The general THDM in gauge-invariant form

Ingo Schienbein
UGA/LPSC Grenoble

Based on:
Lohan Sartore, Markos Maniatis, IS, Björn Herrmann
JHEP12(2022)051, arXiv:2208.13719

IRN Terasclae, LPSC Grenoble, April 24-26, 2023

The general THDM

- Two Higgs doublets carrying hypercharge $y = +1/2$

$$\varphi_1(x) = \begin{pmatrix} \varphi_1^+(x) \\ \varphi_1^0(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_1^1(x) + i\sigma_1^1(x) \\ \pi_1^2(x) + i\sigma_1^2(x) \end{pmatrix}$$
$$\varphi_2(x) = \begin{pmatrix} \varphi_2^+(x) \\ \varphi_2^0(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_2^1(x) + i\sigma_2^1(x) \\ \pi_2^2(x) + i\sigma_2^2(x) \end{pmatrix}$$

- The most general potential (tree level, gauge invariant)

$$\begin{aligned} V^0_{\mathrm{THDM}}(\varphi_1, \varphi_2) = {} & m_{11}^2(\varphi_1^\dagger\varphi_1) + m_{22}^2(\varphi_2^\dagger\varphi_2) - m_{12}^2(\varphi_1^\dagger\varphi_2) - (m_{12}^2)^*(\varphi_2^\dagger\varphi_1) \\ & + \frac{1}{2}\lambda_1(\varphi_1^\dagger\varphi_1)^2 + \frac{1}{2}\lambda_2(\varphi_2^\dagger\varphi_2)^2 + \lambda_3(\varphi_1^\dagger\varphi_1)(\varphi_2^\dagger\varphi_2) \\ & + \lambda_4(\varphi_1^\dagger\varphi_2)(\varphi_2^\dagger\varphi_1) + \frac{1}{2}\left[\lambda_5(\varphi_1^\dagger\varphi_2)^2 + \lambda_5^*(\varphi_2^\dagger\varphi_1)^2\right] \\ & + \left[\lambda_6(\varphi_1^\dagger\varphi_2) + \lambda_6^*(\varphi_2^\dagger\varphi_1)\right](\varphi_1^\dagger\varphi_1) + \left[\lambda_7(\varphi_1^\dagger\varphi_2) + \lambda_7^*(\varphi_2^\dagger\varphi_1)\right](\varphi_2^\dagger\varphi_2) \end{aligned}$$

- 14 real parameters: $m_{11}^2, m_{22}^2 \in \mathbf{R}$, $m_{12}^2 \in \mathbf{C}$, $\lambda_{1,2,3,4} \in \mathbf{R}$, $\lambda_{5,6,7} \in \mathbf{C}$

Bilinears in the THDM

- 4 **real**, **gauge invariant** bilinears

$$K_0 = \varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2, \qquad K_1 = \varphi_1^\dagger \varphi_2 + \varphi_2^\dagger \varphi_1,$$
$$K_2 = i\big(\varphi_2^\dagger \varphi_1 - \varphi_1^\dagger \varphi_2\big), \qquad K_3 = \varphi_1^\dagger \varphi_1 - \varphi_2^\dagger \varphi_2.$$

- One-to-one map $\varphi_{1,2} \leftrightarrow K_{0,1,2,3}$ (except for unphysical gauge d.o.f)

- Inverse relations

$$\varphi_1^\dagger \varphi_1 = \frac{1}{2}\left(K_0 + K_3\right), \qquad \varphi_1^\dagger \varphi_2 = \frac{1}{2}\left(K_1 + iK_2\right),$$
$$\varphi_2^\dagger \varphi_1 = \frac{1}{2}\left(K_1 - iK_2\right), \qquad \varphi_2^\dagger \varphi_2 = \frac{1}{2}\left(K_0 - K_3\right).$$

- Most general tree level THDM potential in terms of bilinears

$$V^0_{\mathrm{THDM}}(K_0, K_a) = \underbrace{\xi_0 K_0 + \xi_a K_a}_{\text{Linear in } K_{0,1,2,3}} + \underbrace{\eta_{00} K_0^2 + 2K_0 \eta_a K_a + K_a E_{ab} K_b}_{\text{Quadratic in } K_{0,1,2,3}}$$

Bilinears in the THDM

- 4 **real**, **gauge invariant** bilinears

$$K_0 = \varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2, \qquad K_1 = \varphi_1^\dagger \varphi_2 + \varphi_2^\dagger \varphi_1,$$
$$K_2 = i\big(\varphi_2^\dagger \varphi_1 - \varphi_1^\dagger \varphi_2\big), \qquad K_3 = \varphi_1^\dagger \varphi_1 - \varphi_2^\dagger \varphi_2.$$

- Most general tree level THDM potential in terms of bilinears

$$V^0_{\mathrm{THDM}}(K_0, K_a) = \xi_0 K_0 + \xi_a K_a + \eta_{00} K_0^2 + 2K_0 \eta_a K_a + K_a E_{ab} K_b$$

- 14 real parameters:

Masses $\xi_{0,1,2,3}$ **(4)**

$$\xi_0 = \frac{1}{2}\left(m_{11}^2 + m_{22}^2\right), \quad \boldsymbol{\xi} = (\xi_\alpha) = \frac{1}{2}\Big(-2\mathrm{Re}(m_{12}^2),\, 2\mathrm{Im}(m_{12}^2),\, m_{11}^2 - m_{22}^2\Big)^{\mathrm{T}}$$

Couplings $\eta_{00}, \eta_{1,2,3}$ **(4)**

$$\eta_{00} = \frac{1}{8}(\lambda_1 + \lambda_2) + \frac{1}{4}\lambda_3, \quad \boldsymbol{\eta} = (\eta_a) = \frac{1}{4}\Big(\mathrm{Re}(\lambda_6 + \lambda_7),\, -\mathrm{Im}(\lambda_6 + \lambda_7),\, \tfrac{1}{2}(\lambda_1 - \lambda_2)\Big)^{\mathrm{T}}$$

Couplings E_{ab} **(6)**

$$E = (E_{ab}) = \frac{1}{4}\begin{pmatrix} \lambda_4 + \mathrm{Re}(\lambda_5) & -\mathrm{Im}(\lambda_5) & \mathrm{Re}(\lambda_6 - \lambda_7) \\ -\mathrm{Im}(\lambda_5) & \lambda_4 - \mathrm{Re}(\lambda_5) & -\mathrm{Im}(\lambda_6 - \lambda_7) \\ \mathrm{Re}(\lambda_6 - \lambda_7) & -\mathrm{Im}(\lambda_6 - \lambda_7) & \frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_3 \end{pmatrix}$$

4-component notation

Bilinears: $$\widetilde{\boldsymbol{K}} = \begin{pmatrix} K_0 \\ \boldsymbol{K} \end{pmatrix}, \quad \text{with } \boldsymbol{K} = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}$$

Parameters: $$\tilde{\boldsymbol{\xi}} = \begin{pmatrix} \xi_0 \\ \boldsymbol{\xi} \end{pmatrix}, \qquad \tilde{E} = \begin{pmatrix} \eta_{00} & \boldsymbol{\eta}^{\mathrm{T}} \\ \boldsymbol{\eta} & E \end{pmatrix}$$

Potential: $$V^0_{\mathrm{THDM}}(\widetilde{\boldsymbol{K}}) = \boxed{\widetilde{\boldsymbol{K}}^{\mathrm{T}} \tilde{\boldsymbol{\xi}}} + \boxed{\widetilde{\boldsymbol{K}}^{\mathrm{T}} \tilde{E} \widetilde{\boldsymbol{K}}}$$

Very simple, real structure!
Gauge symmetry manifest

Linear in $K_{0,1,2,3}$

Quadratic in $K_{0,1,2,3}$

compare to

$$\begin{aligned} V^0_{\mathrm{THDM}}(\varphi_1, \varphi_2) = {} & m_{11}^2(\varphi_1^\dagger \varphi_1) + m_{22}^2(\varphi_2^\dagger \varphi_2) - m_{12}^2(\varphi_1^\dagger \varphi_2) - (m_{12}^2)^*(\varphi_2^\dagger \varphi_1) \\ & + \frac{1}{2}\lambda_1(\varphi_1^\dagger \varphi_1)^2 + \frac{1}{2}\lambda_2(\varphi_2^\dagger \varphi_2)^2 + \lambda_3(\varphi_1^\dagger \varphi_1)(\varphi_2^\dagger \varphi_2) \\ & + \lambda_4(\varphi_1^\dagger \varphi_2)(\varphi_2^\dagger \varphi_1) + \frac{1}{2}\Big[\lambda_5(\varphi_1^\dagger \varphi_2)^2 + \lambda_5^*(\varphi_2^\dagger \varphi_1)^2\Big] \\ & + \Big[\lambda_6(\varphi_1^\dagger \varphi_2) + \lambda_6^*(\varphi_2^\dagger \varphi_1)\Big](\varphi_1^\dagger \varphi_1) + \Big[\lambda_7(\varphi_1^\dagger \varphi_2) + \lambda_7^*(\varphi_2^\dagger \varphi_1)\Big](\varphi_2^\dagger \varphi_2) \end{aligned}$$

EW symmetry breaking

- **Domain:**
  - $K_0 \geq 0$ (obvious car $K_0 = |\varphi_1|^2 + |\varphi_2|^2$)
  - $K_0^2 - K_1^2 - K_2^2 - K_3^2 \geq 0$ (with Cauchy-Schwarz inequality)

- **EW symmetry at minimum:**
  - **Unbroken** for $K_0 = 0 \Rightarrow V = 0$
  - **Charge breaking** for $K_0 > 0$ and $K_0^2 - K_a^2 > 0$

    CB condition: $\partial_\mu V \equiv \dfrac{\partial V}{\partial K^\mu} = 0$
  - **Charge conserving** for $K_0 > 0$ and $K_0^2 - K_a^2 = 0$

    CC condition: $\partial_\mu [V - u(K_0^2 - k_a^2)] = 0$

    Lagrange multiplier u, minimum at $K_0, K_1, K_2, K_3, K_4, u$

Spaces and bases

Reminder:

$$\varphi_1(x) = \begin{pmatrix} \varphi_1^+(x) \\ \varphi_1^0(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_1^1(x) + i\sigma_1^1(x) \\ \pi_1^2(x) + i\sigma_1^2(x) \end{pmatrix} \qquad \varphi_2(x) = \begin{pmatrix} \varphi_2^+(x) \\ \varphi_2^0(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_2^1(x) + i\sigma_2^1(x) \\ \pi_2^2(x) + i\sigma_2^2(x) \end{pmatrix}$$

Spaces:

- $\varphi_1 \in \mathbf{C}^2$, $\varphi_2 \in \mathbf{C}^2$, space of the $SU(2)_L$ doublets
- $\varphi = (\varphi_1, \varphi_2)^T \in C_2$, grouping the two doublets together which can mix
- $\widetilde{K} = (K_0, K_1, K_2, K_3)^T \in R_4$, space of the 4 real, gauge-invariant bilinears
- $\phi = (\pi_1^1, \pi_1^2, \sigma_1^1, \sigma_1^2, \pi_2^1, \pi_2^2, \sigma_2^1, \sigma_2^2)^T \in R_8$, space of the 8 real component fields

Spaces and bases

- $\varphi_1 \in \mathbf{C}^2$, $\varphi_2 \in \mathbf{C}^2$, $\varphi = (\varphi_1, \varphi_2)^T \in C_2, \widetilde{K} = (K_0, K_1, K_2, K_3)^T \in R_4$
  $\phi = (\pi_1^1, \pi_1^2, \sigma_1^1, \sigma_1^2, \pi_2^1, \pi_2^2, \sigma_2^1, \sigma_2^2)^T \in R_8$

- Note that the parameters in the different spaces are linked.

  For example consider the mixing of two doublets:

  - $\varphi' = U\varphi$ with a unitary 2×2 matrix U

  - The bilinears transform as $K_0' = K_o$, $K_a' = R_{ab}(U)K_b$
    with $R \in SO(3)$ a proper rotation matrix defined by
    $U^\dagger \sigma^a U = R^a_b(U)\sigma^b$

  - The potential stays invariant under a change of basis of the bilinears if we simultaneously transform the parameters:
    $\xi_0' = \xi_0$, $\overrightarrow{\xi}' = R\overrightarrow{\xi}, \eta_{00}' = \eta_{00}$, $\overrightarrow{\eta}' = R\overrightarrow{\eta}, E' = RER^T$
    (Can be used to diagonalise the symmetric matrix E)

Symmetries

- Symmetries straightforward to study (see, e.g., [1,2])

- Example: Standard CP symmetry

$$\varphi_{1,2}(x) \overset{CP}{\to} \varphi^*_{1,2}(x') \text{ with } x = (t, \mathbf{x})^T,\ x' = (t, -\mathbf{x})^T$$

- In terms of bilinears [3,4]

$$K_0 \overset{CP}{\to} K_0, \qquad (K_1, K_2, K_3)^T \overset{CP}{\to} (K_1, -K_2, K_3)^T$$

[1] Ferreira, Maniatis, Nachtmann, Silva, JHEP08(2010); [2] Bento, Boto, Silva, Trautner, JHEP21(2020)
[3] Maniatis, Manteuffel, Nachtmann, EPJC57(2008); [4] Ferreira, Haber, Maniatis, Nachtmann, Silva, IJMPA26(2011)

The complete THDM gauge invariantly

Lohan Sartore, Markos Maniatis, IS, Björn Herrmann,
JHEP12(2022)051, arXiv:2208.13719

- Up to now only potential considered
- How to study gauge couplings, Yukawa couplings?
- How to study potential beyond tree level?
- **Key:**
  - **study mass matrices gauge invariantly**
  - **derive couplings from mass matrices**

Mass matrices

- Mass matrix in terms of the 8 component fields
$\phi = (\pi_1^1, \pi_1^2, \sigma_1^1, \sigma_1^2, \pi_2^1, \pi_2^2, \sigma_2^1, \sigma_2^2)^T$

$$(M_s^2)_{ij} = \frac{\partial^2}{\partial \phi_i \partial \phi_j} V, \qquad i, j \in \{1,\ldots,8\}$$

- Mass matrix in terms of bilinears $K^\mu = (K_0, K_1, K_2, K_3)^T$

$$\mathscr{M} = (\mathscr{M}_{\mu\nu}) = \frac{\partial^2}{\partial K^\mu \partial K^\nu} V \equiv \partial_\mu \partial_\nu V, \qquad \mu, \nu \in \{0,\ldots,3\}$$

- How to change between M_s^2 and $\mathscr{M}$?

Mass matrices

- Establish first connection between component fields and bilinears

- Write bilinears in terms of the component fields:

$$K^\mu \equiv \frac{1}{2}\Delta^\mu_{ij}\phi^i\phi^j, \qquad i,j \in \{1,\ldots,8\}$$

with constants Δ^μ_{ij} (four 8×8 matrices, depending on basis R_8)

- Connection from (gauge-dependent) components to bilinears:

$$\Gamma^\mu_i \equiv \frac{\partial K^\mu}{\partial \phi^i} = \partial_i K^\mu = \Delta^\mu_{ij}\phi^j$$ where Γ is a 8×4 matrix

Mass matrices

- Now we can write the mass matrix in terms of bilinears:

$$(M_s^2)_{ij} = \partial_i \partial_j V = \partial_i (\Gamma_j^{\mu} \partial_\mu V) = \Delta_{ij}^{\mu} \partial_\mu V + \Gamma_i^{\mu} \Gamma_j^{\nu} \partial_\mu \partial_\nu V$$

- In matrix notation:

$M_s^2 = \Delta^{\mu} \partial_\mu V + \Gamma \mathscr{M} \Gamma^T$ with $\mathscr{M} = (\mathscr{M}_{\mu\nu})$

- Use a canonical basis in the space R_8 where Γ is very simple (A rotation in R_8 such that bilinears in R_4 unchanged and where the first 4 rows in Γ vanish)

- Diagonalize $M_s^2 \to \bar{M}_s^2$

Mass matrices

- In the charge conserving (CC) case, we find after some non-trivial algebra that the mass matrix takes a block-diagonal form:

$$\widehat{M}_s^2 \overset{CC}{=} \begin{pmatrix} 0_{3\times 3} & & \\ & \widehat{\mathcal{M}}^2_{\text{charged}} & \\ & & \widehat{\mathcal{M}}^2_{\text{neutral}} \end{pmatrix}$$

- The charged matrix takes the final form

$$\bar{\mathcal{M}}^2_{\text{charged}} = \text{diag}\left(m^2_{H^\pm},\, m^2_{H^\pm}\right) = \text{diag}\left(4uK_0,\, 4uK_0\right)$$

- The neutral matrix takes the form with $\tilde{g} = diag(1, -1, -1, -1)$ and the 3×4 matrix $\gamma_3 = (\mathbf{K}, \mathbf{I}_3)\sqrt{2/K_0}$

$$\widehat{\mathcal{M}}^2_{\text{neutral}} = \gamma_3 \left(\mathcal{M} - 2u\tilde{g}\right) \gamma_3^{\mathrm{T}}$$

- This can be diagonalised by a rotation in bilinear space R_4 for **any potential** V

$$\bar{\mathcal{M}}^2_{\text{neutral}} = R\widehat{\mathcal{M}}^2_{\text{neutral}}R^{\mathrm{T}} = \text{diag}\left(m_1^2,\, m_2^2,\, m_3^2\right)$$

Mass matrices: What have we found?

- Mass matrix manifestly gauge invariant (dependence only on bilinears K_μ)

- Potential V not specified in $\mathscr{M} = \partial_\mu \partial_\nu V$

- Valid in any THDM at any perturbation order

- In particular the charged part reads $m_{H^\pm}^2 = 4uK_0$

- K_μ and u to be determined from stationary point equation and the CC condition depending on the parameters in V

THDM couplings

- Compute the couplings gauge invariantly using the mass matrix. For instance for the scalar cubic and quartic couplings:

$$\lambda_{ijk} = \left(\partial_i M_s^2\right)^{jk} = \left(\Delta_{ij}^{\mu}\Gamma_k^{\nu} + \Delta_{ik}^{\mu}\Gamma_j^{\nu} + \Delta_{jk}^{\mu}\Gamma_i^{\nu}\right)\mathcal{M}_{\mu\nu}\,,$$

$$\lambda_{ijkl} = \left(\partial_i\partial_j M_s^2\right)^{kl} = \left(\Delta_{ij}^{\mu}\Delta_{kl}^{\nu} + \Delta_{ik}^{\mu}\Delta_{jl}^{\nu} + \Delta_{il}^{\mu}\Delta_{jk}^{\nu}\right)\mathcal{M}_{\mu\nu}\,.$$

- Example (using the tree level potential)

$h - H^{\pm} - H^{\pm}$

$$\lambda_{h^a H^\pm H^\pm} = \frac{1}{\sqrt{2K_0}}\left(8K_0(\eta_{00}k^a + \eta^a) - k^a m_a^2\right)$$ **with** $k^a = K^a/K_0$

How to deal with Yukawa couplings?

- Yukawa couplings linear in Higgs doublets, bilinears not

$$-\mathcal{L}_Y = \left[\overline{Q}_L\, \mathcal{U}^a\, \widetilde{\varphi}_a\, u_R + \overline{Q}_L\, \mathcal{D}_a\, \varphi^a\, d_R + \overline{L}\, \mathcal{E}_a\, \varphi^a\, e_R\right] + \text{h.c.}$$

With $\left(\widetilde{\varphi}_a\right)^i = \varepsilon^{ij}\left(\varphi_a^*\right)_j$

$$\mathcal{U}^a = \begin{pmatrix} y_u \\ \epsilon_u \end{pmatrix} \qquad \mathcal{D}_a = \begin{pmatrix} y_d & \epsilon_d \end{pmatrix} \qquad \mathcal{E}_a = \begin{pmatrix} y_e & \epsilon_e \end{pmatrix}$$

- Under a change of basis in the space C_2 the Yukawa couplings transform as

$$\varphi^a \to U^a{}_b\, \varphi^b \quad \Rightarrow \quad \mathcal{D}_a \to \mathcal{D}_b \left(U^\dagger\right)^b{}_a, \qquad \mathcal{E}_a \to \mathcal{E}_b \left(U^\dagger\right)^b{}_a, \qquad \mathcal{U}^a \to U^a{}_b\, \mathcal{U}^b$$

How to deal with Yukawa couplings?

- However, Yukawa couplings are gauge invariant and hence expressible in terms of bilinears:

$$\mathcal{U}^a \mathcal{U}_b^\dagger \equiv \frac{1}{2} Y_u^\mu \left(\sigma_\mu\right)^a{}_b \qquad \mathcal{D}_a \mathcal{D}^{\dagger\, b} = \frac{1}{2} Y_d^\mu \left(\sigma_\mu\right)^b{}_a \qquad \mathcal{E}_a \mathcal{E}^{\dagger\, b} = \frac{1}{2} Y_e^\mu \left(\sigma_\mu\right)^b{}_a$$

Similar to $\underline{K}^a{}_b = \varphi^a \varphi_b^\dagger = \frac{1}{2} K^\mu \left(\sigma_\mu\right)^a{}_b$

- Explicitly we find 4-component bilinear Yukawa couplings:

$$Y_u = \begin{pmatrix} y_u y_u^\dagger + \epsilon_u \epsilon_u^\dagger \\ y_u \epsilon_u^\dagger + \epsilon_u y_u^\dagger \\ i\left(y_u \epsilon_u^\dagger - \epsilon_u y_u^\dagger\right) \\ y_u y_u^\dagger - \epsilon_u \epsilon_u^\dagger \end{pmatrix}, \quad Y_d = \begin{pmatrix} y_d y_d^\dagger + \epsilon_d \epsilon_d^\dagger \\ y_d \epsilon_d^\dagger + \epsilon_d y_d^\dagger \\ -i\left(y_d \epsilon_d^\dagger - \epsilon_d y_d^\dagger\right) \\ y_d y_d^\dagger - \epsilon_d \epsilon_d^\dagger \end{pmatrix}, \quad Y_e = \begin{pmatrix} y_e y_e^\dagger + \epsilon_e \epsilon_e^\dagger \\ y_e \epsilon_e^\dagger + \epsilon_e y_e^\dagger \\ -i\left(y_e \epsilon_e^\dagger - \epsilon_e y_e^\dagger\right) \\ y_e y_e^\dagger - \epsilon_e \epsilon_e^\dagger \end{pmatrix}$$

- They transform in the same way as K_μ under a change of basis

How to deal with Yukawa couplings?

- Observation: for a neutral vacuum the rank of the bilinear matrix is one such that it can be 'linearized' in terms of a vector κ^a. In the mass basis:

$$\underline{\bar{K}}^a{}_b = \frac{1}{2}\bar{K}^\mu \left(\sigma_\mu\right)^a{}_b \equiv \bar{\kappa}^a \bar{\kappa}^*_b$$

- The Yukawa couplings can then be expressed in terms of the vector $\bar{\kappa}^a$

$$\bar{\kappa} = \sqrt{\frac{K_0}{2}} \frac{1}{\sqrt{1+\bar{k}_3}} \begin{pmatrix} 1+\bar{k}_3 \\ \bar{k}_1 + i\bar{k}_2 \end{pmatrix} = \sqrt{\frac{K_0}{2}} \begin{pmatrix} \sqrt{1+\bar{k}_3} \\ \sqrt{1-\bar{k}_3}\, e^{i\zeta} \end{pmatrix}$$

- For example, the neutral Higgs coupling to up-type quarks: $h^a - u_L^\dagger - u_R$ can be written in very compact form:

$$\frac{1}{\sqrt{2K_0}} \left(\sigma_a\right)^\alpha{}_\beta \, \bar{\kappa}^*_\alpha \bar{\mathcal{U}}^\beta$$

Conclusions

- Mass matrices to all orders computed
- Complete THDM formulated gauge invariantly:
  Scalar, Yukawa, gauge boson couplings
- Program of bilinears completed!
  Sartore, Maniatis, Schienbein, Herrmann, JHEP12(2022)051, arXiv:2208.13719
- ToDo:
  - Phenomenological applications
  - Further work out dictionnary between bilinear formalism and conventional formalism with concrete examples
  - Go beyond tree level