m=0}\text{''}$

[Binder, Chester, Pufu, Wang; Chester, Pufu]

$\mathcal{N}$=4 SYM Integrated Correlators: Large N

With this in hand, the p=2 case was systematically studied in the very strongly coupled limit.

Combining various constraints + flat space limit yielded the previous unintegrated formula.

(Idea: integrate general amplitude and match.)

~ No protection beyond

At higher-orders, can only get SUSY *part* of the correction: not enough localization constraints…

But, here's a new thing:

$$\frac{1}{c^2}\partial_m^4 \log Z|_{m=0} \supset \frac{1}{c^{5/2}}\delta_1 F_{3;\frac{3}{2},\frac{3}{2}}(\tau) + \frac{1}{c^3}\left(\delta_2 F_{6;\frac{5}{2},\frac{3}{2}}(\tau) + \delta_3 F_{4;\frac{5}{2},\frac{3}{2}}(\tau)\right)$$
$$+ \frac{1}{c^{7/2}}\left(\delta_4 F_{3;\frac{3}{2},\frac{3}{2}}(\tau) + \sum_{r=5,7,9} \alpha_r F_{r;\frac{3}{2},\frac{3}{2}}(\tau) + \beta_r F_{r;\frac{5}{2},\frac{5}{2}}(\tau) + \gamma_r F_{r;\frac{7}{2},\frac{3}{2}}(\tau)\right) + \dots$$

 Generalized Eisensteins appear to <u>all</u> orders in the AdS_5 and string effective actions.

[Chester, Green, Pufu, Wang, Wen]

$\mathcal{N}$=4 SYM Integrated Correlators: Finite N

What about finite N?

A solution may be inferred from SUSY localization, for all and !

This was pioneered by [Dorigoni, Green, Wen], who conjectured/computed this for p=2:

$$\mathcal{G}_2^{(N)}(\tau) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty d\xi\, B_N(\xi) \exp\left(-\pi\xi \frac{|m+n\tau|^2}{y}\right)$$

Rational function involving Jacobi polynomials, e.g.

$$B_2(\xi) = \frac{3\xi(3-10\xi+3\xi^2)}{(1+\xi)^5}$$

N>2 recursively defined by “Laplace difference equation”:

$$-(\Delta_\tau + 2)\,\mathcal{G}_2^{(N)}(\tau) = N^2\left[\mathcal{G}_2^{(N+1)}(\tau) - 2\mathcal{G}_2^{(N)}(\tau) + \mathcal{G}_2^{(N-1)}(\tau)\right] - N\left[\mathcal{G}_2^{(N+1)}(\tau) - \mathcal{G}_2^{(N-1)}(\tau)\right]$$

This recursion was later proven by [Dorigoni, Green, Wen, Xie] directly from the matrix model,

with N=2 as an initial condition.

$\mathcal{N}$=4 SYM Integrated Correlators: Finite N

The DGW result is manifestly SL(2,$\mathbb{Z}$)-invariant.

There is another SL(2,$\mathbb{Z}$)-invariant presentation of this object, which arose from some parallel developments in the broader CFT literature.

$\mathcal{N}$=4 SYM Integrated Correlators: Finite N

The DGW result is manifestly SL(2,$\mathbb{Z}$)-invariant.

There is another SL(2,$\mathbb{Z}$)-invariant presentation of this object, which arose from some parallel developments in the broader CFT literature.

How do we better understand modularity in CFT?

Use SL(2,$\mathbb{Z}$) spectral theory = Harmonic analysis on the fundamental domain

Applying to the integrated correlators leads to great simplifications, and makes physical features transparent.

Spectral Theory

A square-integrable, -invariant function admits a unique decomposition into an -invariant complete eigenbasis of the hyperbolic Laplacian.

$$L^2(\mathcal{F}) = L^2_{\text{const}}(\mathcal{F}) \oplus L^2_{\text{cont}}(\mathcal{F}) \oplus L^2_{\text{disc}}(\mathcal{F})$$

[See e.g. Terras]

Spectral Theory

A square-integrable, -invariant function admits a unique decomposition into an -invariant complete eigenbasis of the hyperbolic Laplacian.

$$L^2(\mathcal{F}) = L^2_{\text{const}}(\mathcal{F}) \oplus L^2_{\text{cont}}(\mathcal{F}) \oplus L^2_{\text{disc}}(\mathcal{F})$$

1. *Constant*: “Modular average”

$$\Delta_\tau f = 0$$

2. *Continuous*: Eisenstein series

$$\Delta_\tau E_s(\tau) = s(1-s)E_s(\tau)$$

$$s = \frac{1}{2} + it\,, \quad t \in \mathbb{R}$$

3. *Discrete*: Maass cusp forms

$$\Delta_\tau \phi_n(\tau) = \left(\frac{1}{4} + t_n^2\right)\phi_n(\tau)$$

Smooth

Chaotic

Spectral Theory

$$(f,g) := \int_{\mathcal{F}} \frac{dxdy}{y^2} f(\tau)\overline{g(\tau)}$$

A square-integrable, -invariant function admits a unique decomposition into an -invariant complete eigenbasis of the hyperbolic Laplacian.

$$\mathbb{O}(\tau) = \overline{\mathbb{O}} + \frac{1}{4\pi i}\int_{\mathrm{Res}=\frac{1}{2}} ds\,\{\mathbb{O}, E_s\}E_s^*(\tau) + \sum_{n=1}^{\infty}(\mathbb{O},\phi_n)\phi_n(\tau)$$

1. *Constant*: “Modular average”

$$\Delta_\tau f = 0$$

2. *Continuous*: Eisenstein series

$$\Delta_\tau E_s(\tau) = s(1-s)E_s(\tau)$$

$$s = \frac{1}{2} + it\,, \quad t \in \mathbb{R}$$

3. *Discrete*: Maass cusp forms

$$\Delta_\tau \phi_n(\tau) = \left(\frac{1}{4} + t_n^2\right)\phi_n(\tau)$$

Smooth

Chaotic

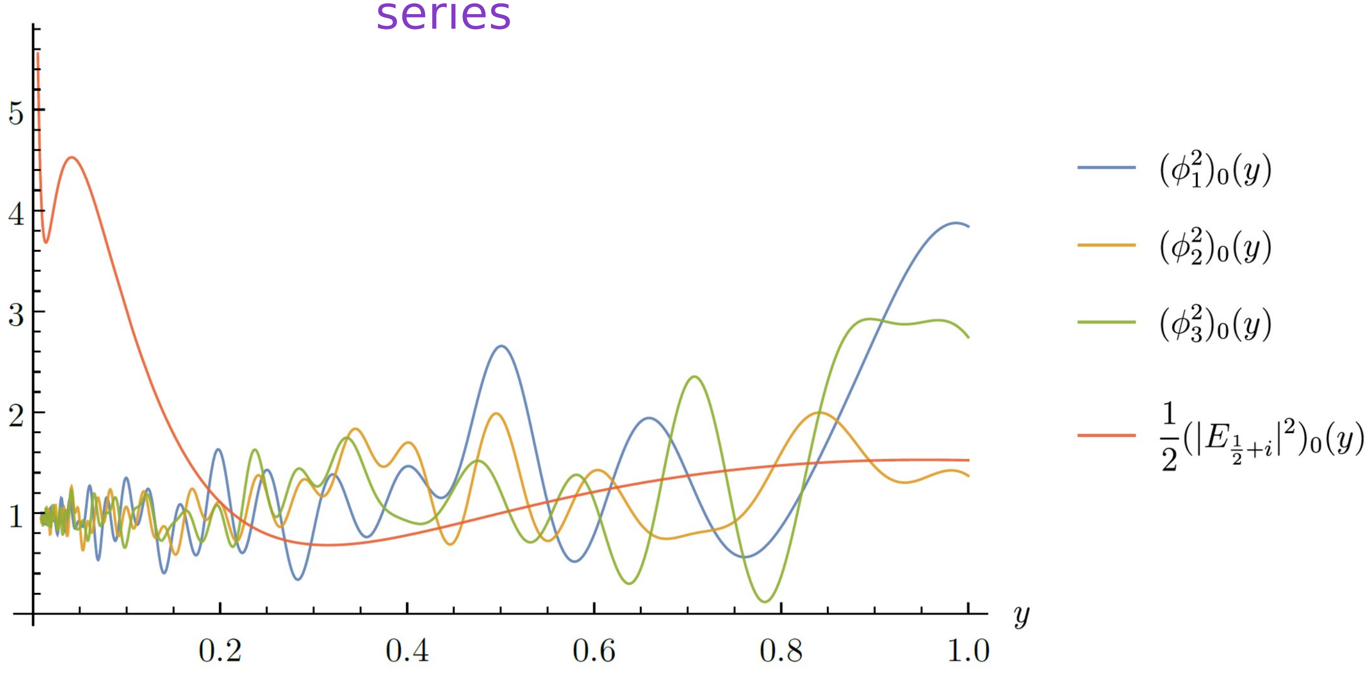
Spectral Theory

$$(f,g) := \int_{\mathcal{F}} \frac{dxdy}{y^2} f(\tau)\overline{g(\tau)}$$

A square-integrable, -invariant function admits a unique decomposition into an -invariant complete eigenbasis of the hyperbolic Laplacian.

$$\mathbb{O}(\tau) = \overline{\mathbb{O}} + \frac{1}{4\pi i}\int_{\mathrm{Re}s=\frac{1}{2}} ds\,\{\mathbb{O}, E_s\} E_s^*(\tau) + \sum_{n=1}^{\infty} (\mathbb{O},\phi_n)\phi_n(\tau)$$

1. *Constant*: “Modular average”
2. *Continuous*: Eisenstein series
3. *Discrete*: Maass cusp forms



Spectral Theory

$$(f,g) := \int_{\mathcal{F}} \frac{dxdy}{y^2} f(\tau)\overline{g(\tau)}$$

A square-integrable, -invariant function admits a unique decomposition into an -invariant complete eigenbasis of the hyperbolic Laplacian.

$$\mathbb{O}(\tau) = \overline{\mathbb{O}} + \frac{1}{4\pi i}\int_{\text{Re}s=\frac{1}{2}} ds\, \{\mathbb{O}, E_s\} E_s^*(\tau) + \sum_{n=1}^{\infty} (\mathbb{O}, \phi_n)\phi_n(\tau)$$

1. *Constant*: "Modular average"

$$\overline{\mathbb{O}} := \text{vol}(\mathcal{F})^{-1} \int_{\mathcal{F}} \frac{dxdy}{y^2} \mathbb{O}(\tau)$$

2. *Continuous*: Eisenstein series

3. *Discrete*: Maass cusp forms

Spectral Theory

$$(f,g) := \int_{\mathcal{F}} \frac{dxdy}{y^2} f(\tau)\overline{g(\tau)}$$

A square-integrable, -invariant function admits a unique decomposition into an -invariant complete eigenbasis of the hyperbolic Laplacian.

$$\mathbb{O}(\tau) = \overline{\mathbb{O}} + \frac{1}{4\pi i}\int_{\mathrm{Re}s=\frac{1}{2}} ds\, \{\mathbb{O}, E_s\} E_s^*(\tau) + \sum_{n=1}^{\infty} (\mathbb{O}, \phi_n)\phi_n(\tau)$$

1. *Constant*: "Modular average"

2. *Continuous*: **Eisenstein series**

3. *Discrete*: Maass cusp forms

- Fourier decomposition: $E_s^*(\tau) := \Lambda(s) E_s(\tau)$

$$= \Lambda(s) y^s + \Lambda(1-s) y^{1-s} + \sum_{k=1}^{\infty} 4\cos(2\pi k x) \frac{\sigma_{2s-1}(k)}{k^{s-\frac{1}{2}}} \sqrt{y} K_{s-\frac{1}{2}}(2\pi k y)$$

- Functional equations: $E_s^*(\tau) := E_{1-s}^*(\tau)$

Completed Riemann zeta

$\Lambda(s) := \pi^{-s}\Gamma(s)\zeta(2s)$

- Overlap is a Mellin integral of **zero mode**:

$$(\mathbb{O}, E_s) = \int_0^{\infty} dy\, y^{-s-1} \mathbb{O}_0(y) := \Lambda(s)\{\mathbb{O}, E_s\}$$

[Rankin, Selberg]

Spectral Theory

$$(f,g) := \int_{\mathcal{F}} \frac{dxdy}{y^2} f(\tau)\overline{g(\tau)}$$

A square-integrable, -invariant function admits a unique decomposition into an -invariant complete eigenbasis of the hyperbolic Laplacian.

$$\mathbb{O}(\tau) = \overline{\mathbb{O}} + \frac{1}{4\pi i}\int_{\mathrm{Res}=\frac{1}{2}} ds\,\{\mathbb{O}, E_s\}E_s^*(\tau) + \sum_{n=1}^{\infty}(\mathbb{O},\phi_n)\phi_n(\tau)$$

1. *Constant*: "Modular average"

2. *Continuous*: Eisenstein series

3. *Discrete*: Maass cusp forms

Maass cusp forms are the most interesting eigenfunctions.

- Functionally similar to Eisenstein series…
- … but they vanish at the cusp…
- … and are infinite in number, but **none** known analytically
  - *Small n*: numerics (~10^3 digits)
  - *Large n*: universality

$$\phi_n(\tau) = \sum_{k=1}^{\infty} a_k^{(n)} \cos(2\pi kx)\sqrt{y}K_{it_n}(2\pi ky)$$

$$\phi_n(\tau) \sim e^{-2\pi y} \qquad (y\to\infty)$$

[Hejhal; Then; Booker, Strombergsson, Venkatesh; Sarnak; …]

Maass form on $\Gamma_0(1)$ with $R = 13.7797513519$

The even Maass form on $SL(2, \mathbb{Z})$ with the smallest eigenvalue.

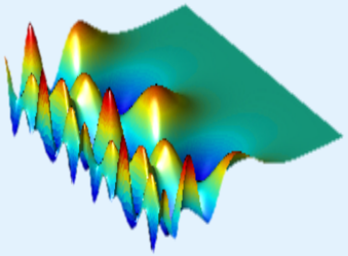
Maass form invariants

Level: 1
Weight: 0
Character: 1.1
Symmetry: even
Fricke sign: $+1$
Spectral parameter: 13.7797513519

Maass form coefficients

$a_{1} = +1.000000000$	$a_{2} = +1.549304478$	$a_{3} = +0.246899772$	$a_{4} = +1.400344365$	$a_{5} = +0.737060385$
$a_{6} = +0.382522923$	$a_{7} = -0.261420076$	$a_{8} = +0.620255318$	$a_{9} = -0.939040503$	$a_{10} = +1.141930962$
$a_{11} = -0.953564653$	$a_{12} = +0.345744705$	$a_{13} = +0.278827029$	$a_{14} = -0.405019294$	$a_{15} = +0.181980041$
$a_{16} = -0.439380024$	$a_{17} = +1.307341715$	$a_{18} = -1.454859655$	$a_{19} = +0.092558583$	$a_{20} = +1.032138358$
$a_{21} = -0.064544557$	$a_{22} = -1.477361986$	$a_{23} = +1.138068521$	$a_{24} = +0.153140897$	$a_{25} = -0.456741988$
$a_{26} = +0.431987965$	$a_{27} = -0.478748659$	$a_{28} = -0.366078130$	$a_{29} = +0.752113845$	$a_{30} = +0.281942493$
$a_{31} = +0.024851954$	$a_{32} = -1.300988756$	$a_{33} = -0.235434896$	$a_{34} = +2.025470373$	$a_{35} = -0.192682382$
$a_{36} = -1.314980076$	$a_{37} = +0.199265656$	$a_{38} = +0.143401426$	$a_{39} = +0.068842330$	$a_{40} = +0.457165624$
$a_{41} = -0.304032997$	$a_{42} = -0.099999172$	$a_{43} = +0.783239364$	$a_{44} = -1.335318888$	$a_{45} = -0.692129555$
$a_{46} = +1.763214656$	$a_{47} = +0.360568411$	$a_{48} = -0.108482828$	$a_{49} = -0.931659544$	$a_{50} = -0.707632408$
$a_{51} = +0.322782372$	$a_{52} = +0.390453860$	$a_{53} = +1.398065718$	$a_{54} = -0.741727442$	$a_{55} = -0.702834729$
$a_{56} = -0.162147187$	$a_{57} = +0.022852693$	$a_{58} = +1.165253349$	$a_{59} = -1.587730962$	$a_{60} = +0.254834726$

Properties



Level	1
Weight	0
Character	1.1
Symmetry	even
Fricke sign	$+1$

Related objects

L-function

Downloads

Coefficients to text
All stored data to text
Underlying data

Learn more

Source and acknowledgments
Completeness of the data
Reliability of the data

Spectral Theory

Why are the cusp forms so elusive?

They are chaotic. (“Arithmetic chaos”, not RMT.)

Leads to sporadic behavior and large n universality.

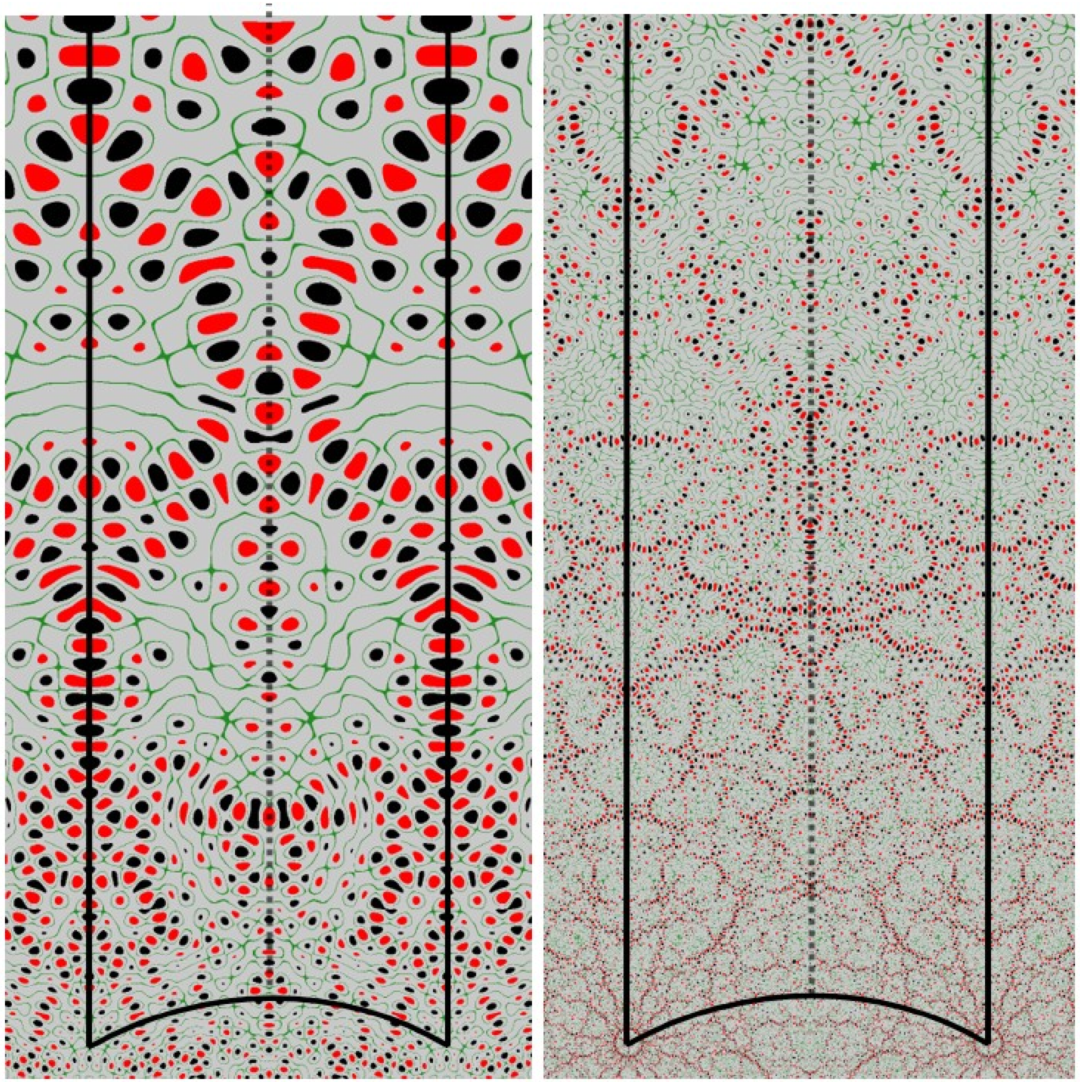
Spectral Theory

Why are the cusp forms so elusive?

They are **chaotic**. (“Arithmetic chaos”, not RMT.)

Leads to sporadic behavior and large n universality.

Spectral parameter Spectral parameter



[Hejhal, Rackner; Then]

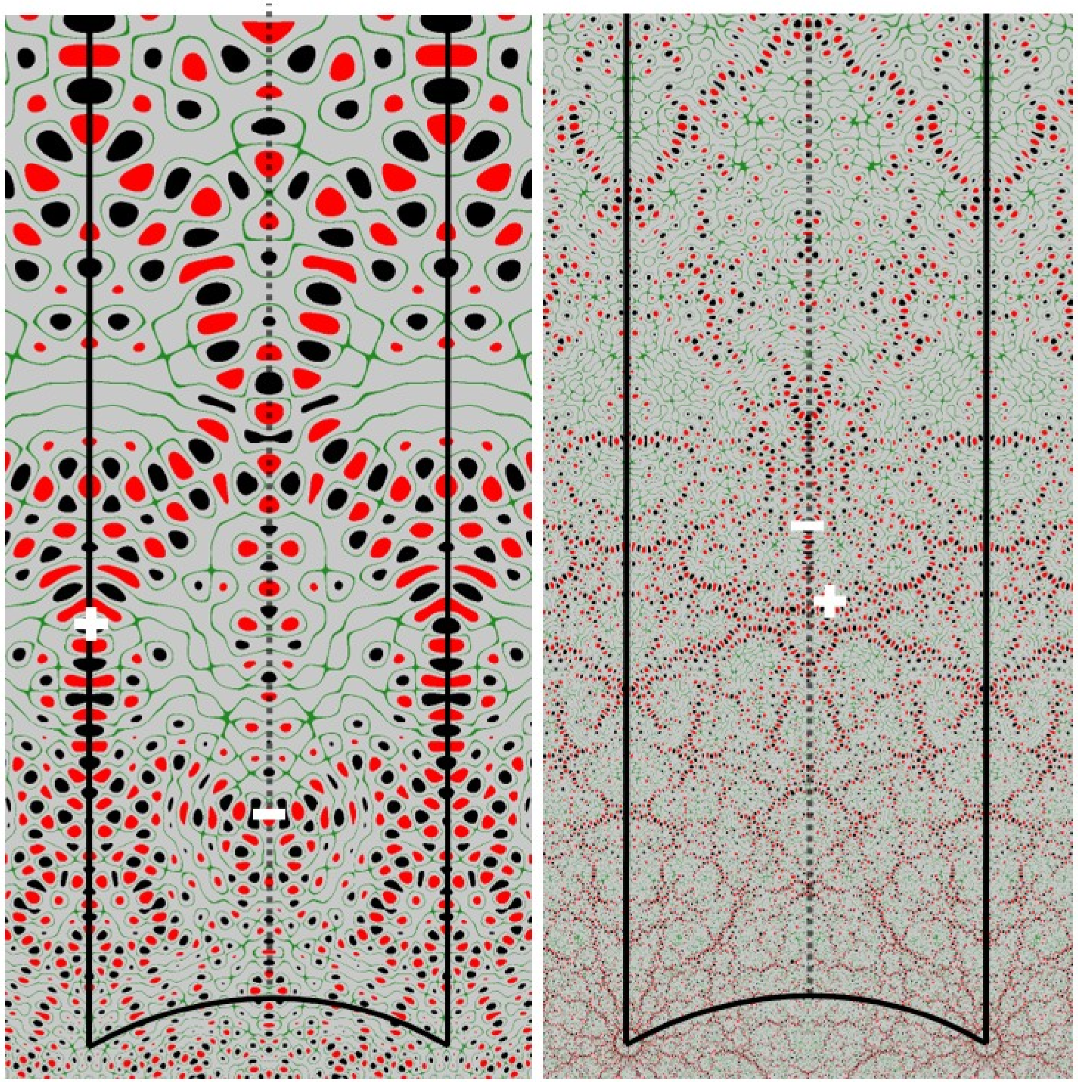
Spectral Theory

Why are the cusp forms so elusive?

They are chaotic. (“Arithmetic chaos”, not RMT.)

Leads to **sporadic** behavior and large n universality.

Spectral parameter　　Spectral parameter



[Hejhal, Rackner; Then]

Spectral Theory

Why are the cusp forms so elusive?

They are chaotic. ("Arithmetic chaos", not RMT.)

Leads to sporadic behavior and large n **universality**.

Random Wave Conjecture

Gaussian Moments Conjecture

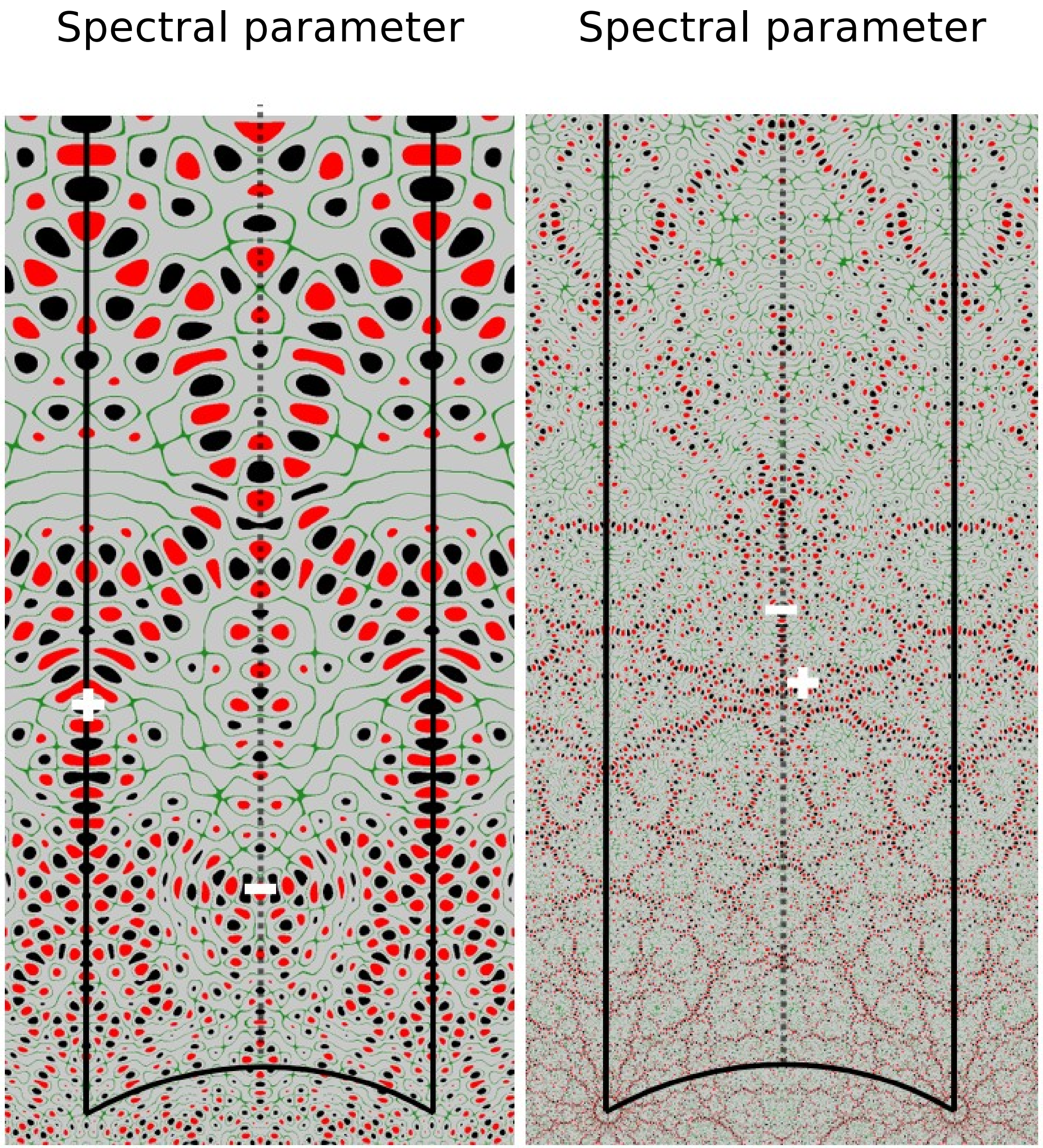
[Sarnak et al; …]

Quantum Unique Ergodicity

Ramanujan Conjecture

…

Understanding this chaos in CFT: still in progress!



[Hejhal, Rackner; Then]

Spectral Decomposition in CFT

The power of modularity is its ability to relate “UV” data to “IR” data.

In 2d CFT, relates low-E and high-E spectral densities (Cardy formula)

Normally, modularity of Z() is obscured while character expansion is manifest.

The spectral decomposition reverses this.

[Benjamin, Collier, Fitzpatrick, Maloney, EP]

Beware: Z(τ) is not square-integrable. Apply with care!

$$Z_{\text{primary}}(y \to \infty) \sim e^{\frac{\pi(c-1)}{6}y} \qquad Z_{\text{primary}}(y \to \infty) \sim y^{c/2}$$

Virasoro CFT — U(1)^c Narain CFT

[Benjamin, Chang; Haehl, Marteau, Reeves, Rozali; Benjamin, Collier, Kruthoff, Verlinde, Zhang; Di Ubaldo, EP]

Interesting connections to random matrix statistics and wormholes in AdS$_3$ gravity.

[Luo, Wang]

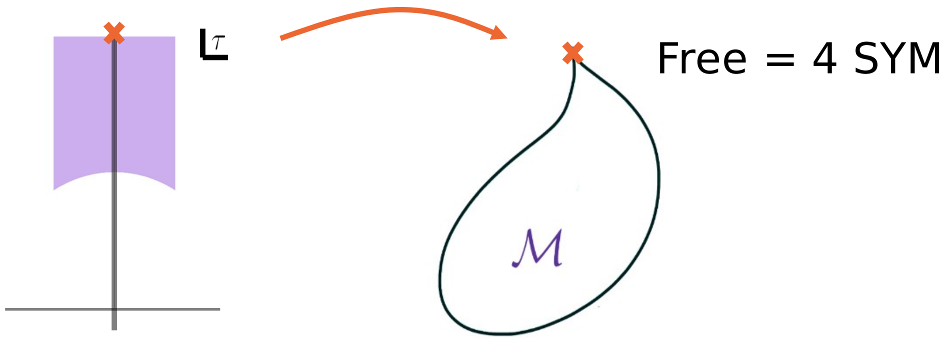
Many further CFT applications await

Spectral Decomposition of = 4 SYM

The spectral decomposition fits like a glove in = 4 SYM.

In any CFT at finite coupling, well-defined observables are finite, modulo possible divergences at boundaries of moduli space.

In = 4 SYM, the cusp just maps to the free theory, where observables converge to their free values.



Therefore, = 4 SYM observables admit a spectral decomposition.

[Collier, EP]

Spectral Decomposition of = 4 SYM

$$\mathbb{O}(\tau) = \overline{\mathbb{O}} + \frac{1}{4\pi i}\int_{\mathrm{Re}s=\frac{1}{2}} ds\,\{\mathbb{O}, E_s\}E_s^*(\tau) + \sum_{n=1}^{\infty}(\mathbb{O}, \phi_n)\phi_n(\tau)$$

This is a **complete basis**. The overlaps *are* the observable.

$$\{\mathbb{O}, E_s\}, \quad (\mathbb{O}, \phi_n)$$

Resist the temptation to revert to -space!

The Analytic Structure of = 4 SYM

In = 4 SYM, Fourier number = Instanton number:

$$\mathbb{O}(\tau) = \mathbb{O}_0(y) + \sum_{k\neq 0}^{\infty} e^{2\pi i k x}\, \mathbb{O}_k(y)$$

k = total instanton number

$$(q^n \bar{q}^{n+k} + \text{c.c.})$$

An obvious constraint: consistent perturbation theory.

(N.B. insensitive to cusp forms.)

$$\mathbb{O}_0(y) = \overline{\mathbb{O}} + \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds\, \{\mathbb{O}, E_s\}\Lambda(s) y^s$$

To develop large y (small g²) expansion, deform to left. Demand no logs, only integer powers of

$$\{\mathbb{O}, E_s\} = \frac{\pi}{\sin \pi s} s(1-s) f_{\mathrm{p}}(s) + f_{\mathrm{np}}(s)$$

Perturbative, ~ :
Values on integers encode weak coupling data

Non-perturbative, ~ :
Instanton-anti-instanton corrections

These functions are reflection symmetric (s 1-s) and real (for real s).

[Collier, EP]

SL(2,ℤ) Spectral Decomposition of Integrated Correlators

Back to integrated correlators.

$\mathcal{N}$=4 SYM Integrated Correlators: Finite N

What about finite N?

A solution may be inferred from SUSY localization, for all N, p and τ!

This was pioneered by [Dorigoni, Green, Wen], who conjectured/computed this for p=2:

$$\mathcal{G}_2^{(N)}(\tau) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty d\xi \, B_N(\xi) \exp\left(-\pi\xi \frac{|m+n\tau|^2}{y}\right)$$

Rational function involving Jacobi polynomials, e.g.

$$B_2(\xi) = \frac{3\xi(3-10\xi+3\xi^2)}{(1+\xi)^5}$$

N>2 recursively defined by "Laplace difference equation":

$$-(\Delta_\tau+2)\,\mathcal{G}_2^{(N)}(\tau) = N^2\left[\mathcal{G}_2^{(N+1)}(\tau) - 2\mathcal{G}_2^{(N)}(\tau) + \mathcal{G}_2^{(N-1)}(\tau)\right] - N\left[\mathcal{G}_2^{(N+1)}(\tau) - \mathcal{G}_2^{(N-1)}(\tau)\right]$$

This recursion was later proven by [Dorigoni, Green, Wen, Xie] directly from the matrix model, with N=2 as an initial condition.

Now pass to the SL(2,ℤ) spectral basis. What happens?

SL(2,$\mathbb{Z}$) Spectral Decomposition of Integrated Correlators

The integrated correlators become *polynomials*!

[Collier, EP; Paul, EP, Raj]

SL(2,$\mathbb{Z}$) Spectral Decomposition of Integrated Correlators

The integrated correlators become *polynomials*!

1 Cusp overlap vanishes $$(\mathcal{G}_2^{(N)}, \phi_n) = 0$$

[Collier, EP; Paul, EP, Raj]

SL(2,$\mathbb{Z}$) Spectral Decomposition of Integrated Correlators

The integrated correlators become *polynomials*!

1. Cusp overlap vanishes

$$(\mathcal{G}_2^{(N)}, \phi_n) = 0$$

2. Non-perturbative overlap vanishes

$$\{\mathcal{G}_2^{(N)}, E_s\} = \frac{\pi}{\sin \pi s} s(1-s) f_{2,p}^{(N)}(s) + \cancel{f_{2,\mathrm{np}}^{(N)}(s)}$$

[Collier, EP; Paul, EP, Raj]

SL(2,ℤ) Spectral Decomposition of Integrated Correlators

The integrated correlators become *polynomials*!

1. Cusp overlap vanishes $$(\mathcal{G}_2^{(N)}, \phi_n) = 0$$

2. Non-perturbative overlap vanishes $$\{\mathcal{G}_2^{(N)}, E_s\} = \frac{\pi}{\sin \pi s} s(1-s) f_{2,p}^{(N)}(s) + \cancel{f_{2,\text{np}}^{(N)}(s)}$$

3. Perturbative overlap is closed-form polynomial $$f_{2,p}^{(N)}(s) = \frac{N(N-1)}{2}(2s-1)^2 {}_3F_2(2-N, s, 1-s; 3, 2; 1)$$

The SU(2) integrated correlator is literally the simplest possible non-trivial overlap for a SYM observable.

[Collier, EP; Paul, EP, Raj]

SL(2,$\mathbb{Z}$) Spectral Decomposition of Integrated Correlators

This polynomial structure extends to all 22pp integrated correlators:

$$\mathcal{G}_p^{(N)}(\tau) = \overline{\mathcal{G}_p^{(N)}} + \frac{1}{4\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds\, \frac{\pi}{\sin \pi s} s(1-s)(2s-1)^2 g_p^{(N)}(s) E_s^*(\tau)$$

Even polynomial of degree $2N + 2\left\lfloor \frac{p}{2} \right\rfloor - 6$

Behold the power of polynomiality:

The perturbative expansion from the zero mode is

$$\mathcal{G}_{p,0}^{(N)}(y) = \overline{\mathcal{G}_p^{(N)}} + \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds\, \frac{\pi}{\sin \pi s} s(1-s)(2s-1)^2 g_p^{(N)}(s) \Lambda(s) y^s$$

⟹ *The integrated correlator is completely* ***fixed*** *by the first* $N + \left\lfloor \frac{p}{2} \right\rfloor - 2$ *orders in perturbation theory.*

“Instanton redundancy”: *very* strong form of resurgence

[Collier, EP; Paul, EP, Raj]

SL(2,$\mathbb{Z}$) Spectral Decomposition of Integrated Correlators

A nice choice for : $$\mathcal{O}_p := [\mathcal{O}_2]^{p/2}\,, \quad p \in 2\mathbb{Z}_+$$ ("maximal trace") [Paul, EP, Raj]

Exact solution for all and … just hypergeometric functions in spectral basis (cf. Hynek Paul's talk)

SL(2,ℤ) Spectral Decomposition of Integrated Correlators

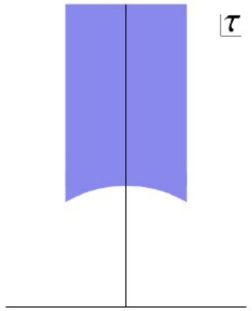
A nice choice for : $\mathcal{O}_p := [\mathcal{O}_2]^{p/2}\,, \quad p \in 2\mathbb{Z}_+$ (“maximal trace”) [Paul, EP, Raj]

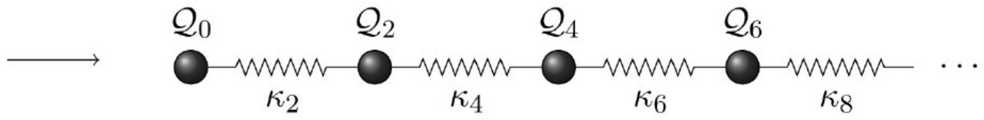
Exact solution for all and … just hypergeometric functions in spectral basis (cf. Hynek Paul’s talk)

Integrated correlators = Simple harmonic oscillators evolving over

$$\Delta_\tau\, \mathcal{Q}^{(N)}_{p-2}(\tau) = -\kappa_p \left(\mathcal{Q}^{(N)}_{p}(\tau) - \mathcal{Q}^{(N)}_{p-2}(\tau)\right) + \kappa_{p-2} \left(\mathcal{Q}^{(N)}_{p-2}(\tau) - \mathcal{Q}^{(N)}_{p-4}(\tau)\right)$$

(Evocative of Toda chain for SQCD extremal correlators)





Coupling

$$\kappa_p := \frac{p}{4}\left(N^2 + p - 3\right)$$

Shifted correlator

$$\mathcal{Q}^{(N)}_p(\tau) := \mathcal{G}^{(N)}_p(\tau) - \frac{N^2-1}{2}\,\Delta_\tau^{-1}\mathcal{G}^{(N)}_2(\tau)$$

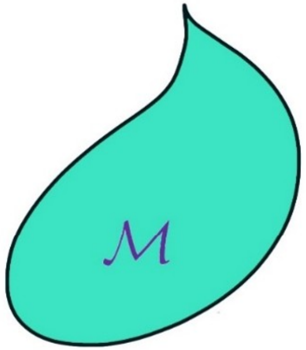
Ensemble averages in = 4 SYM

What about the constant term?

$$\mathbb{O}(\tau) = \overline{\mathbb{O}} + \frac{1}{4\pi i}\int_{\mathrm{Res}=\frac{1}{2}} ds\,\{\mathbb{O}, E_s\}E_s^*(\tau) + \sum_{n=1}^{\infty}(\mathbb{O},\phi_n)\phi_n(\tau)$$

In a general CFT with a conformal manifold , we may define an average over exactly marginal couplings.

$$\langle \mathbb{O} \rangle := \int d\mu_{\mathcal{M}}(\lambda)\,\mathbb{O}(\lambda)\,, \quad \text{where} \quad d\mu_{\mathcal{M}}(\lambda) := g^{Z}_{\mu\nu}(\lambda)d\lambda^\mu d\lambda^\nu$$



A natural choice of measure is the Zamolodchikov measure.

In = 4 SYM, thanks to maximal SUSY, the Zamolodchikov metric is the hyperbol

Ensemble average

$$\langle \mathbb{O} \rangle = \overline{\mathbb{O}}$$

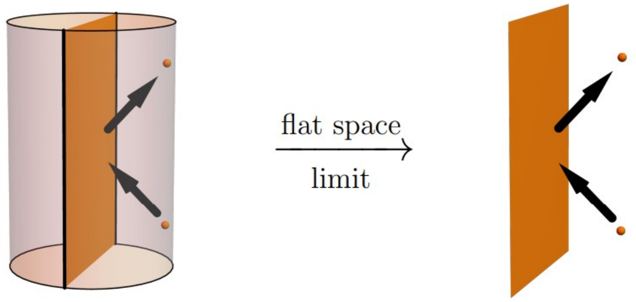
Modular average

$$\langle \mathcal{G}_2^{(N)} \rangle = \frac{N(N-1)}{4} \qquad \langle \mathcal{G}_p^{(N)} \rangle = \langle \mathcal{G}_2^{(N)} \rangle \left(H_{\frac{N^2+p-3}{2}} - H_{\frac{N^2-3}{2}} \right)$$

For maximal trace integrated correlators, This application of $\mathrm{SL}(2,\mathbb{Z})$ spectral theory suggests a rethinking of coupling-dependence in $\mathcal{N} = 4$ SYM.

Integrated Correlators: Extensions

Much more has been done…

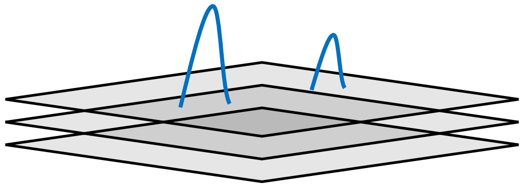


- Integrated correlators on defects (two-point functions)

  [Drukker, Kong, Sakkas; Cavaglia, Gromov, Julius, Preti; Pufu, Rodriguez, Wang]

- Integrated gluon correlators Worldvolume gauge vertices

  [Behan, Chester, Ferrero]



$\sim F^4 f(\tau)$

… and can be done (!?):

- Integrated four-point functions in SQCD

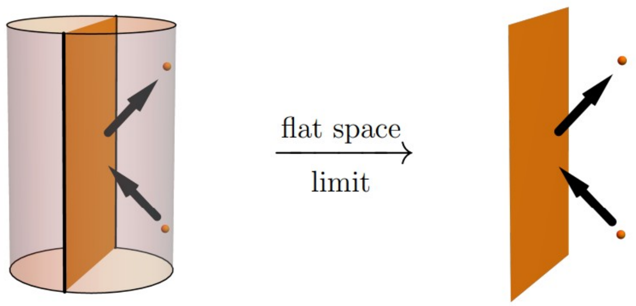
  [Chester; Fiol, Kong]

- Integrated four-point functions in d=2,3 SCFTs (cf. Val Reys' talk)

- Beyond vacuum correlators

Integrated Correlators: Extensions

Much more has been done…

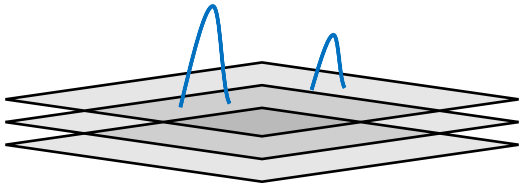


- Integrated correlators on defects (two-point functions)

  [Drukker, Kong, Sakkas; Cavaglia, Gromov, Julius, Preti; Pufu, Rodriguez, Wang]

- Integrated gluon correlators Worldvolume gauge vertices

  [Behan, Chester, Ferrero]



$$\sim F^4 f(\tau)$$

… and can be done (!?):

- Integrated four-point functions in SQCD

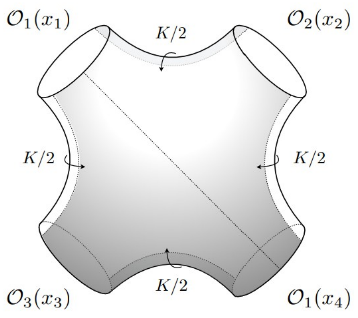
  [Chester; Fiol, Kong]

- Integrated four-point functions in d=2,3 SCFTs (cf. Val Reys' talk)

- **Beyond vacuum correlators**

Exact Large Charge & Semiclassical String Theory

Let us study large charge, p >> 1. Why?



1) Large quantum number expansions are useful and interesting…

- $\mathcal{N}$=4 SYM octagon

$$\langle pppp \rangle \sim [\mathbb{O}(z, z^{-1})]^2 \sim -\frac{\log^2 z}{2\pi^2}\left[\log\cosh\left(\frac{\lambda}{8\pi}\right)\right] + \ldots \quad (p \sim \sqrt{N} \to \infty)$$

[Coronado; Belitsky, Korchemsky; Kostov, Petkova, Serban; …]

- Large charge EFT in CFT_d

[Hellerman, …]

- Extremal two-point functions <pp> in $\mathcal{N}$=2 SQCD ("'t Hooft-like limit")

$$\log\left(\frac{\langle \mathrm{Tr}(\phi^2)^n \mathrm{Tr}(\phi^2)^n \rangle}{\mathcal{N}_n}\right) = \sum_{g=0}^{\infty} n^{-g}\mathcal{F}_g^{(N)}(\lambda_n) + (\text{non-pert in } n) \quad (n \to \infty\,,\ \lambda_n := g_{\mathrm{YM}}^2 n \text{ fixed})$$

[Bourget, Rodriguez-Gomez, Russo; Beccaria; Maeda, Hellerman; Grassi, Komargodski, Tizzano; …]

2) We can probe string theory in backreacted geometries

Exact Large Charge & Semiclassical String Theory

How? Integrated <22pp> at large p, with $\mathcal{O}_p := [\mathcal{O}_2]^{p/2}\,, \quad p \in 2\mathbb{Z}_+$

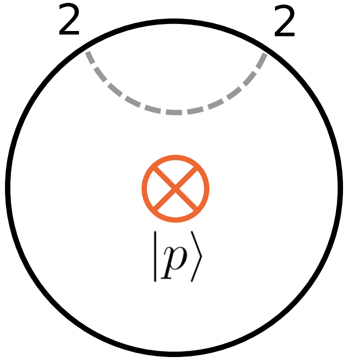
[Paul, EP, Raj]
[Brown, Wen, Xie]

Explore solution in the "*Gravity Regime*":

$$p \gg 1\,, \quad N \gg 1\,, \quad \alpha := \frac{p}{N^2} \text{ fixed}$$

Becomes an integrated + supersymmetric **heavy-heavy-light-light** correlator (HHLL)

$$\int dz d\bar{z}\, \mu(z,\bar{z})\, \langle \mathcal{O}_p(0)\mathcal{O}_2(z,\bar{z})\mathcal{O}_2(1)\mathcal{O}_p(\infty)\rangle = \int dz d\bar{z}\, \mu(z,\bar{z})\, \langle p|\mathcal{O}_2(z,\bar{z})\mathcal{O}_2(1)|p\rangle$$



 Predictions for *string* theory (finite α') in backreacted geometry

e.g. instanton effects ~ $e^{-2\sqrt{\lambda/R_\alpha}}\,, \quad R_\alpha := 1 + 2\alpha - 2\sqrt{\alpha(\alpha+1)}$

Large charge screening factor of AdS_5 x S^5 semiclassical string theory

Closing thoughts

Are we any closer to the full AdS_5 x $\boldsymbol{S^5}$ tree-level string amplitude?

[Chen, Elvang, Herderschee]

[Schlotterer, Stieberger]

We have everything we need: 10d string *n*-point functions + exact AdS_5 x $\mathbf{S^5}$ string background.

Suggestion: planar 4-point functions are more likely to be solved in closed-form than 3-point functions.

[Maldacena, Ooguri; Dei, Eberhardt; Bufalini, Iguri, Kovensky]

(Side note: we **do** know the "AdS-Virasoro-Shapiro" amplitude in AdS_3 pure-NS case.)

More generally in = 4 SYM, it seems wise to try to harness S-duality as efficiently as possible.

Perhaps SL(2,$\mathbb{Z}$) spectral theory and superconformal bootstrap can be fruitfully combined.

Closing thoughts

Are we any closer to the full AdS_5 x $\boldsymbol{S^5}$ *tree-level string amplitude?*

[Chen, Elvang, Herderschee]

[Schlotterer, Stieberger]

We have everything we need: 10d string *n*-point functions + exact AdS_5 x $\mathbf{S^5}$ string background.

Suggestion: planar 4-point functions are more likely to be solved in closed-form than 3-point functions.

[Maldacena, Ooguri; Dei, Eberhardt; Bufalini, Iguri, Kovensky]

(Side note: we **do** know the "AdS-Virasoro-Shapiro" amplitude in AdS_3 pure-NS case.)

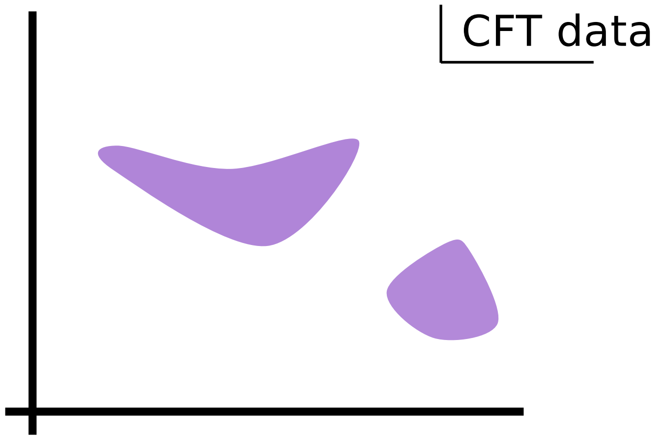
More generally in = 4 SYM, it seems wise to try to harness S-duality as efficiently as possible.

Perhaps SL(2,$\mathbb{Z}$) spectral theory and superconformal bootstrap can be fruitfully combined.

Thank You!

Why Ensemble Averages?

Answer #1:



The set of bootstrap solutions within some CFT universality class is a (generalized) kind of of ensemble.

What are *statistics* of CFT data within these islands?

What is a *typical* theory?

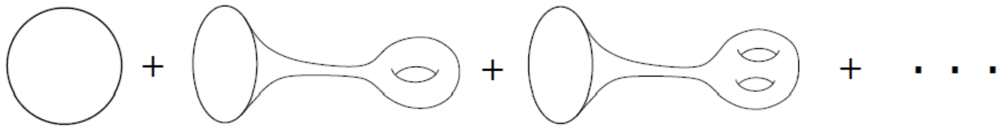
CFTs with exactly marginal couplings have a more literal notion of ensemble.

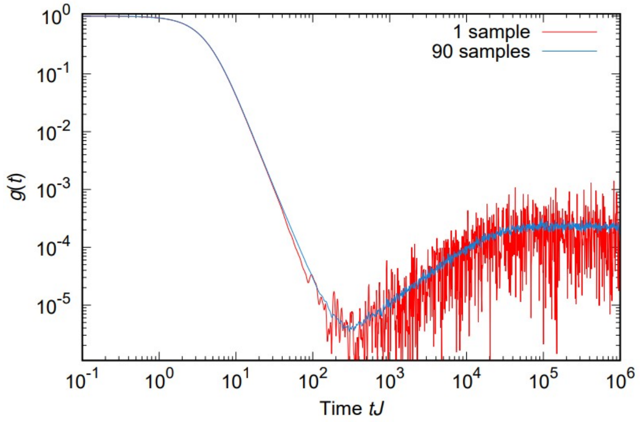
- What are the statistics of CFT ensembles defined by conformal manifolds ?
- How do these statistics relate to other properties of (e.g. compactness)?
- …

Why Ensemble Averages?

Answer #2: Recent developments in low-dimensional AdS/CFT establish a duality between random matrix ensembles and simple theories of d=2 gravity (JT/RMT duality).

[Saad, Shenker, Stanford]





Even in d>2, semiclassical Einstein gravity may, in a statistical/universality sense, be a theory of averages.

How this is compatible with 25 years of successful checks of AdS/CFT at large N, *without* averaging, has been a subject of much recent work.

[Schlenker, Witten; Chandra, Collier, Hartman, Maloney; Collier, EP]

In $\mathcal{N} = 4$ SYM at large N, there is a rigorous version of these statements.

Ensemble Averages at Large N

Consider the ‘t Hooft double-scaling limit, $N \to \infty\,,\ g^2 \to 0\,,\ \lambda := g^2 N$ fixed

Genus expansion: $$\mathbb{O}(N,\lambda) = \sum_{g=0}^{\infty} N^{2-2g}\,\mathbb{O}_0^{(g)}(\lambda) + (\mathrm{NP})$$

(Nonzero modes and, therefore, cusp forms are exponentially suppressed in N.)

In the spectral decomposition, $$\mathbb{O}(N,\lambda) = \langle \mathbb{O} \rangle + \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\frac{1}{2}} ds\,\{\mathbb{O}, E_s\} E^*_{s,0}(y)$$

Develop genus expansion

Write in terms of N and

Result: everything from spectral integral is suppressed at large N and large !

$$\mathbb{O}(\lambda \gg 1) \approx \langle \mathbb{O} \rangle + (\text{subleading in } 1/N \text{ and } 1/\lambda)$$

By standard AdS/CFT duality, the LHS is equivalent to the $AdS_5 \times S^5$ supergravity value.

Supergravity as an Average

An equivalence between large N ensemble averaging and strong coupling in planar = 4 SYM:

$$\mathbb{O}(\lambda \to \infty) = \langle \mathbb{O} \rangle = \mathbb{O}_{\text{sugra}}$$

- *The traditional holographic correspondence still holds.*
- The ensemble average is emergent at strong coupling and large N.
- Applies to observables with a genus expansion – double-trace dimensions, KK correlators, …

e.g. Integrated correlator: $\mathcal{G}_p^{(g=0)}(\lambda) = \frac{p-1}{2p}\left(1 - O(\lambda^{-3/2})\right)$ [Binder, Chester, Pufu, Wang]

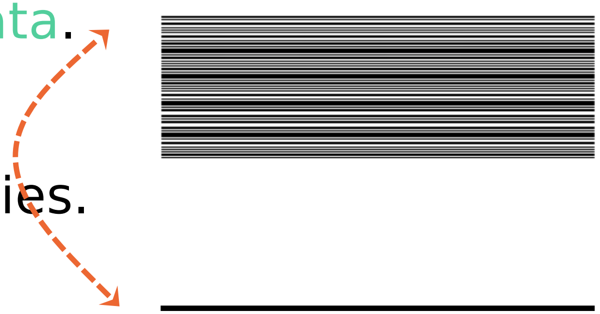
$$\langle \mathcal{G}_p^{(N)} \rangle = \frac{p-1}{2p} - \frac{(p-1)}{N} - \frac{(p^2-1)(p^2-4)}{48N^2} + \dots$$ [Paul, EP, Raj]

- Extends to all genera:

$$\mathbb{O}^{(g)}(\lambda \to \infty) = \langle\!\langle \mathbb{O}^{(g)} \rangle\!\rangle := \lim_{N\to\infty} N^{2g-2} \langle \mathbb{O}^{(g)} \rangle$$

This is the finite term remaining after string theory regularization of UV divergences of g-loop supergravity.

The power of modularity is its ability to relate “UV” data to “IR” data.



In 2d CFT, -invariance relates low- and high-energy spectral densities.

In gauge theory, maps strong coupling to “dual” weakly coupled description.

Solitons become elementary particles, black holes become vacua, …

Grand canonical partition functions for families of 2d CFTs can have modular symmetries acting on the potential conjugate to the number of d.o.f.

These symmetries relate CFTs with small and large central charge.

What is the “fundamental domain” of the space of conformal field theories?