Symmetries of nuclear models

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Symmetries in nuclei

Symmetries of the nuclear shell model

- Isospin and SU(2)
- Deformation and SU(3)
- Seniority and SU(2)

An application of seniority: effective operators in a single-j shell.

PhyNuBe meeting, Aussois, March 2023

Symmetry in quantum mechanics

Assume a Hamiltonian *H* which commutes with operators g_i that form a Lie algebra *G*:

$$\forall \hat{g}_i \in G: \quad \left[\hat{H}, \hat{g}_i\right] = 0$$

∴ *H* has symmetry *G* or is invariant under *G*.

Lie algebra: a set of (infinitesimal) operators that closes under commutation.

Consequences of a symmetry:

Degeneracies in the energy spectrum

Eigenstates that carry conserved quantum numbers

Symmetries of nuclear models

Heisenberg (1932): isospin SU(2) → Lenzi, Sorlin

Wigner (1937): spin-isospin SU(4) → Shen

Racah (1943): seniority SU(2) → Valiente-Dobon, PVI

Elliott (1958): rotation SU(3) → Nowacki

Arima & Iachello (1976): IBM U(6) → nobody

Dynamical symmetry: conserved labels but no degeneracy.

Isospin symmetry in nuclei

Empirical observations:

About equal masses of n(eutron) and p(roton).

n and p have isospin 1/2.

Equal (to about 1%) nn, np, pp strong forces.

This suggests an isospin SU(2) symmetry of the nuclear Hamiltonian:

$$\left[\hat{H}_{\text{nucl}}, \hat{T}_\nu\right] = 0, \quad \hat{T}_\nu = \sum_{k=1}^{A} \hat{t}_\nu(k)$$

$$\text{n}: \quad t = \tfrac{1}{2}, m_t = +\tfrac{1}{2}; \qquad \text{p}: \quad t = \tfrac{1}{2}, m_t = -\tfrac{1}{2}$$

$$\Rightarrow \quad \hat{t}_+\text{n} = 0, \quad \hat{t}_+\text{p} = \text{n}, \quad \hat{t}_-\text{n} = \text{p}, \quad \hat{t}_-\text{p} = 0, \quad \hat{t}_z n = \tfrac{1}{2}\text{n}, \quad \hat{t}_z p = -\tfrac{1}{2}\text{p}$$

W. Heisenberg, Z. Phys. **77** (1932) 1

E.P. Wigner, Phys. Rev. **51** (1937) 106

Supermultiplet/SU(4) model

Two assumptions:

The forces between nucleons are independent of spin and isospin => SU(4) symmetry.

The n-n interaction is short-range attractive.

Consequences:

Many-nucleon states can be classified according to their spatial or spin-isospin symmetry.

States with highest spatial symmetry are lowest in energy.

E.P. Wigner, Phys. Rev. **51** (1937) 106
F. Hund, Z. Phys. **105** (1937) 202

Supermultiplet/SU(4) model

Particle number	Spatial symmetry	L	Spin–isospin symmetry	$(\lambda\mu\nu)$	(S, T)
1	$\square$	0, 2	$\square$	(100)	$(\frac{1}{2}, \frac{1}{2})$
2	$\square\square$ (S)	$0^2, 2^2, 4$	$\begin{array}{c}\square\\\square\end{array}$ (A)	(010)	(0,1) (1,0)
	$\begin{array}{c}\square\\\square\end{array}$ (A)	1, 2, 3	$\square\square$ (S)	(200)	(0,0) (1,1)

Example: one nucleon & two nucleons in the *sd* shell.

Wigner binding energy

Introduce favoured (i.e., lowest-energy) SU(4) labels in the eigenvalue expression:

$$(N-Z)^2+8|N-Z|+8\delta_{N,Z}\pi_{\mathrm{np}}+6\delta_{\mathrm{pairing}}(N,Z)$$

with $\delta_{\mathrm{pairing}}(N,Z)$=0, 1, 2 for even-even, odd-mass, odd-odd, and π_{np}=1 for odd-odd.

Compare with Wigner binding energy:

$$B_{\mathrm{W}}(N,Z)=-W(A)|N-Z|-d(A)\delta_{N,Z}\pi_{\mathrm{np}}$$

Breaking of SU(4) symmetry

- SU(4) symmetry breaking as a consequence of
  - *Spin-orbit term in nuclear mean field.*
  - *Coulomb interaction.*
  - *Spin-dependence of the nuclear interaction.*
- Evidence for SU(4) symmetry breaking from masses and from Gamow-Teller β decay.

Elliott's SU(3) model of rotation

Harmonic oscillator (*no* spin-orbit coupling) plus a residual quadrupole interaction:

$$\hat{H} = \sum_{k=1}^{A}\left[\frac{p_k^2}{2m} + \frac{1}{2}m\omega^2 r_k^2\right] - g_2 \hat{Q}\cdot\hat{Q}, \quad \hat{Q}_\mu = \sum_{k=1}^{A} r_k^2 Y_{2\mu}(\hat{\boldsymbol{r}}_k)$$

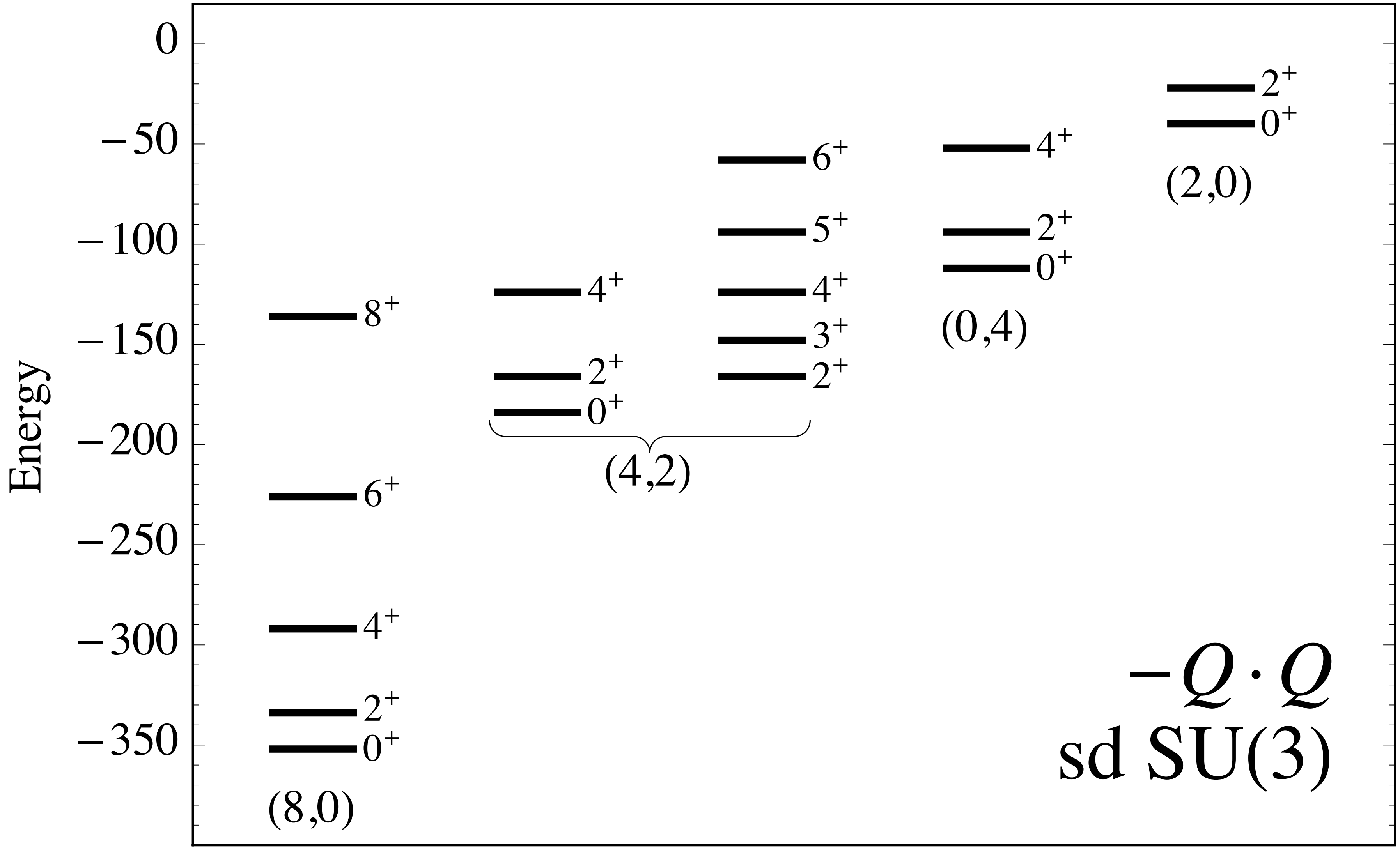
Solvable for major shells N=1,2,... (*p*, *sd*,...).

For the group-theory aficionados:

$$\begin{array}{ccccccc} \mathrm{U}(\Gamma_N) & \supset & \mathrm{U}(3) & \supset & \mathrm{SU}(3) & \supset & \mathrm{SO}(3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [h] & & [\tilde{h}] & & (\lambda,\mu) & K & L \end{array}$$

J.P. Elliott, Proc. Roy. Soc. A **245** (1958) 128; 562

Example: 2n+2p in *sd* shell



Importance & limitations of SU(3)

Historical importance:

Bridge between the spherical shell model and the liquid-drop model through mixing of orbits.

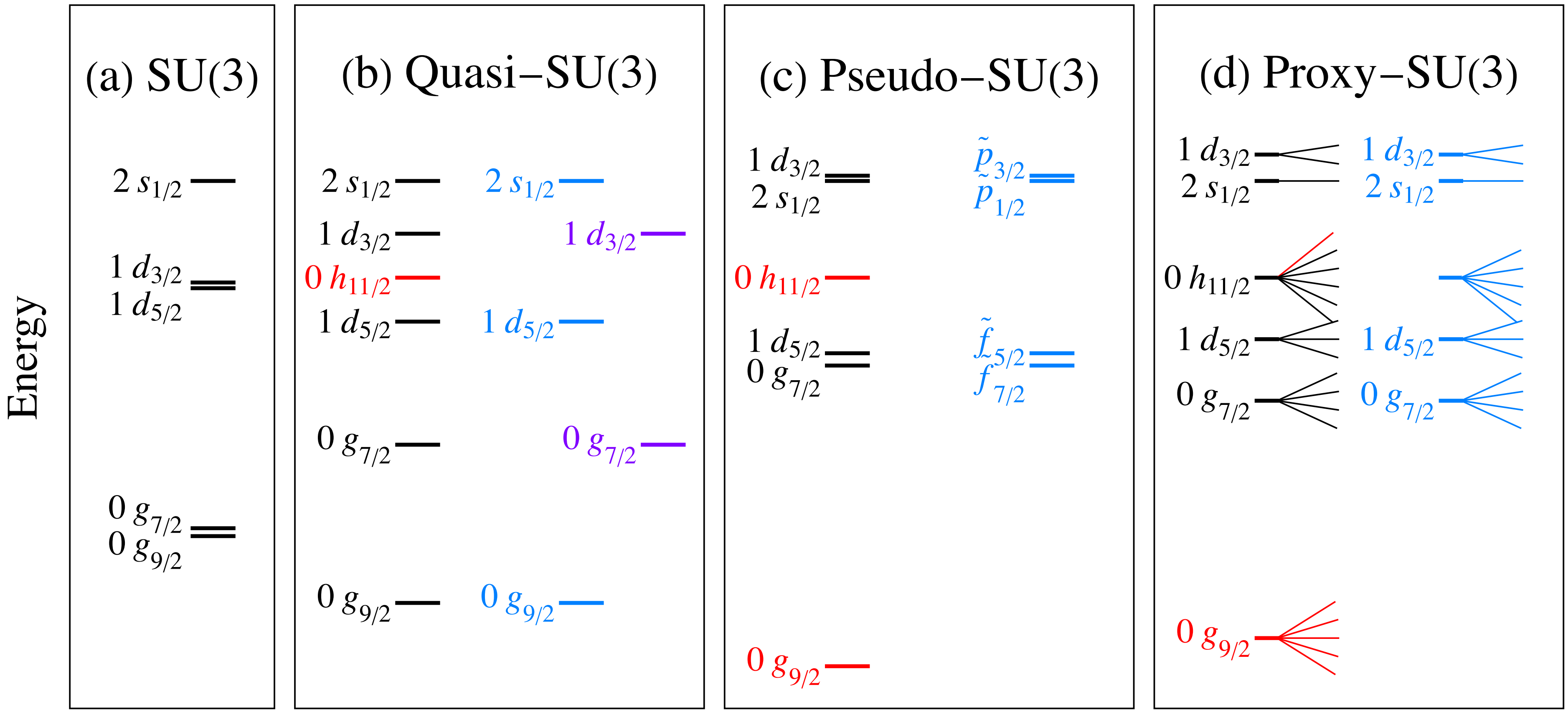
Spectrum generating algebra of Wigner's SU(4) model.

Limitations:

LS *(Russell-Saunders) coupling, not* jj *coupling (no spin-orbit splitting)* $\Rightarrow$ *(beginning of) sd shell.*

Q *is the algebraic quadrupole operator* $\Rightarrow$ *no major-shell mixing.*

SU(3): quasi, pseudo & proxy



Theory of complex spectra

In the 1940s Racah published a series of seminal papers on the application of group theory to atomic spectra. The third of the series (primarily concerned with coefficients of fractional parentage) contains the first mention of seniority.

PHYSICAL REVIEW VOLUME 63, NUMBERS 9 AND 10 MAY 1 AND 15, 1943

Theory of Complex Spectra. III

GIULIO RACAH
The Hebrew University, Jerusalem, Palestine
(Received February 8, 1943)

The consideration of the phases of the fractional-parentage coefficients allows the extension of the matrix methods to configurations with more than two equivalent electrons. Tables are given for the parentages of the terms of p^n and d^n. Applications are made to the spin-orbit interaction of the d^n terms and to the electrostatic interaction between the configurations d^n, $d^{n-1}s$, and $d^{n-2}s^2$. Errata in Part II are indicated.

Racah's "seniority number"

In this section we shall classify the terms of the configuration l^n according to the eigenvalues of

$$Q=\sum_{i<j} q_{ij}, \qquad (34)$$

where q_{ij} is a scalar operator which operates on the two equivalent electrons i and j and is defined by the relation

$$(l^2LM|q_{ij}|l^2LM)=(2l+1)\delta(L,0). \qquad (35)$$

It will be shown that to every term of l^n with non-vanishing Q a term of the same kind corresponds in l^{n-2}, and this fact will allow us to assign to each term a "seniority number" according to the value of n for which the term appeared for the first time. Some useful relation between the fractional parentages of corresponding terms will be obtained and it will also be shown that the classification of the terms of l^{2l+1} according to the two possibilities of (76)II depends only on the seniority of the term.

We may thus assign to each term in the QSL scheme a "seniority number" v, which indicates the number of electrons of the first member of its chain; it follows immediately from (45) that Q depends only on n and v and that its values are given by

$$Q(n,v)=\tfrac{1}{4}(n-v)(4l+4-n-v). \qquad (50)$$

Confronting (41) and (50) we see that conjugate terms have the same seniority.

The seniority number suffices for distinguishing the different terms of the same kind in the configurations d^n but not in f^n, since there are in f^n terms of the same kind which have also the same seniority. For such configurations an unspecified parameter α must be maintained besides v; terms corresponding according to (49) will have the same values of v and of α.

Pairing

Definition of pairing interaction in a single-j shell:

$$\left\langle j^2;J\left|\hat{V}_{\text{pairing}}(1,2)\right|j^2;J\right\rangle = -\tfrac{1}{2}(2j+1)\,g\,\delta_{J0}$$

Analytic solution of pairing hamiltonian for identical nucleons in a single-j shell:

$$\left\langle j^n \upsilon J\left|\sum_{1\le k<l}^{n}\hat{V}_{\text{pairing}}(k,l)\right|j^n \upsilon J\right\rangle = -\tfrac{1}{4}\,g\,(n-\upsilon)(2j-n-\upsilon+3)$$

Seniority υ (number of nucleons not in pairs coupled to J=0) is a good quantum number.

Correlated ground-state solution (*cfr.* BCS).

G. Racah, Phys. Rev. **63** (1943) 367

Algebraic definition of seniority

Classification of n identical fermions with spin j :

$$
\begin{array}{ccccccc}
\mathrm{U}(2j+1) & \supset & \mathrm{USp}(2j+1) & \supset & \cdots & \supset & \mathrm{SU}(2) \\
\downarrow & & \downarrow & & & & \downarrow \\
[1^n] & & [1^v] & & & & J
\end{array}
$$

G. Racah, Phys. Rev. **76** (1949) 1352

Algebraic definition of seniority

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An equivalent representation exists with `quasi-spin' algebras.

A generic mechanism: dual representations.

D.J. Rowe *et al.*, Rev. Mod. Phys. **84** (2012) 711

Quasi-spin algebra

Pair operators:

$$\hat{S}_+ = \tfrac{1}{2}\sqrt{2j+1}\left(a_j^+ a_j^+\right)_0^{(0)}, \quad \hat{S}_- = \left(\hat{S}_+\right)^+$$

Second-quantised form of pairing Hamiltonian:

$$\hat{V}_{\text{pairing}} = -g_0 \hat{S}_+ \hat{S}_- = -g_0\left(\hat{S}^2 - \hat{S}_z^2 + \hat{S}_z\right)$$

The pairing Hamiltonian is solvable due to an SU(2) quasi-spin symmetry:

$$\left[\hat{S}_+, \hat{S}_-\right] = \tfrac{1}{2}\left(2\hat{n} - 2j - 1\right) \equiv 2\hat{S}_z, \quad \left[\hat{S}_z, \hat{S}_\pm\right] = \pm \hat{S}_\pm$$

A. Kerman, Ann. Phys. (NY) **12** (1961) 300
K. Helmers, Nucl. Phys. **23** (1961) 595

Conservation of seniority

Seniority υ is the number of particles not in pairs coupled to J=0 (Racah).

Conditions for the conservation of seniority by a an interaction can be derived in general

Any two-body interaction between identical fermions with spin j conserves seniority if j≤7/2.

A. de-Shalit & I. Talmi, *Nuclear Shell Theory*
I. Talmi, *Simple Models of Complex Nuclei*

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A. de-Shalit & I. Talmi, *Nuclear Shell Theory*
I. Talmi, *Simple Models of Complex Nuclei*

Conservation of seniority

Necessary and sufficient conditions for a two-body interaction to conserve seniority:

$$\sum_J (2J+1)\sqrt{2I+1}\, a_{jI}^{J} \left\langle j^2;J \middle| \hat{V} \middle| j^2;J \right\rangle = 0, \quad I = 2,4,\ldots,2\lfloor j \rfloor,$$

$$a_{jI}^{J} = \frac{\delta_{JI}}{\sqrt{(2J+1)(2I+1)}} + 2\begin{Bmatrix} j & j & J \\ j & j & I \end{Bmatrix} - \frac{4}{(4j^2-1)}$$

Valid for identical fermions.

I. Talmi, *Simple Models of Complex Nuclei*

Is seniority conserved in nuclei?

The interaction between nucleons is “short range”.

A δ interaction is therefore a reasonable approximation to the nucleon two-body force.

The δ interaction between identical nucleons conserves seniority.

$\therefore$ In semi-magic nuclei seniority is conserved to a good approximation.

Generalised seniority models

- Generalisation of pairing from a single-*j* shell to several degenerate *j* shells.
- Non-degenerate shells:
  - *Generalised seniority (Talmi).*
  - *Integrable pairing models (Richardson, Gaudin).*
- Pairing with neutrons and protons (isospin):
  - *SO(5) T=1 pairing (Racah, Flowers; Hecht).*
  - *SO(8) T=0 & T=1 pairing (Flowers and Szpikowski).*

Particle-hole (ph) conjugation

A long history, cfr. Condon & Shortley (1935).

In atomic and nuclear physics: Racah and Bell.

The ph conjugation operator Γ transforms a problem of *n* fermions in a *j* shell into one with 2j+1-*n* fermions.

In the language of second quantisation:

$$\hat{\Gamma}|0\rangle = a^{+}_{j,m=j}a^{+}_{j,m=j-1}\dots a^{+}_{j,m=-j}|0\rangle$$

$$\hat{\Gamma}a^{+}_{jm}\hat{\Gamma}^{+} = (-)^{j+m}a_{j,-m} \equiv \tilde{a}_{jm}, \quad \hat{\Gamma}\tilde{a}_{jm}\hat{\Gamma}^{+} = a^{+}_{jm}$$

E.U. Condon & G.H. Shortley, *The Theory of Atomic Spectra*
G. Racah, Phys. Rev. **62** (1942) 438
J.S. Bell, Nucl. Phys. **12** (1959) 117

Seniority and ph conjugation

A representation of the ph transformation

$$\hat{\Gamma} = \exp\left[\tfrac{1}{2}\pi\left(\hat{S}_+ - \hat{S}_-\right)\right]$$

where $S_\pm$ are the quasi-spin operators

$$\hat{S}_+ = \tfrac{1}{2}\sqrt{2j+1}\left(a_j^+ a_j^+\right)_0^{(0)}, \quad \hat{S}_- = \left(\hat{S}_+\right)^+$$

The relation with seniority:

$$\hat{\Gamma}\left(\hat{S}_+\right)^p \hat{\Gamma}^+ = (-)^p \left(\hat{S}_-\right)^p$$

$$\hat{\Gamma}\left|j^n \upsilon J\right\rangle = (-)^{(n-\upsilon)/2}\left|j^{2j+1-n}\upsilon J\right\rangle$$

A. Müller-Arnke, Nucl. Phys. A **215** (1973) 205

A geometric phase

The action of ph conjugation on a seniority state:

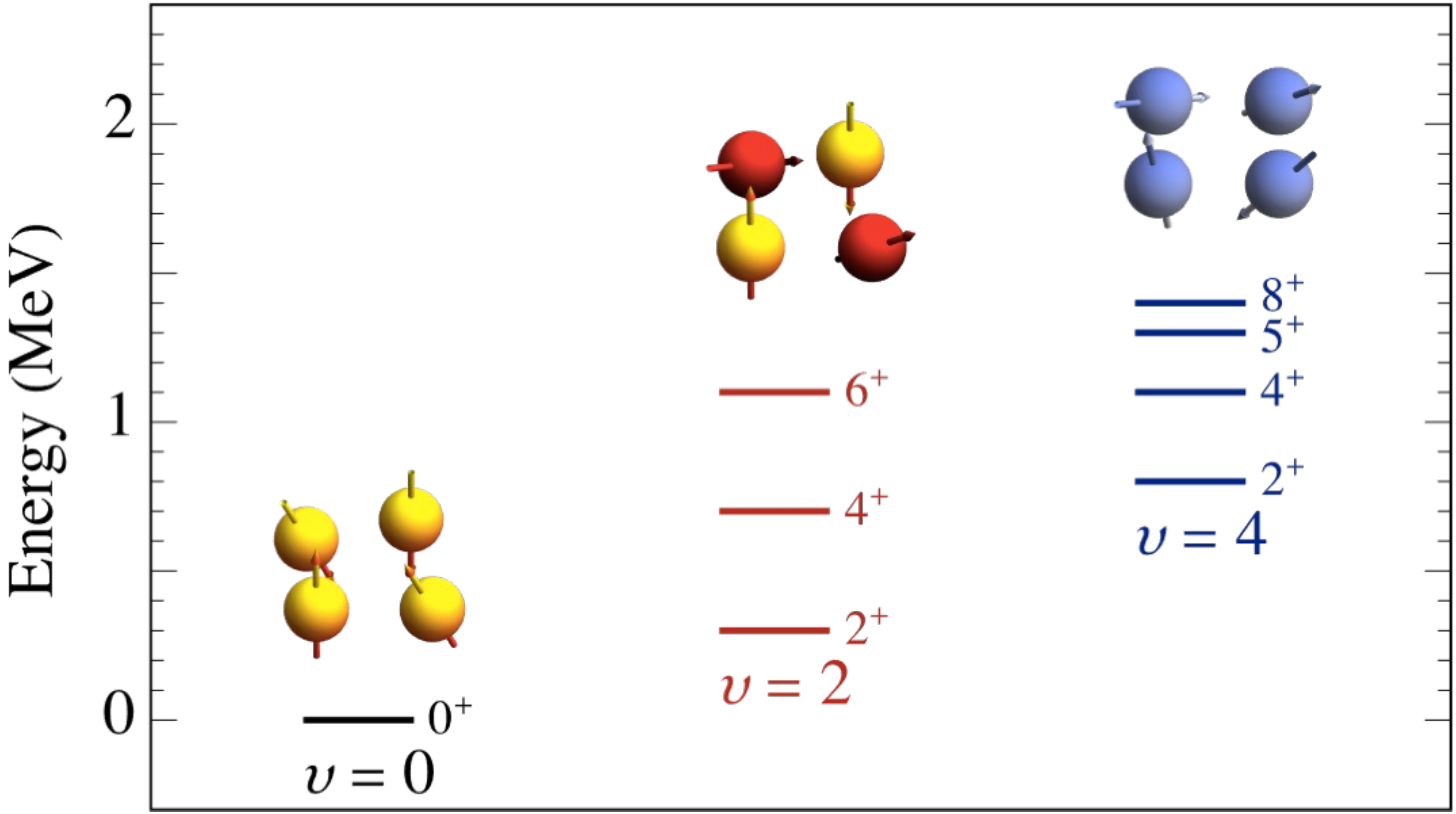
$$\hat{\Gamma}\left|j^n \upsilon J\right\rangle = (-)^{(n-\upsilon)/2}\left|j^{2j+1-n} \upsilon J\right\rangle$$

The sign is without any consequence *except* if the left and right states are the same, that is for a half-filled shell, $n=2j+1-n$.

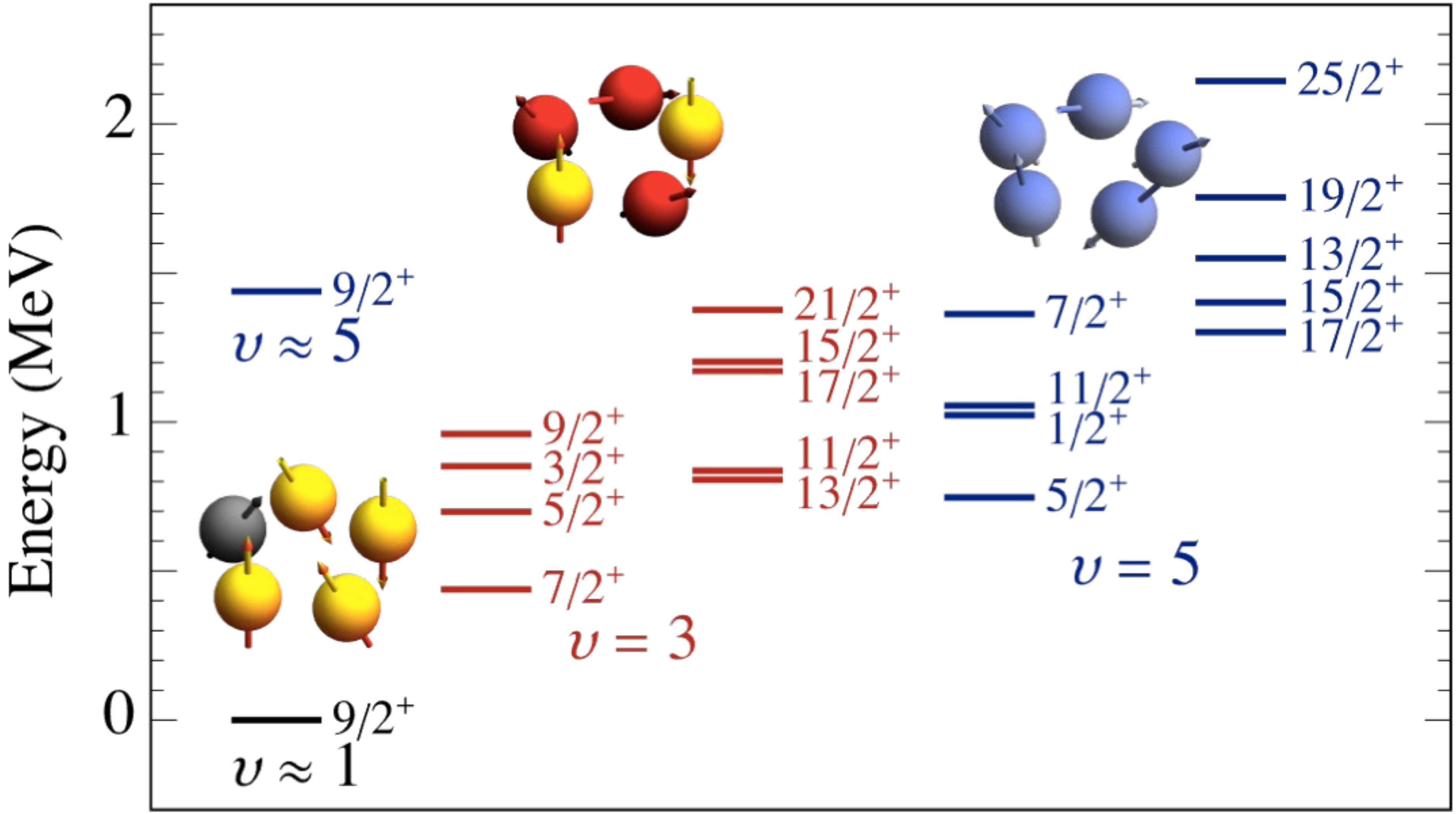
The observable consequence of this phase is that $\Delta\upsilon=\pm 2$ seniority mixing is forbidden.

R.D. Lawson, *Theory of the Nuclear Shell Model*
J.J. Valiente-Dobón *et al.*, Phys. Lett. B **816** (2021) 136183

Four nucleons in a j=7/2 shell



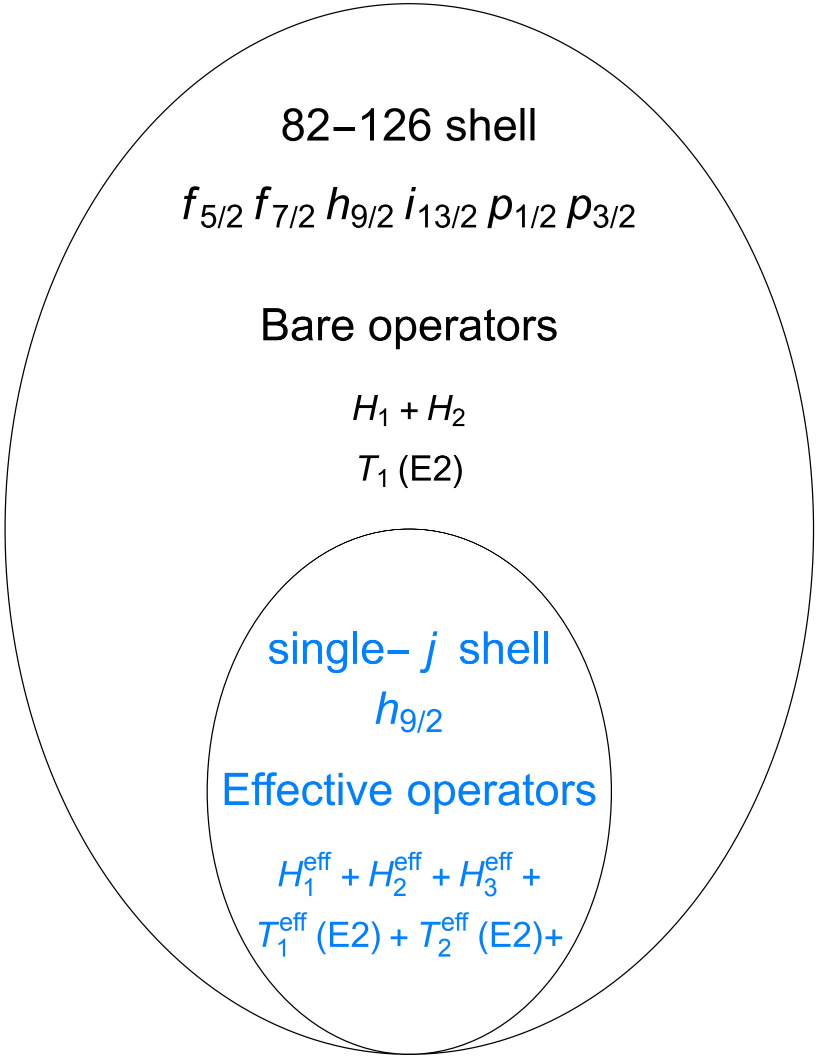
Five nucleons in a j=9/2 shell



An application: effective operators

- Higher-order operators in a single-j shell. Here: higher-order effective charges in E2 operator.
- Application to $N = 126$ isotones with protons in the $0h_{9/2}$ orbital. E2 data are available in ^{210}Po and ^{211}At.
- Analysis profits from seniority considerations.

Bare and effective operators



One-body E2

One kind of nucleon in a single-*j* shell

$$\hat{T}_1(E2) = e_1\left(a_j^+ \tilde{a}_j\right)^{(2)}$$

Matrix elements between *n*-nucleon states are calculated cursively until

$$\left\langle j \left\| \hat{T}_1(E2) \right\| j \right\rangle = e_1\sqrt{5}$$

One effective charge.

One+two-body E2

In a single-j shell

$$\hat{T}_{1+2}(E2) = e_1 \left(a_j^+ \tilde{a}_j\right)^{(2)} + \sum_{J=2,4,\ldots}^{2j-1} e_2(J,J) \left[\left(a_j^+ a_j^+\right)^{(J)} \left(\tilde{a}_j \tilde{a}_j\right)^{(J)}\right]^{(2)}$$

$$+ \sum_{J=2,4,\ldots}^{2j-1} e_2(J,J-2) \frac{1}{2} \left\{\left[\left(a_j^+ a_j^+\right)^{(J-2)} \left(\tilde{a}_j \tilde{a}_j\right)^{(J)}\right]^{(2)} + \text{h.c.}\right\}$$

Many effective charges, here taken from E2 data.

State-dependent one-body E2

In a single-*j* shell

$$\hat{T}_1(E2) = e_1(J_\mathrm{i}, J_\mathrm{f})\left(a_j^+ \tilde{a}_j\right)^{(2)}$$

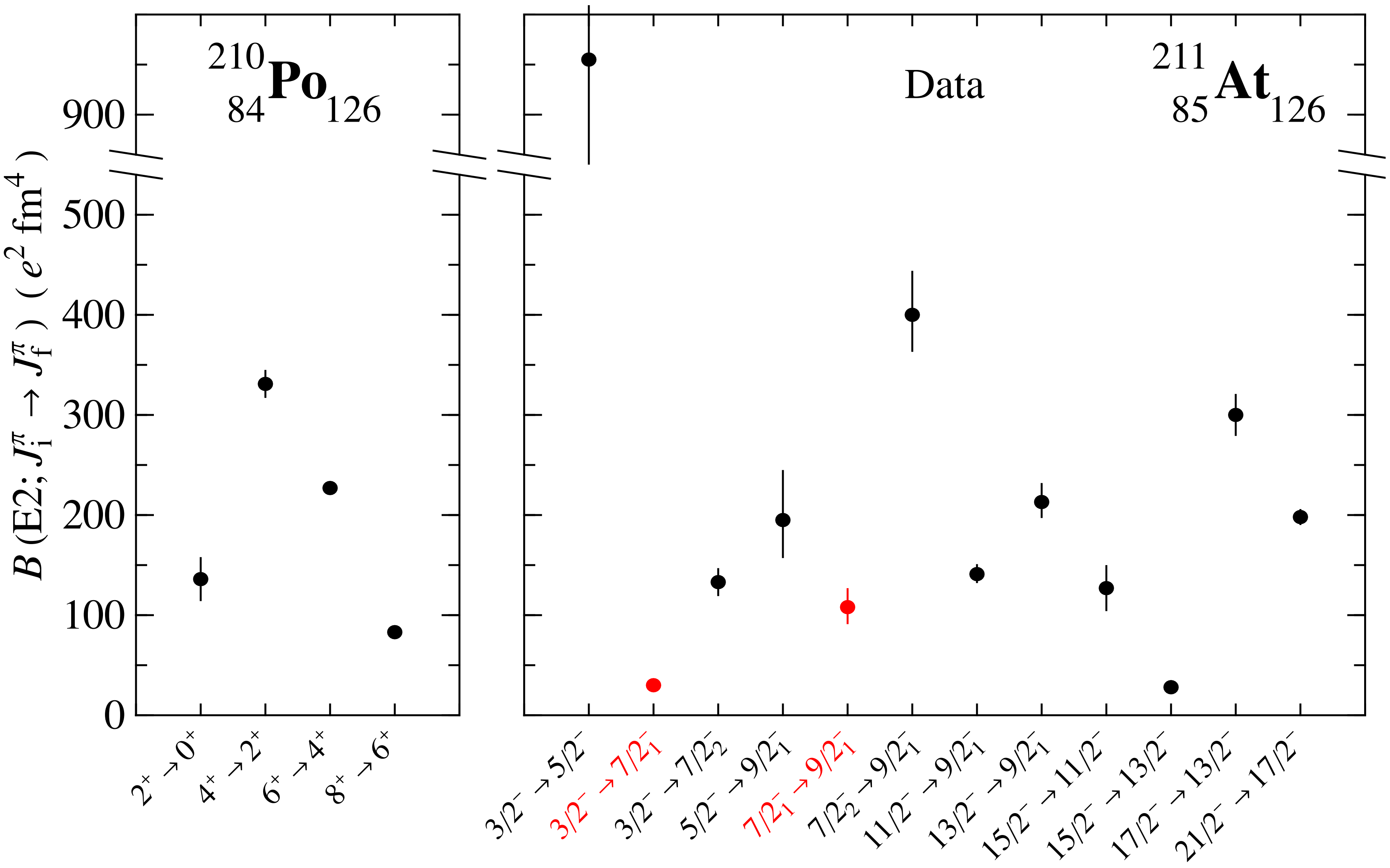
Matrix elements between *n*-body states are calculated cursively until

$$\left\langle j^2 J_\mathrm{f} \left\| \hat{T}_1(E2) \right\| j^2 J_\mathrm{i} \right\rangle$$

$$= -\sqrt{20(2J_\mathrm{i}+1)(2J_\mathrm{f}+1)} \begin{Bmatrix} j & j & 2 \\ J_\mathrm{i} & J_\mathrm{f} & j \end{Bmatrix} e_1(J_\mathrm{i}, J_\mathrm{f})$$

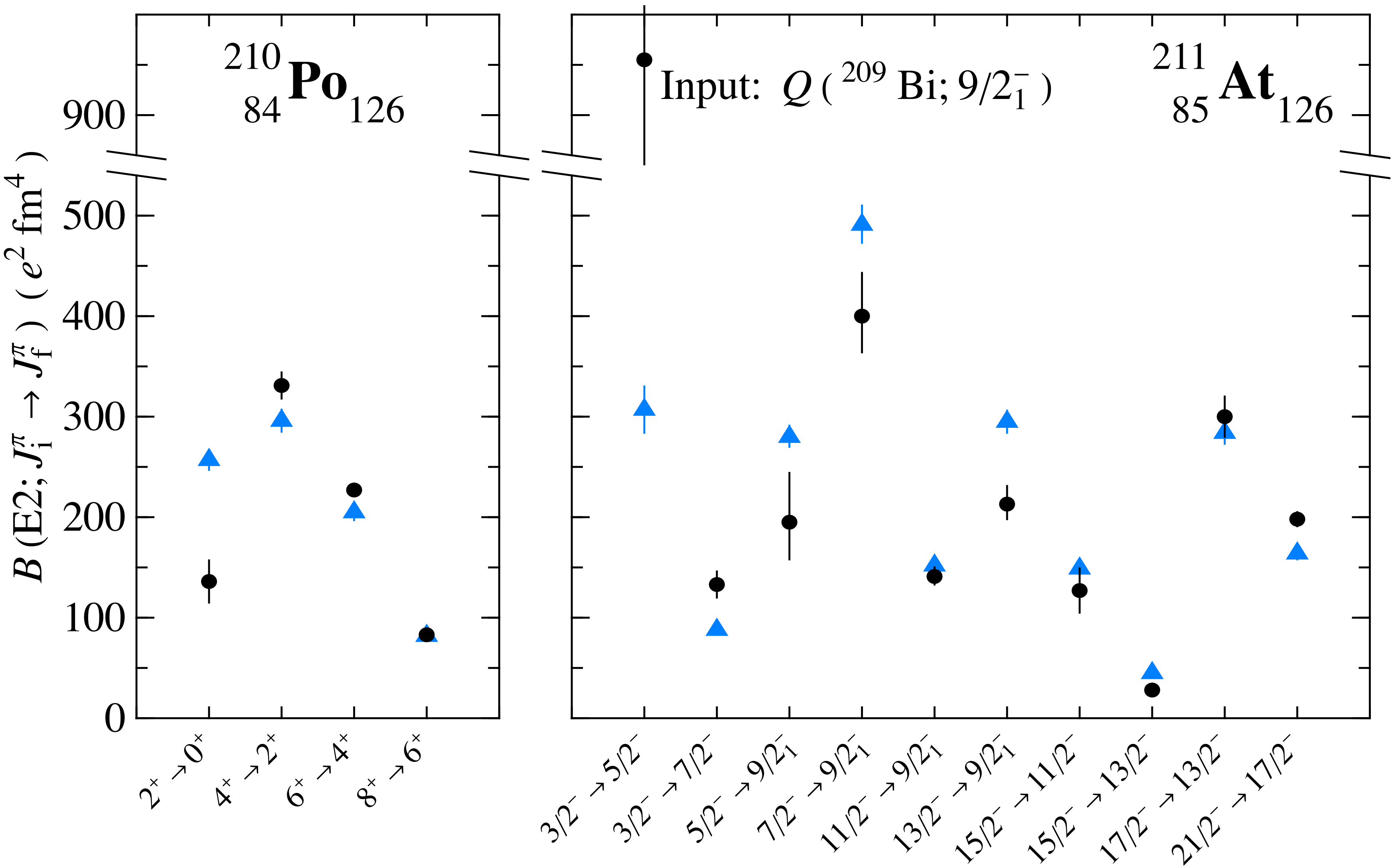
Many effective charges, here taken from E2 data.

E2 data in ^{210}Po and ^{211}At



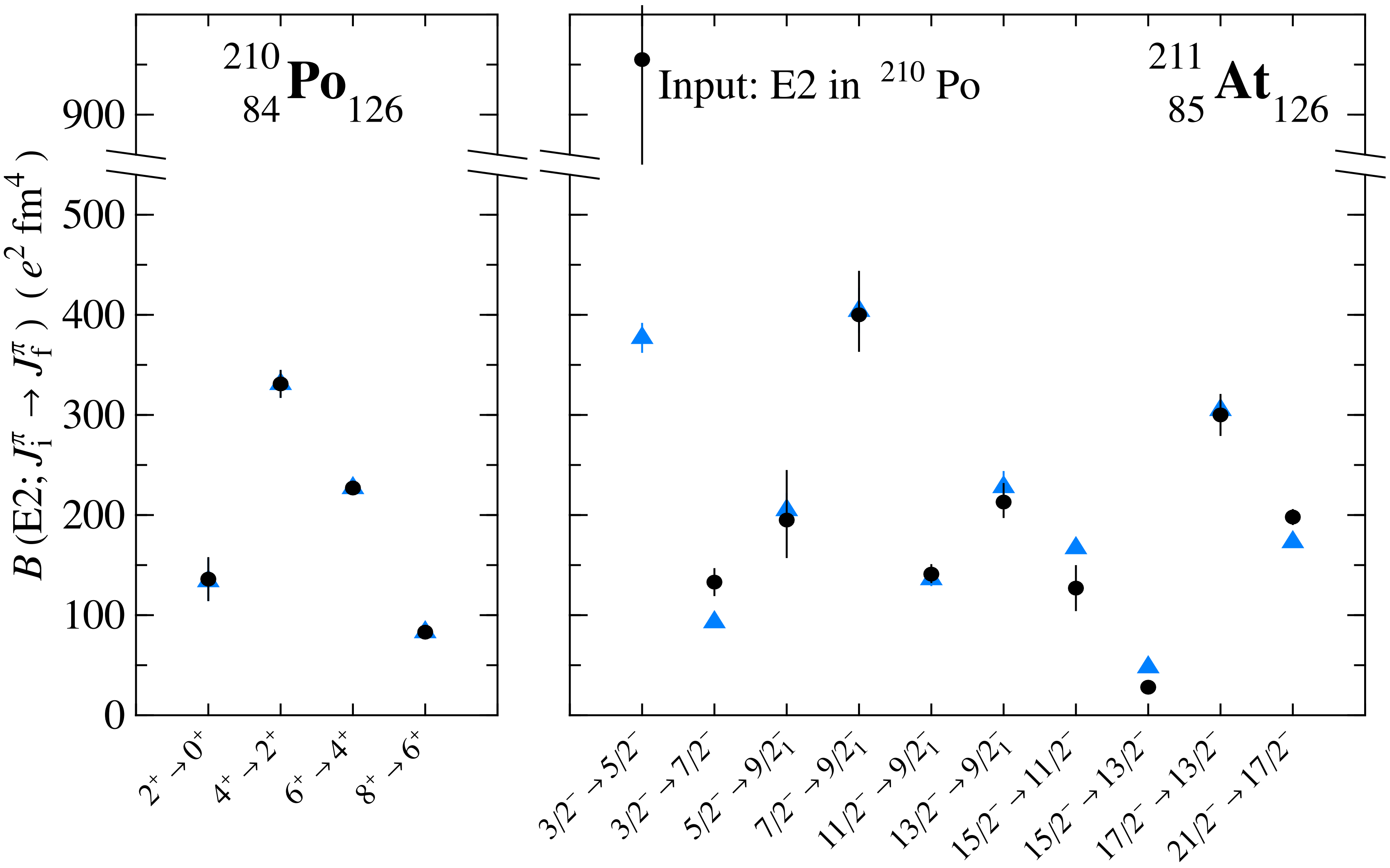
V. Karayonchev *et al.*, Phys. Rev. C **99** (2019) 024326
V. Karayonchev *et al.*, Phys. Rev. C **106** (2022) 044321

One-body E2

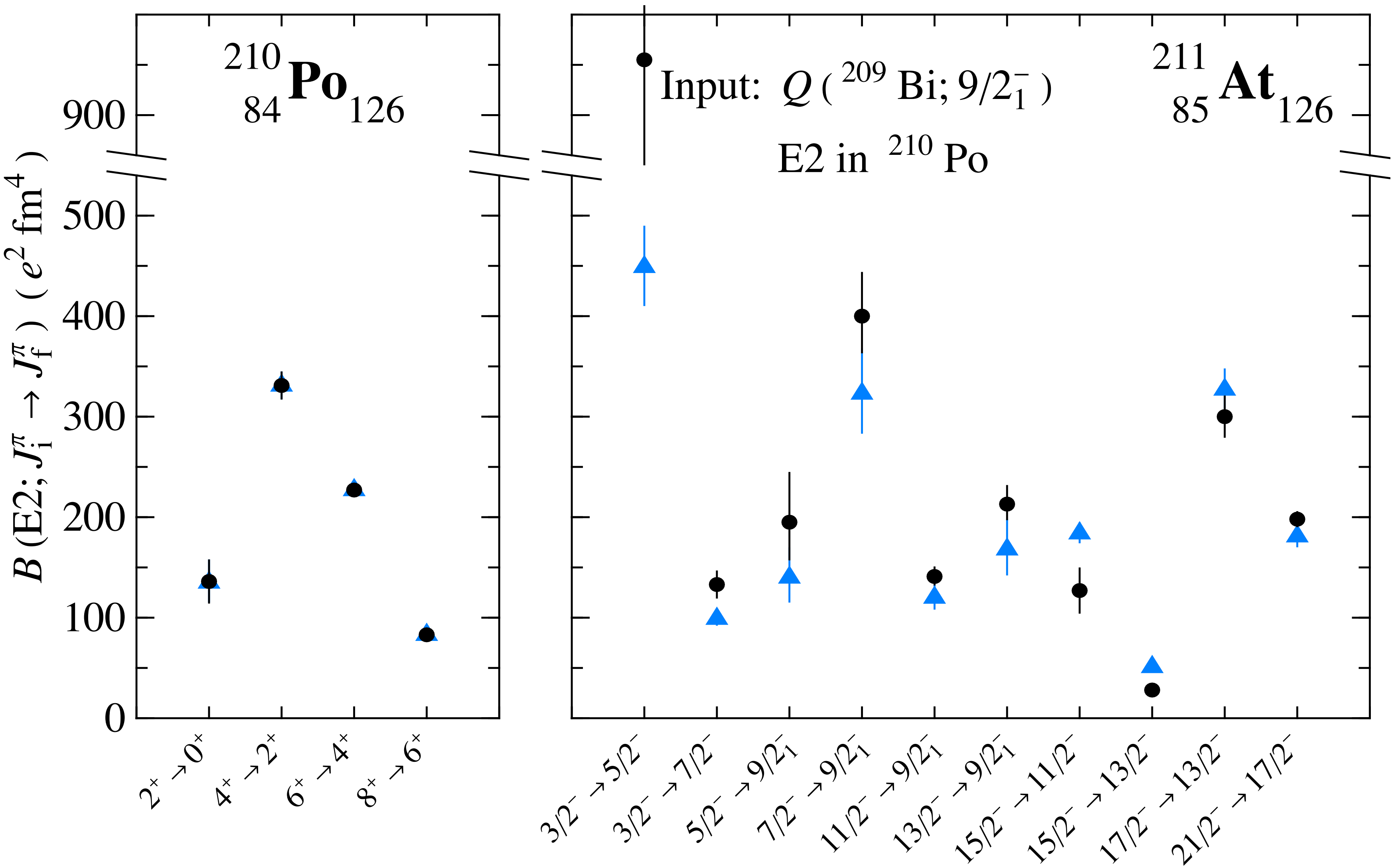


$^{210}_{84}\mathbf{Po}_{126}$
$^{211}_{85}\mathbf{At}_{126}$
Input: $Q\,(^{209}\mathrm{Bi}; 9/2^-_1)$
$B\,(\mathrm{E2}; J^\pi_i \to J^\pi_f)\ (e^2\,\mathrm{fm}^4)$
0
100
200
300
400
500
900
$2^+ \to 0^+$
$4^+ \to 2^+$
$6^+ \to 4^+$
$8^+ \to 6^+$
$3/2^- \to 5/2^-$
$3/2^- \to 7/2^-$
$5/2^- \to 9/2^-_1$
$7/2^- \to 9/2^-_1$
$11/2^- \to 9/2^-_1$
$13/2^- \to 9/2^-_1$
$15/2^- \to 11/2^-$
$15/2^- \to 13/2^-$
$17/2^- \to 13/2^-$
$21/2^- \to 17/2^-$

State-dependent one-body E2



One+two-body E2



Conclusions

Symmetry considerations are useful to obtain insight in the structure of nuclear models.

Seniority is a relevant quantum number in semi-magic nuclei.

Seniority conservation in mid-shell nuclei is the consequence of a geometric phase associated with particle-hole conjugation.

Application: effective charges in a single-j shell.