In Memory of Peter Schuck

A journey along semiclassical approximations to the pairing problem

(1992 -2022)

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The Wigner-Kirkwood $\hbar$ expansion of the density matrix in inhomogeneous Fermi systems (I)

According to K. Taruishi and P. Schuck, Z. Phys **A3342**, 397 (1992) and J.C. Pei at al, Phys. Rev.**C92** 064316 (2015) we start from the Bloch propagator

$$\hat{\mathcal{C}} = exp(-\beta\hat{\mathcal{H}}) \qquad \hat{\mathcal{H}} = \begin{pmatrix} \hat{h} & \hat{\Delta} \\ \hat{\Delta}^{\dagger} & -\hat{h}^{*} \end{pmatrix} \qquad \frac{\partial\hat{\mathcal{C}}}{\partial\beta} + \{\hat{\mathcal{H}}, \hat{\mathcal{C}}\}_{+} = 0$$

After performing the Wigner transform of the Bloch propagator $\hat{\mathcal{C}}$ we obtain

$$\mathcal{C} = \mathcal{C}_0 + \hbar\mathcal{C}_1 + \hbar^2\mathcal{C}_2$$

$$\mathcal{C}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh(\beta E) + \begin{pmatrix} h & \Delta \\ \Delta & -h \end{pmatrix} \frac{\sinh(\beta E)}{E}$$

$$\mathcal{R} = \mathcal{L}^{-1}\frac{1}{\beta}\mathcal{C} = \mathcal{R}_0 + \hbar\mathcal{R}_1 + \hbar^2\mathcal{R}_2 \qquad \mathcal{R} = \begin{pmatrix} \rho & \kappa \\ \kappa^{*} & 1-\rho^{*} \end{pmatrix}$$

$$\mathcal{R}_0 = \frac{1}{2}[1 + \frac{\mathcal{H}}{E}] \quad \mathcal{R}_2 = g_1\frac{\mathcal{H}}{E} - g_2\frac{\mathcal{F}}{E} \quad \mathcal{H} = \begin{pmatrix} h & \Delta \\ \Delta & -h \end{pmatrix} \quad \mathcal{F} = \begin{pmatrix} -\Delta & h \\ h & \Delta \end{pmatrix}$$

The Wigner-Kirkwood $\hbar$ expansion of the density matrix in inhomogeneous Fermi systems (II)

P. Schuck, M. Urban and X. Viñas, work in progress

$$\kappa_2(\mathbf{R},\mathbf{p}) = -\frac{\Delta(2h^2-\Delta^2)}{16E^5}\frac{\hbar^2\nabla^2 U}{m} + \frac{h(h^2-2\Delta^2)}{16E^5}\frac{\hbar^2\nabla^2\Delta}{m}$$
$$+\frac{h\Delta(2h^2-3\Delta^2)}{16E^7}\frac{\hbar^2(\nabla U)^2}{m} + \frac{h\Delta(2h^2-3\Delta^2)}{48E^7}\frac{p^2\hbar^2\nabla^2 U}{m^2}$$
$$-\frac{h^4+\Delta^4-3h^2\Delta^2}{8E^7}\frac{\hbar^2\nabla U\cdot\nabla\Delta}{m} - \frac{h\Delta(3h^2-2\Delta^2)}{16E^7}\frac{\hbar^2(\nabla\Delta)^2}{m}$$
$$+\frac{5h^2\Delta^2}{48E^7}\frac{p^2\hbar^2\nabla^2\Delta}{m^2} - \frac{\Delta}{48E^5}\frac{p^2\hbar^2(\nabla\Delta)^2}{m^2}$$

where $h = \frac{p^2}{2m} + U(R)$

$$\begin{aligned}\kappa_2(\mathbf{r}) &= Y_1(\mathbf{r})\nabla^2\Delta + Y_2(x_0)(\nabla\Delta)^2 + Y_3(\mathbf{r})\nabla^2 U \\ &+ Y_4(\mathbf{r})(\nabla U)^2 + Y_7(\mathbf{r})\nabla U\cdot\nabla\Delta\end{aligned}$$

The Bogoliubov-de Gennes equation for finite systems (I)

Within a suitable coarse graining approach of the Bogoliubov-de Gennes equation (Local Phase Density Approximation (LPDA)), for the gap parameter as a function of the distance can be written as
(see S. Simonucci and G.C. Strinati, Phys. Rev. **B89**, 054511 (2014))

$$-\frac{m}{4\pi\hbar^2 a_F}\Delta(\mathbf{r}) = I_0(\mathbf{r})\Delta(\mathbf{r}) + I_1(\mathbf{r})\frac{\hbar^2}{4m}\nabla^2\Delta(\mathbf{r}),$$

where the functions $I_0(x_0)$ and $I_1(x_0)$ are given by

$$I_0(x_0) = \frac{1}{2\pi^2}\int dk\left(\frac{k^2}{2E} - \frac{m}{\hbar^2}\right) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\Delta(\mathbf{r})}\left[x_0 I_6(x_0) - I_5(x_0)\right]$$

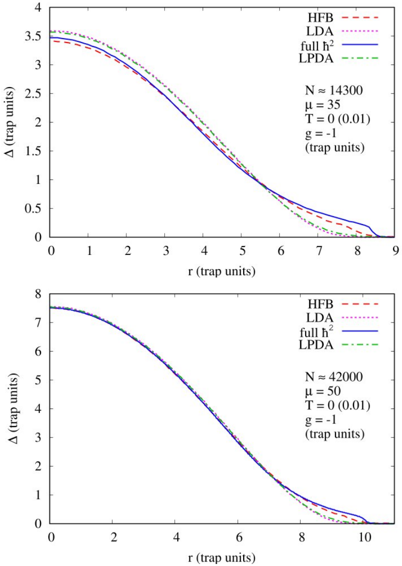
and

$$I_1(x_0) = \frac{1}{4\pi^2}\int dk\frac{k^2}{E^3}\left(\frac{\hbar^2 k^2}{2m} - \mu(\mathbf{r})\right) = \frac{1}{8\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{I_6(x_0)}{\sqrt{\Delta(\mathbf{r})}}.$$

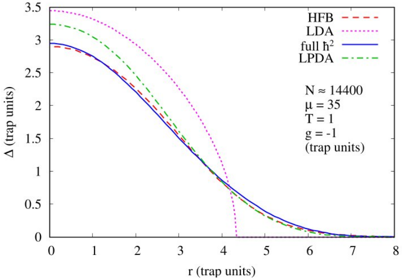
where the quasiparticle energy E reads:

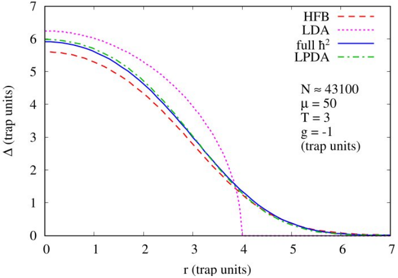
$$E = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu(\mathbf{r})\right)^2 + \Delta(\mathbf{r})^2} \quad \mu(\mathbf{r}) = \mu - U(\mathbf{r}) \quad x_0 = \frac{\mu(\mathbf{r})}{\Delta(\mathbf{r})}$$

Some results: zero temperature



Some results: finite temperature





INTRODUCTION

STABLE NUCLEI

177 even-even; 58 even-odd; 54 odd-even; 10 odd-odd

$$\Delta \sim \frac{1}{2}\left(E(A+1) - 2E(A) + E(A-1)\right) \qquad A \quad even$$

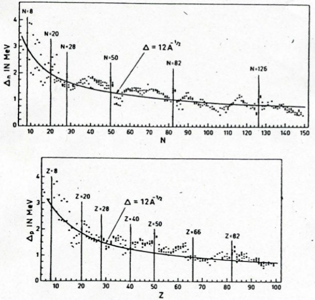


Fig. 15.5 Neutron and proton pairing energies as a function of N and Z, respectively (Zeldes *et al.* 1967).

THEORETICAL FRAMEWORK

BCS THEORY

$$\Delta_{n_c} = -\sum_{n'_c} V_{n_c n'_c} \frac{\Delta_{n'_c}}{2E_{n'_c}}, \quad N = \sum_{n_c} \left(1 - \frac{\epsilon_{n_c} - \mu}{E_{n_c}}\right)$$

where

$$V_{n_c n'_c} = \langle n_c \bar{n}_c | v | n'_c \bar{n}'_c \rangle \quad and \quad E_n = [(\epsilon_n - \mu)^2 + \Delta_n^2]^{1/2},$$

FURTHER IMPROVMENTS

Hartree-Fock-Bogolibov theory

Particle-vibration coupling

SEMICLASSICAL APPROXIMATIONS

LDA is the standard semiclassical limit to the pairing problem.

H.Kuchareck, P.Ring, P.Schuck, R.Bengtsson amd M.Girod, Phys. Lett.**B216**, 240 (1989).

$$\Delta(\mathbf{R},\mathbf{p}) = -\int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} v(\mathbf{p}-\mathbf{p}')\kappa(\mathbf{R},\mathbf{p}')$$

$$\kappa(\mathbf{R},\mathbf{p}) = \frac{\Delta(\mathbf{R},\mathbf{p})}{2\sqrt{((p^2-p_F^2(\mathbf{R}))/2m^*)^2+(\Delta(\mathbf{R},\mathbf{p}))^2}}$$

The pairing matrix elements are evaluated using plane waves.

THE THOMAS-FERMI APPROXIMATION IN WEAK COUPLING

In the weak cooupling regime $\Delta/\mu << 1$. In this case the canonical basis can be replaced by the HF one:

$$H|n\rangle = \epsilon_n|n\rangle.$$

At equilibrium and for time reversal invariant systems canonical conjugation and time reversal operation are related by

$$\langle \mathbf{r}|\bar{n}\rangle = \langle n|\mathbf{r}\rangle \Rightarrow \langle \mathbf{r}_1\mathbf{r}_2|n\bar{n}\rangle = \langle \mathbf{r}_1|\hat{\rho}_n|\mathbf{r}_2\rangle,$$

$$V_{nn'} = \langle n\bar{n}|v|n'\bar{n}'\rangle = \int \langle \mathbf{r}_2|\hat{\rho}_n|\mathbf{r}_1\rangle\langle \mathbf{r_1}\mathbf{r_2}|v|\mathbf{r_1'}\mathbf{r_2'}\rangle\langle \mathbf{r}'_1|\hat{\rho}_n|\mathbf{r}'_2\rangle d\mathbf{r_1}d\mathbf{r_2}d\mathbf{r_1'}d\mathbf{r_2'}$$

The gap equation in semiclassical TF approximation reads:

$$\Delta(E) = -\int_0^\infty dE' g^{TF}(E') V(E,E') \kappa(E'),$$

with

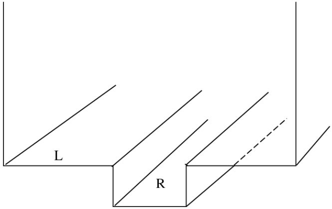
$$\kappa(E) = \frac{\Delta(E)}{2\sqrt{(E-\mu)^2 + \Delta^2(E)}}.$$

and

$$V(E,E') = \int \frac{d\mathbf{R} d\mathbf{p}}{(2\pi\hbar)^3} \int \frac{d\mathbf{R}' d\mathbf{p}'}{(2\pi\hbar)^3} f_E(\mathbf{R},\mathbf{p}) f_{E'}(\mathbf{R}',\mathbf{p}') v(\mathbf{R},\mathbf{p};\mathbf{R}',\mathbf{p}'),$$

with $v(\mathbf{R},\mathbf{p};\mathbf{R}',\mathbf{p}')$ the double Wigner transform of A$< \mathbf{r_1 r_2}|v|\mathbf{r'_1 r'_2} >$ what for a local translationally invariant force yields $v(\mathbf{R},\mathbf{p};\mathbf{R}',\mathbf{p}') = \delta(\mathbf{R}-\mathbf{R}')v(\mathbf{p}-\mathbf{p}')$

A SIMPLE EXEMPLE: SLAB GEOMETRY



Mean field potential

$$V(x) = 0 \quad -L \leq x \leq -R \quad or \quad R \leq x \leq L; \quad V(x) = V_0 \quad -R \leq x \leq R$$

$$V_0 = -40 MeV \quad L = 100 fm \quad R = 10 fm$$

Pairing force

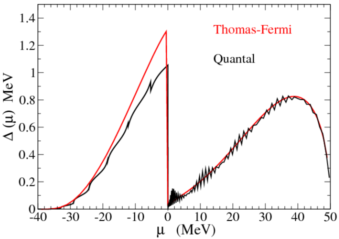
$$V_p = -g\delta(\mathbf{r} - \mathbf{r}') \quad g = 150 MeV fm^3 \quad \Lambda = 50 MeV$$

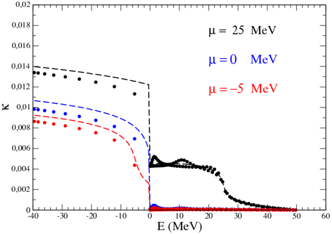
Gap and Pairing Tensor

$$\Delta_n = -\sum_{n'} \Theta(\Lambda - \varepsilon_{n'}) V_{nn'} K_{n'}; \quad K_n = \frac{m}{4\pi\hbar^2}\Delta_n \ln \frac{\Lambda - \mu + \sqrt{(\Lambda-\mu)^2 + \Delta_n^2}}{\varepsilon_n - \mu + \sqrt{(\varepsilon_n - \mu)^2 + \Delta_n^2}}$$

$$\Delta(z) = -gK(z)$$

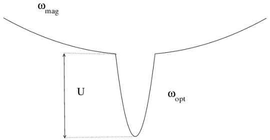
$$K(z) = \sum K_n |\varphi_n(z)|^2 \quad K(z) = \int_{V_0}^{\Lambda} dE g^{TF}(E) K(E) \rho_E^{TF}(z)$$

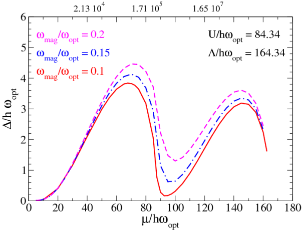




COLD ATOMS

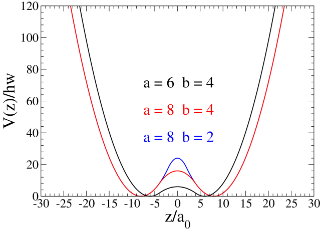
Ketterle's trap, Phys. Rev. Lett. **81**, 2194 (1998)

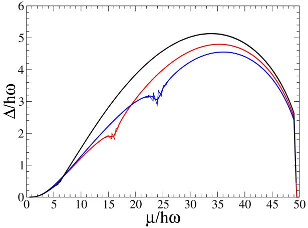




$$\omega_{opt} = 2\pi \times 1000Hz; \quad g = -\hbar\omega_{opt} \quad \Lambda = 164.34\hbar\omega_{opt}$$

DOUBLE POTENTIAL WELL





$$V(z) = \frac{1}{2}(z+a)^2 \qquad z \leq b$$

$$V(z) = \frac{a(a-b)}{2} + \frac{1}{2}\left(1 - \frac{a}{b}\right) \qquad |z| \leq b$$

$$V(z) = \frac{1}{2}(z-a)^2 \qquad z \geq b$$

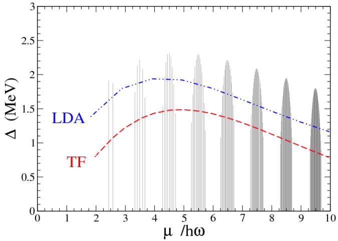
HARMONIC OSCILLATOR POTENTIAL

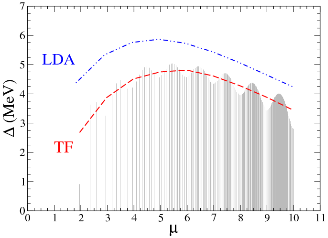
$\hbar\omega = 8.65$ MeV m*/m =0.8

M. Prakash, S. Shlomo and V.A. Kolomietz, Nucl. Phys. **370**, 30 (1981).

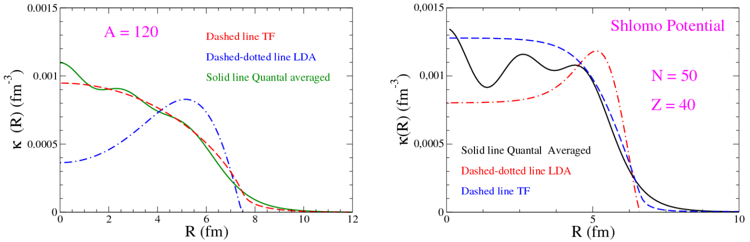
$$\kappa(\mathbf{R},\mathbf{p}) = \sum_N \kappa_N f_N(H_{cl.}) = \sum_N u_N v_N f_N(H_{cl.})$$

$$f_N(H_{cl.}) = [\sum_{nlm} \varphi_{nlm}(\mathbf{r})\varphi_{nlm}(\mathbf{r}')]_W = 8(-1)^N e^{-\frac{2H_{cl}}{\hbar\omega}} L_N^{(2)}(\tfrac{4H_{cl}}{\hbar\omega})$$



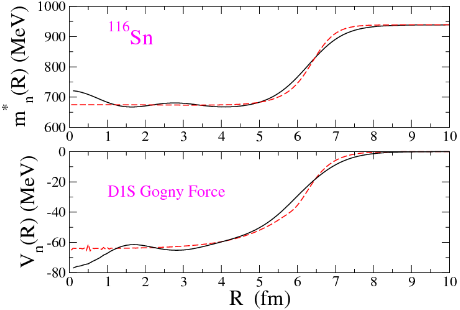


COMPARISON BETWEEN TF AND LDA



$$\kappa(\mathbf{R}) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3}\kappa(\mathbf{R},\mathbf{p}); \quad \kappa(\mathbf{R},\mathbf{p}) = \int dE g(E)\kappa(E) f_E(\mathbf{R},\mathbf{p})$$

REALISTIC APPROACH TO AVERAGE PAIRING GAPS

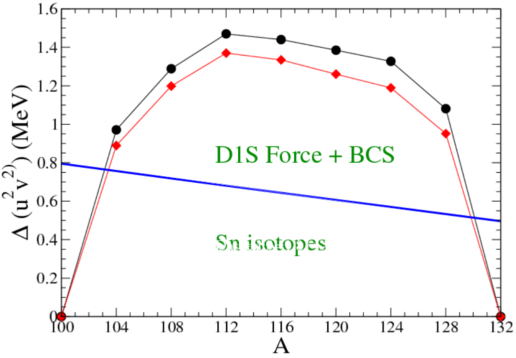


The main ingredient to solve the TF gap equation is

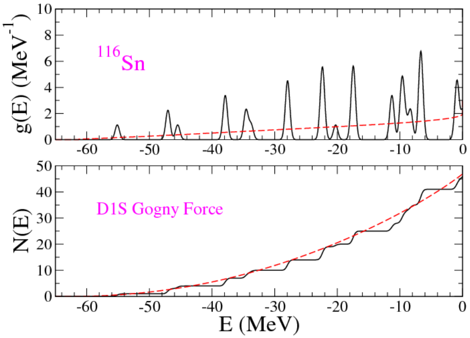
$$f_E(\mathbf{R}, \mathbf{p}) \quad with \quad H_{cl} = \frac{p^2}{2m^*(\mathbf{R})} + V(\mathbf{R})$$

Nucl. Phys.**A665**, 291 (2000); Phys. Rev.**C67**, 014324 (2003); Phys. Rev.**C74**, 064310 (2006)

AVERAGE PAIRING GAPS ALONG SN ISOTOPIC CHAIN

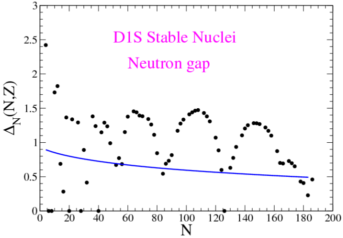


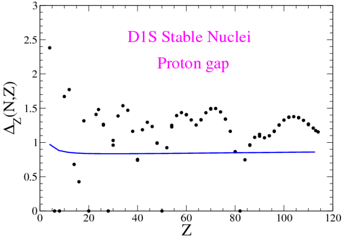
RECOVERING THE ARCH STRUCTURE



$$\tilde{g}(E) = \sum_{i=1}^{n_{tot}} g_{0,i} e^{-(\frac{E-\varepsilon_i}{\sigma})^2} \quad with \quad \sigma = 0.5$$

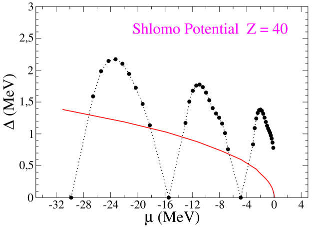
AVERAGE ALONG THE STABILITY LINE

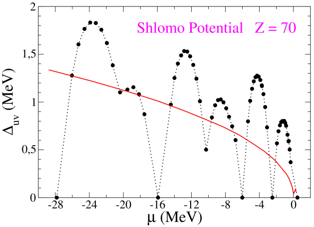




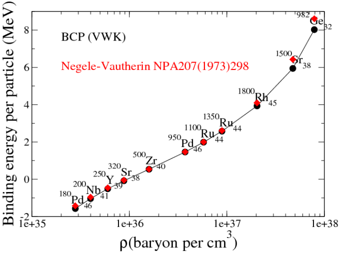
PAIRING ALONG ISOTOPIC CHAINS

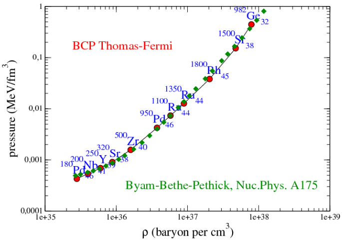
S. Shlomo, Nucl. Phys. **539**, 30 (1992).





TF approach to the inner crust of neutron stars
BCP Energy Density Functional
Finite Temperature





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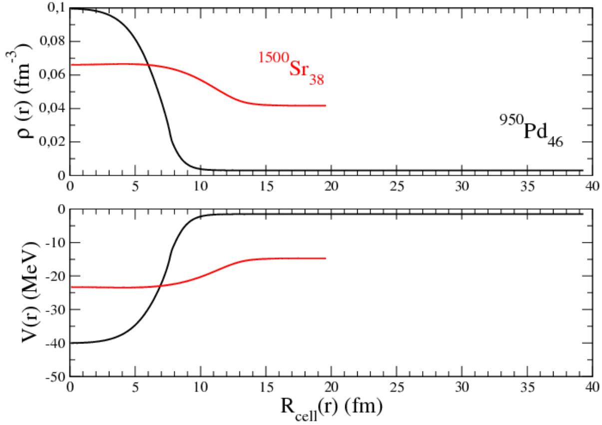
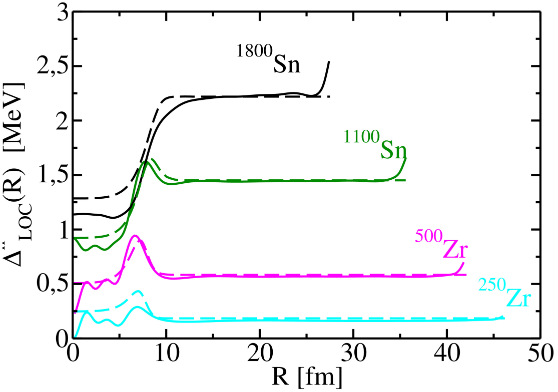
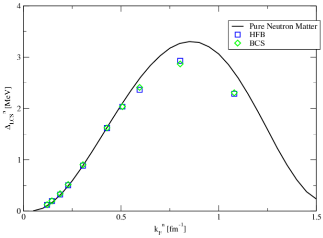


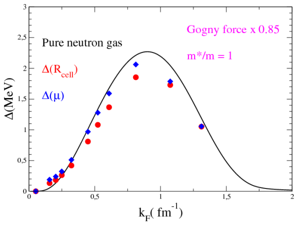
Semiclassical pairing in Wigner-Seitz cells

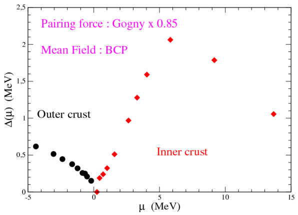


$$\Delta(\mathbf{R}) = \frac{1}{\kappa(\mathbf{R})} \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \Delta(\mathbf{R}, \mathbf{p}) \kappa(\mathbf{R}, \mathbf{p}); \quad \Delta(\mathbf{R}, \mathbf{p}) = -\int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} v(\mathbf{p} - \mathbf{p}') \kappa(\mathbf{R}, \mathbf{p}')$$

Semiclassical pairing in Wigner-Seitz cells

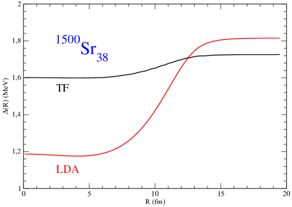


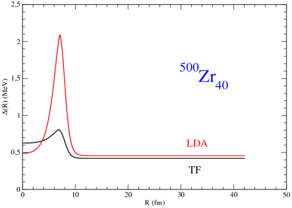




Semiclassical pairing in Wigner-Seitz cells

Comparison between TF and LDA





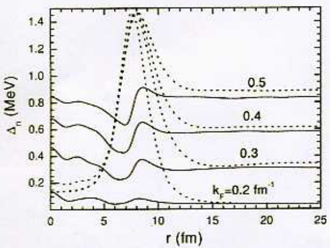


FIG. 6. The neutron gap functions for various k_F values.

N. Chamel et al, Phys. Rev. **C81**, 045804 (2010)

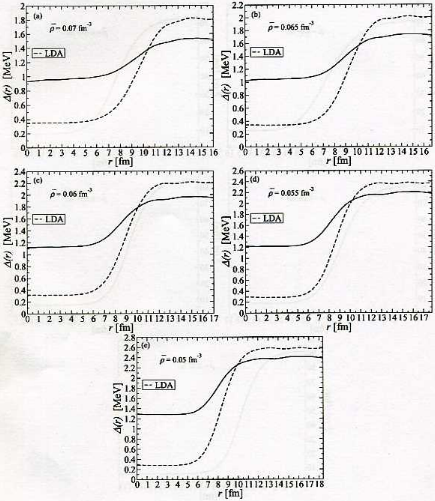
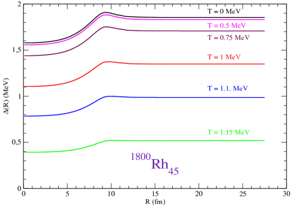
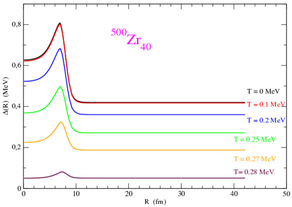


FIG. 4. Angle-averaged neutron pairing field $\Delta(r)$ (solid line) and LDA pairing field (dashed line) for the different crustal layers at $T = 0$.

Semiclassical pairing in Wigner-Seitz cells
Finite Temperature





$$T_c = \frac{e^{\zeta}}{\pi}\Delta_F; \qquad \zeta = 0.577$$

Peter, thank you very much for all these years that I have had the opportunity of sharing your friendship and working with you !!!

Rest in peace