Low vs moderate $x_{\rm Bj}$ matching: how x is strictly 0 and what we can do about it

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Saturation at the EIC

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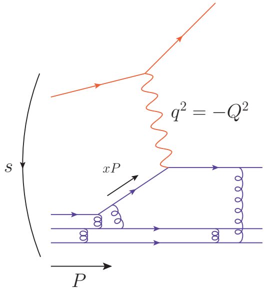
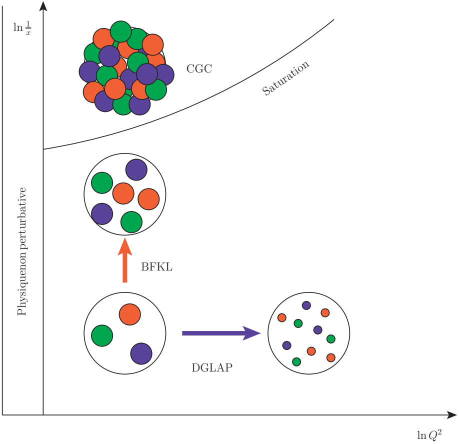
Accessing the partonic content of hadrons with an electromagnetic probe

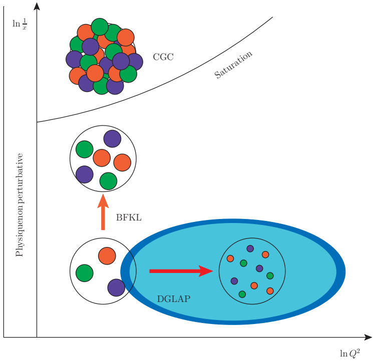


Electron-proton collision (parton model)



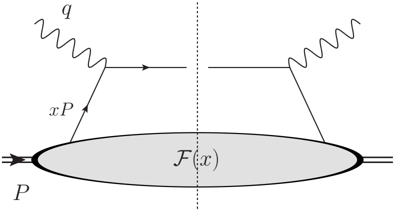
QCD at moderate $x_{\mathrm{Bj}} \sim Q^2/s$

Bjorken limit: $Q^2 \sim s$



QCD factorization

processes with a hard scale $Q \gg \Lambda_{QCD}$



$$\sigma = \mathcal{F}(x, \mu) \otimes \mathcal{H}(x, \mu)$$

At a scale μ, the process is factorized into:

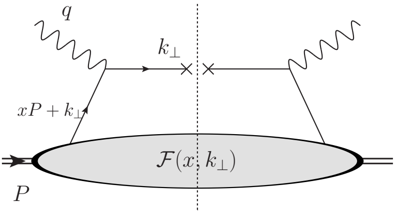
- A hard scattering subamplitude $\mathcal{H}(x, \mu)$
- A Parton Distribution Function (PDF) $\mathcal{F}(x, \mu)$

μ independence: DGLAP renormalization equation for $\mathcal{F}$

Transverse Momentum Dependent (TMD) factorization:

semi-inclusive processes with one hard and one semihard scale

$$Q \sim \sqrt{s} \gg k_\perp$$



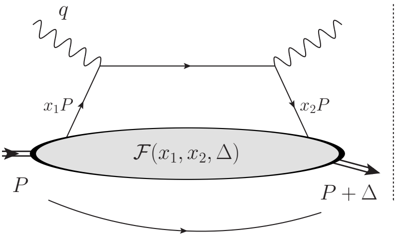
$$\sigma = \mathcal{F}(x, k_\perp, \zeta, \mu) \otimes \mathcal{H}(\mu) \otimes \hat{\mathcal{F}}(\hat{x}, \hat{k}_\perp, \hat{\zeta}, \mu)$$

At a scale μ, the process is factorized into:

- A hard scattering subamplitude $\mathcal{H}(\mu)$
- A TMD PDF $\mathcal{F}(x, k_\perp, \zeta, \mu)$
- A TMD FF $\hat{\mathcal{F}}(\hat{x}, \hat{k}_\perp, \hat{\zeta}, \mu)$

$\mu, \zeta, \hat{\zeta}$ independence: TMD evolution for $\mathcal{F}, \hat{\mathcal{F}}$

Factorization with Generalized Parton Distributions (GPD): exclusive processes with one hard scale $Q \sim \sqrt{s}$



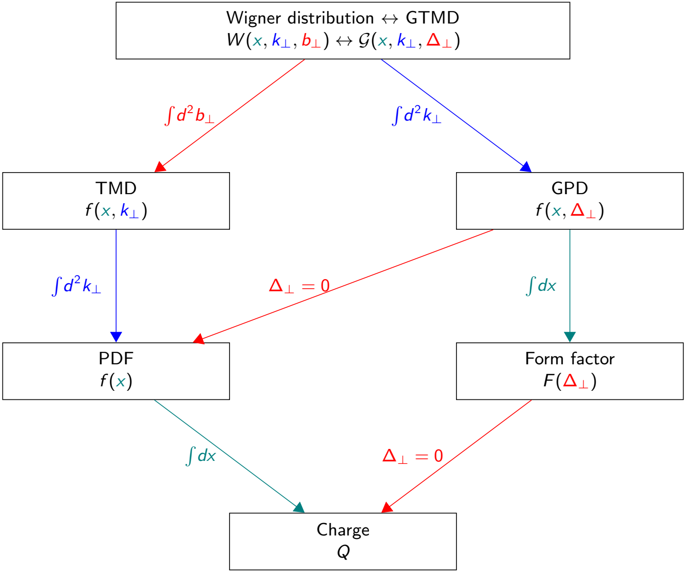
$$\sigma = \mathcal{F}(x_1, x_2, |\Delta_\perp|, \mu) \otimes \mathcal{H}(x_1, x_2, \mu)$$

At a scale μ, the process is factorized into:

- A hard scattering subamplitude $\mathcal{H}(x_1, x_2, \mu)$
- A Generalized Parton Distribution (GPD) $\mathcal{F}(x_1, x_2, |\Delta_\perp|, \mu)$

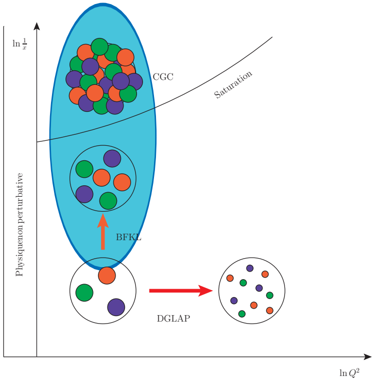
μ independence: DGLAP/ERBL renormalization equation for $\mathcal{F}$

The family tree of parton distributions

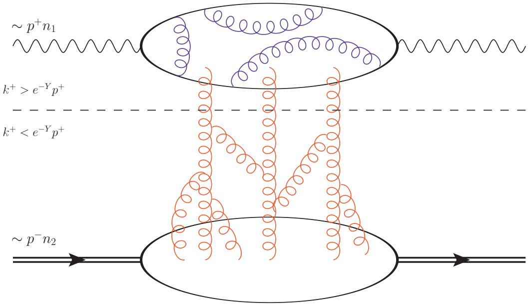


QCD at small $x_{\mathrm{Bj}} \sim Q^2/s$

Regge limit: $Q^2 \ll s$



Rapidity separation

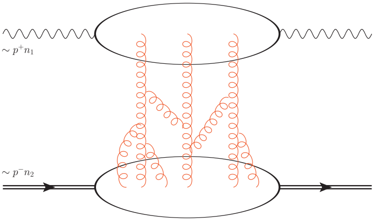


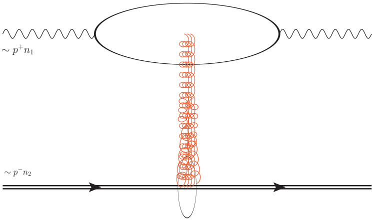
Let us split the gluonic field between "fast" and "slow" gluons

$$\begin{aligned}\mathcal{A}^{\mu a}(k^+, k^-, \boldsymbol{k}) &= A^{\mu a}(|k^+| > e^{-Y} p^+, k^-, \boldsymbol{k}) \\ &+ A^{\mu a}_{\rm cl}(|k^+| < e^{-Y} p^+, k^-, \boldsymbol{k})\end{aligned}$$

$e^{-Y} \ll 1$

Large longitudinal boost to the projectile frame





$$A^+_{\rm cl}(x^+, x^-, \boldsymbol{x}) \qquad \longrightarrow \qquad \frac{1}{\Lambda} A^+_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x})$$

$$A^-_{\rm cl}(x^+, x^-, \boldsymbol{x}) \qquad \longrightarrow \qquad \Lambda A^-_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x})$$

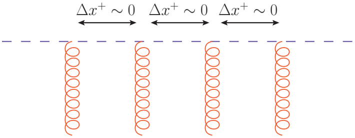
$$A^i_{\rm cl}(x^+, x^-, \boldsymbol{x}) \qquad \Lambda \sim \sqrt{\frac{s}{m_t^2}} \qquad A^i_{\rm cl}(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x})$$

$$A^\mu_{\rm cl}(x) \to A^-_{\rm cl}(x)\, n_2^\mu = \delta(x^+)\, \mathbf{A}(\boldsymbol{x})\, n_2^\mu + O(\sqrt{\frac{m_t^2}{s}})$$

Shock wave approximation

Effective Feynman rules in the slow background field

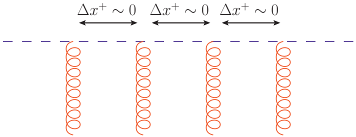
Effective fermion propagator in the external classical field



- $A^i_{\rm cl} = 0, \; A^+_{\rm cl} = 0$: the Dirac structure factorizes
- $A_{\rm cl}$ does not depend on x^-: conservation of $+$ momentum
- $A_{\rm cl}$ is peaked around $x^+ = 0$:
  - Most external propagators get factorized out
  - Gaussians $\sim \delta$ functions: conservation of transverse position
  - Possibility to extend Wilson lines to infinity $[x^+, y^+]_x = [\infty^+, -\infty^+]_x \equiv U_x$

Effective Feynman rules in the slow background field

The interactions with the background field can be exponentiated

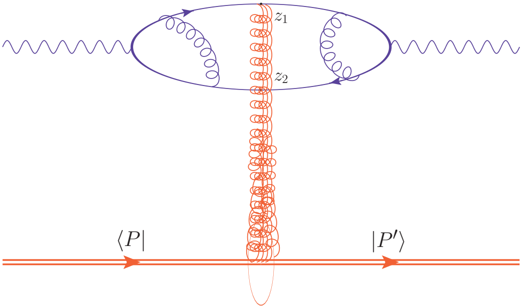


$$D_F(x_2, x_0)|_{x_2^+>0, x_0^+<0} = \int \mathrm{d}^D x_1 \, \delta(x_1^+) \, D_0(x_2, x_1) \, \gamma^+ \, U_{\boldsymbol{x}_1} \, D_0(x_1, x_0)$$

Each fast parton is dressed by an infinite Wilson line

$$U_x \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} \mathrm{d}x \cdot A_{\mathrm{cl}}(x) \right]$$

Factorized picture



Factorized amplitude

$$\mathcal{S} = \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2\, \Phi^Y(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \langle P'|[\mathrm{Tr}(U^Y_{x_1} U^{Y\dagger}_{x_2}) - N_c]|P\rangle$$

Written similarly for any number of Wilson lines in any color representation!

Y independence: B-JIMWLK, BK equations. Resums logarithms of s

The seemingly incompatible nature of the distributions

Two different kinds of gluon distributions

Moderate x distributions

TMD, PDF...

$$\langle P | F^{-i} W F^{-j} W | P \rangle$$

Low x distributions

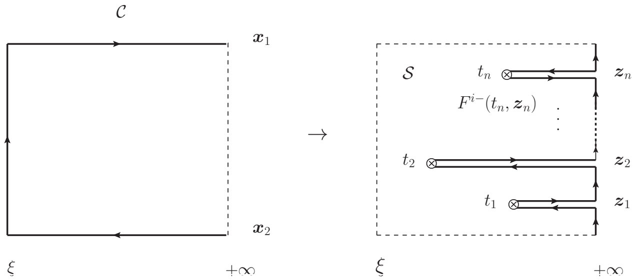
Dipole scattering amplitude

$$\langle P | \mathrm{tr}(U_1 U_2^\dagger) | P \rangle$$

The Wilson line ↔ parton distribution equivalence

Most general equivalence: use the Non-Abelian Stokes theorem

[RB, Mehtar-Tani]



$$\mathcal{P}\exp\left[\oint_{\mathcal{C}} dx_\mu A^\mu(x)\right] = \mathcal{P}\exp\left[\int_{\mathcal{S}} d\sigma_{\mu\nu}\, W F^{\mu\nu} W^\dagger\right]$$

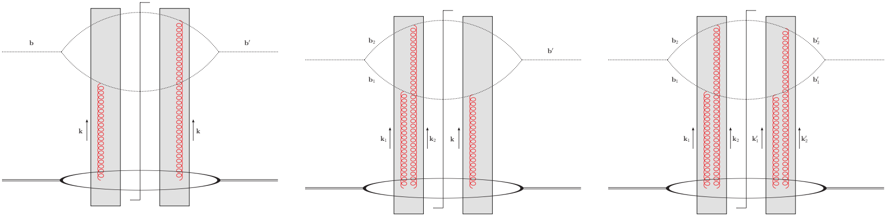
$$U_{x_{1\perp}} U^\dagger_{x_{2\perp}} = [\hat{x}_{1\perp}, \hat{x}_{2\perp}]$$

Inclusive low x cross section

Inclusive low x cross section $=$ TMD cross section

[Altinoluk, RB, Kotko], [Altinoluk, RB]

Generalizes [Dominguez, Marquet, Xiao, Yuan]



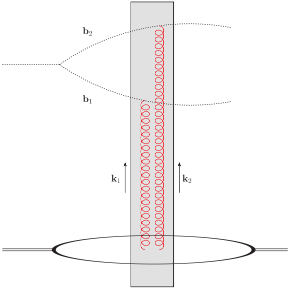
$$\sigma = \mathcal{H}_2^{ij}\left(\boldsymbol{k}\right) \otimes f_2^{ij}(x = 0, \boldsymbol{k})$$
$$+\, \mathcal{H}_3^{ijk}\left(\boldsymbol{k}, \boldsymbol{k}_1\right) \otimes f_3^{ijk}(x = 0, x_1 = 0, \boldsymbol{k}, \boldsymbol{k}_1)$$
$$+\, \mathcal{H}_4^{ijkl}\left(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_1'\right) \otimes f_4^{ijkl}(x = 0, x_1 = 0, x_1' = 0, \boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_1')$$

All distributions are evaluated in the strict $x = 0$ limit

Exclusive low x cross section

Exclusive low x amplitude = GTMD amplitude

[Altinoluk, RB]

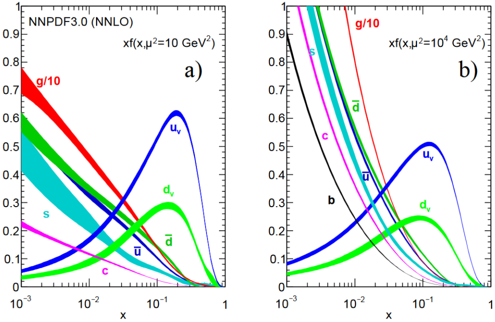


$$\mathcal{H}^{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2) \otimes f^{ij}(x = 0, \xi = 0; \boldsymbol{k}, \Delta)$$

Every exclusive low x process probes a Wigner distribution!

All distributions are evaluated in the strict $x = 0$ limit

All distributions are evaluated in the strict $x = 0$ limit



[NNLO NNPDF3.0 global analysis, taken from PDG2018]

Instabilities in the collinear corner of the phase space

All distributions are evaluated in the strict $x = 0$ limit

Hard part $\mathcal{H}$ and gluon distribution f for an inclusive observable:

Bjorken limit	Leading twist of the CGC
$s \sim Q^2$	$s \gg Q^2, Q^2 \to \infty$
$\int \mathrm{d}x f(x) \mathcal{H}(x)$	$f(0) \int \mathrm{d}x \mathcal{H}(x)$

Strong mismatch beyond LL: the PDF is not a constant in $x \simeq 0$.

Too late to restore a dependence on x via evolution: x is already integrated over

Summary so far

Distributions involved in pQCD observables

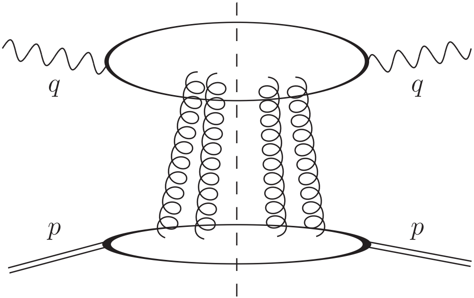
Overarching scheme?

$$f(x_1 \ldots x_n;\, k_{\perp 1} \ldots k_{\perp n})$$

Bjorken limit	Regge limit
$s \sim Q^2$	$s \gg Q^2$
$f(x; 0_\perp) + O(Q^{-2})$	$f(0 \ldots 0, k_{\perp 1} \ldots k_{\perp n}) + O(x_{\mathrm{Bj}})$

Look for an interpolating scheme for simple observables

An interpolating scheme for inclusive DIS



Bjorken limit

$$s \sim Q^2$$

$$f(x, k_\perp = 0) + O(Q^{-2})$$

Regge limit

$$s \gg Q^2$$

$$f(x = 0, k_\perp) + O(x_{\rm Bj})$$

Interpolation?

$$s \gtrsim Q^2$$

$$f(x, k_\perp) + O(x_{\rm Bj} Q^{-2})$$

Basic observation: in both limits, $k^+ \simeq 0$ for t-channel gluons

Factorization in k^+ space is consistent
[Balitsky, Tarasov]

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the Regge limit:

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, x^-, \boldsymbol{x})\, n_2^\mu + A_\perp^\mu(x^+, x^-, \boldsymbol{x})$$

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

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Dependence on x^-: sub-sub-leading in twist counting

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

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Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x})\, n_2^\mu + A_\perp^\mu(x^+, 0^-, \boldsymbol{x})$$

Non-zero $A_\perp$: only two A^i contribute to DDVCS

They can be computed using Ward-Takahashi: only necessary for consistency checks, can be dropped.

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the Regge limit:

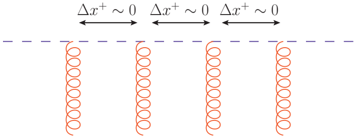
$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit:

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x})\, n_2^\mu$$

Effective Feynman rules in the slow background field

Effective fermion propagator in the external classical field



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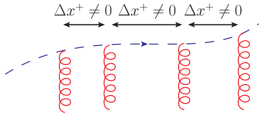

- $A^i_{\mathrm{cl}} = 0,\ A^+_{\mathrm{cl}} = 0$: the Dirac structure factorizes
- A_{cl} does not depend on x^-: conservation of $+$ momentum
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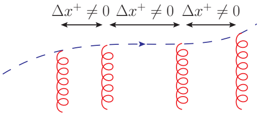
Effective Feynman rules in the slow background field

Effective fermion propagator in the external classical field

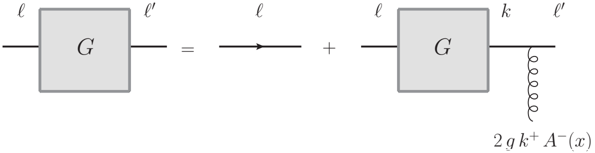


- $A^i_{\rm cl} = 0, \; A^+_{\rm cl} = 0$: the Dirac structure factorizes
- $A_{\rm cl}$ does not depend on x^-: conservation of + momentum

$$D_F(\ell', \ell) = i\frac{\gamma^+}{2\ell^+}(2\pi)^D \delta^D(\ell' - \ell) + i\frac{\not{\ell}'\gamma^+\not{\ell}}{2\ell^+} G_{\rm scal}(\ell', \ell)$$

Effective Feynman rules in the slow background field

Effective scalar propagator in the external classical field



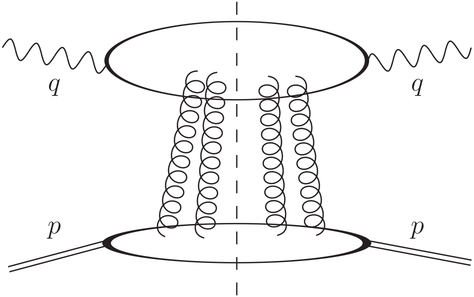
$$G_{\mathrm{scal}}(\ell',\ell) - G_0(\ell')(2\pi)^D\delta^D(\ell'-\ell)$$
$$= 2g\int \mathrm{d}^D z \int \frac{\mathrm{d}^D k}{(2\pi)^D}\mathrm{e}^{i(\ell'-k)\cdot z} G_0(\ell')\,(k\cdot A)(z)\,G_{\mathrm{scal}}(k,\ell).$$

In coordinate space, it satisfies the Klein-Gordon equation in a potential

$$[-\Box_z + 2igA(z)\cdot\partial_z]\,G_{\mathrm{scal}}(z,z_0) = \delta^D(z-z_0)$$

Computation

Application to DIS



$$\mathcal{A} = \frac{e^2}{\mu^{d-2}} \varepsilon_q^{\mu} \varepsilon_q^{\nu *} \sum_f q_f^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \int \frac{\mathrm{d}^D k}{(2\pi)^D}$$

$$\times \langle p | \mathrm{tr} \left[\gamma_\nu D_F(\ell + k, \ell) \gamma_\mu D_F(-q + \ell, -q + \ell + k) \right] | p \rangle$$

(First) final result

Fully general result

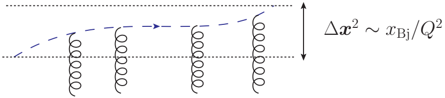
$$\mathcal{A} \propto \mathcal{U}^{ij}(z, q, \ell_1, \ell_2) \otimes_{z,\ell_1,\ell_2} (\partial^i \Phi)(z, \ell_1)(\partial^j \Phi^*)(z, \ell_2)$$

- Φ: standard wave functions
- $\mathcal{U}^{ij}$: generalization of the dipole operator

Contains unnecessary subleading powers of x_{Bj} and Q

Further simplifcations

Partial twist expansion



Typical transverse recoil of a fast parton:
$$\Delta x^2 \sim 1/(2q^+P^-) \sim x_{\mathrm{Bj}}/Q^2$$

x_{Bj}-suppressed in the Regge limit

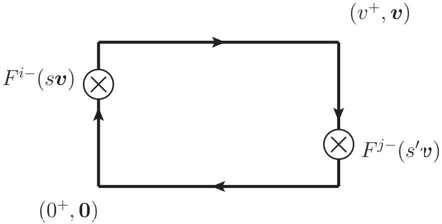
$1/Q^2$-suppressed in in the Bjorken limit.

We can get rid of all corrections from transverse recoils without loss of accuracy

Further simplifications

Partial twist expansion

$$\frac{\langle p|\mathcal{U}^{ij}(z,q,\boldsymbol{\ell}_1,\boldsymbol{\ell}_2)|p\rangle}{\langle p|p\rangle} \simeq -i\frac{(2\pi)^d}{8z\bar{z}(q^+)^2}\int \mathrm{d}x \frac{\mathcal{G}^{ij}(x,\boldsymbol{\ell}_2-\boldsymbol{\ell}_1)}{x-x_{\mathrm{Bj}}-\frac{\left(\frac{\boldsymbol{\ell}_1+\boldsymbol{\ell}_2}{2}\right)^2}{2z\bar{z}q^+P^-}+i0},$$



x-dependent unintegrated PDF

$$\mathcal{G}^{ij}(x,\boldsymbol{k}) \equiv \frac{1}{P^-}\int\frac{\mathrm{d}v^+}{2\pi}\mathrm{e}^{ixP^-v^+}\int\frac{\mathrm{d}^d\boldsymbol{v}}{(2\pi)^d}\mathrm{e}^{-i(\boldsymbol{k}\cdot\boldsymbol{v})}\int_0^1\mathrm{d}s\mathrm{d}s'$$
$$\times\left\langle p\left|\mathrm{tr}_c\left\{\left[v^+,0^+\right]_0 F^{i-}\left(0^+,s\boldsymbol{v}\right)\left[0^+,v^+\right]_{\boldsymbol{v}} F^{j-}\left(v^+,s'\boldsymbol{v}\right)\right\}\right|p\right\rangle$$

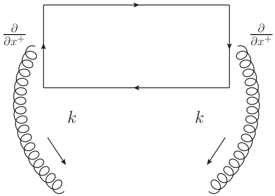
The unintegrated PDF

uPDF as a finite Wilson loop

$$\int \mathrm{d}^2\boldsymbol{k}\,\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\boldsymbol{r}^i\boldsymbol{r}^j\mathcal{G}^{ij}(x,\xi,\boldsymbol{k},\Delta)$$

$$= \frac{1}{\alpha_s}\int\frac{\mathrm{d}^4 v_1\mathrm{d}^4 v_2}{(2\pi)^4}\delta(v_1^-)\delta(v_2^-)\mathrm{e}^{-i(k-\frac{\Delta}{2})\cdot v_1+i(k+\frac{\Delta}{2})\cdot v_2}$$

$$\times\frac{\partial}{\partial v_1^+}\frac{\partial}{\partial v_2^+}\frac{\langle p|\mathrm{tr}[v_1^+,v_2^+]_{\boldsymbol{v}_1}[\boldsymbol{v}_1,\boldsymbol{v}_2]_{v_2^+}[v_2^+,v_1^+]_{\boldsymbol{v}_2}[\boldsymbol{v}_2,\boldsymbol{v}_1]_{v_1^+}|p\rangle}{\langle p|p\rangle}$$



x-dependent unintegrated PDF $\Leftrightarrow$ FT of a finite Wilson loop

(Actual) final result

Final expression for the amplitude

$$\mathrm{Im}\mathcal{A} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{k}$$
$$\times (\partial^i \Phi)(z, \boldsymbol{\ell} - \boldsymbol{k}/2)(\partial^j \Phi^*)(z, \boldsymbol{\ell} + \boldsymbol{k}/2)$$
$$\times \int \mathrm{d}x\, \mathcal{G}^{ij}(x, \boldsymbol{k})\ \delta\left(x - x_{\mathrm{Bj}}\left(1 + \frac{\boldsymbol{\ell}^2}{z\bar{z}Q^2}\right)\right)$$

Standard wave functions Φ

x-dependent unintegrated PDF $\mathcal{G}^{ij}(x, \boldsymbol{k})$

Bjorken limit and Regge limit

Recovering the Bjorken limit

The Bjorken limit is reached by neglecting transverse momentum transfert from the target:

$$|\boldsymbol{\ell}| \sim Q \gg |\boldsymbol{k}|$$

Key observation: $\mathcal{G}^{ij}$ integrates into PDFs

$$\int \mathrm{d}^d \boldsymbol{k} (\partial^i \phi)(z, \boldsymbol{\ell} - \boldsymbol{k}/2)(\partial^j \phi^*)(z, \boldsymbol{\ell} + \boldsymbol{k}/2)\mathcal{G}^{ij}(x, \boldsymbol{k})$$
$$\simeq (\partial^i \phi)(z, \boldsymbol{\ell})(\partial^j \phi^*)(z, \boldsymbol{\ell}) \int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \boldsymbol{k})$$
$$\simeq (\partial^i \phi)(z, \boldsymbol{\ell})(\partial^j \phi^*)(z, \boldsymbol{\ell}) G^{ij}(x)$$

We fully recover the well-known one-loop DIS cross section

The Regge limit

Recovering the Bjorken limit

The Regge limit is reached by neglecting x_{Bj}:

$$\delta\left[x - x_{\mathrm{Bj}}\left(1 + \frac{\ell^2}{z\bar{z}Q^2}\right)\right] \simeq \delta(x) \neq \delta(x - x_{\mathrm{Bj}})$$

Key observation:

$$\int \frac{\mathrm{d}^d \boldsymbol{\ell}_1}{(2\pi)^d} \int \frac{\mathrm{d}^d \boldsymbol{\ell}_2}{(2\pi)^d} \mathrm{e}^{-i(\boldsymbol{\ell}_1 \cdot \boldsymbol{r}_1) + i(\boldsymbol{\ell}_2 \cdot \boldsymbol{r}_2)} \boldsymbol{r}_1^i \boldsymbol{r}_2^j \left[x G^{ij}(x, \boldsymbol{\ell}_2 - \boldsymbol{\ell}_1) \right]_{x=0}$$

$$= \frac{N_c}{2\pi^2 \alpha_s} \delta^d(\boldsymbol{r}_1 - \boldsymbol{r}_2) \int \frac{\mathrm{d}^d \boldsymbol{v}_2}{(2\pi)^d} \mathrm{Re} \frac{\left\langle P \middle| 1 - \frac{1}{N_c} \mathrm{tr}_c \left(U_{\boldsymbol{v}_2 + \boldsymbol{r}_1} U_{\boldsymbol{v}_2}^\dagger \right) \middle| P \right\rangle}{\left\langle P \middle| P \right\rangle}$$

The Regge limit

Recovering the Regge limit

$$(\partial^i \Phi)(z, \ell - \frac{\boldsymbol{k}}{2})(\partial^j \Phi^*)(z, \ell + \frac{\boldsymbol{k}}{2}) \otimes_{\ell, \boldsymbol{k}} xG^{ij}(x, \boldsymbol{k})\delta(x)$$

$$\to \Psi(z, \boldsymbol{r}_1)\Psi^*(z, \boldsymbol{r}_2) \otimes_{\boldsymbol{r}_1, \boldsymbol{r}_2} \boldsymbol{r}_1^i \boldsymbol{r}_2^j \left[xG^{ij}(x, \boldsymbol{k})\right]_{x=0}$$

$$\to \Psi(z, \boldsymbol{r}_1)\Psi^*(z, \boldsymbol{r}_2) \otimes_{\boldsymbol{r}_1, \boldsymbol{r}_2} \delta^d(\boldsymbol{r}_1 - \boldsymbol{r}_2) UU$$

$$\to |\Psi(z, \boldsymbol{r})|^2 \otimes_{\boldsymbol{r}} D(\boldsymbol{r})$$

We fully recover the well-known small-x_{Bj} DIS structure functions.

Rq: $x = 0$ is the reason why wave functions involve the same dipole size in the wave functions

Summary for the DIS case

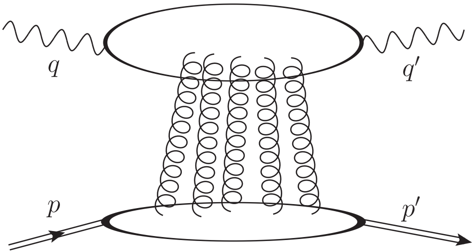
Removing the light cone time scale separation hypothesis
$\Rightarrow$ fully restored dependence on the x variable

However:

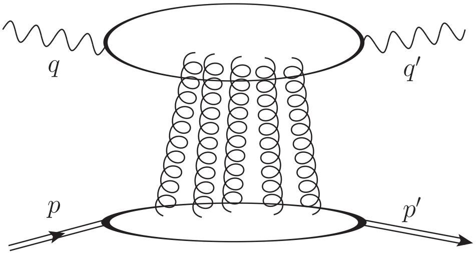
- The x dependence for inclusive DIS is boringly trivial
- Inclusive DIS only involves one longitudinal variable: x

What about an exclusive amplitude?

An interpolating scheme for the $\gamma^{(*)}(q)P(p) \to \gamma^{(*)}(q')P(p')$ amplitude



Double, Spacelike, and Timelike exclusive Compton Scattering



Longitudinal momentum variables:

$$x, \quad \xi \sim \frac{-q^2 + q'^2}{2q \cdot (p + p')}, \quad x_{\mathrm{Bj}} = \frac{-q^2 - q'^2}{2q \cdot (p + p')}$$

Can we restore the dependence on all 3 variables in our CGC-like scheme?

(Actual) final result

Final expression for the amplitude

$$\mathcal{A} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{k}$$
$$\times (\partial^i \Phi)(z, \boldsymbol{\ell} - \boldsymbol{k}/2)(\partial^j \Phi^*)(z, \boldsymbol{\ell} + \boldsymbol{k}/2)$$
$$\times \int \mathrm{d}x \frac{\mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \Delta)}{x - x_{\mathrm{Bj}} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+P^-} + i0}$$

Standard wave functions Φ

(x, ξ)-dependent unintegrated GPD $\mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \Delta)$

Bjorken limit, Regge limit and their non-commutativity

The Bjorken limit

Recovering the Bjorken limit

The Bjorken limit is reached by neglecting transverse momentum transfert from the target:

$$|\boldsymbol{\ell}| \sim Q, Q' \gg |\boldsymbol{k}|$$

Key observation: $\mathcal{G}^{ij}$ integrates into GPDs

$$\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \Delta) = G^{ij}(x, \xi, \Delta)$$

We fully recover the well-known one-loop exclusive Compton scattering amplitudes

The Regge limit

Recovering the Regge limit? What is x?

Naive argument

- In the Regge limit, the amplitude is dominated by its imaginary part
- Leading order amplitude:

$$\mathrm{Im}\mathcal{A}_{LO} \propto \mathrm{Im} \int \mathrm{d}x H^q(x, \xi, t) \frac{1}{x - x_{\mathrm{Bj}} + i\epsilon} = -\pi H^q(x_{\mathrm{Bj}}, \xi, t)$$

- Hence take $x = x_{\mathrm{Bj}}$

Problems

- At NLO, the x cut is way more complicated
- For DDVCS and for TCS, s-channel cuts also contribute to the imaginary part

Recovering the Regge limit

The Regge limit is reached by neglecting $x_{\rm Bj}$ and setting $\frac{\ell^2}{z\bar{z}} \ll q \cdot P$, then taking the x cut:

$$\frac{1}{x - x_{\rm Bj} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \to \frac{1}{x + i0} \to -i\pi\delta(x),$$

then taking $x_{\rm Bj}, \xi \ll 1$.
Rq: the $x = 0$ limit of the uGPD matches the dipole operator.

We recover the small x description of exclusive Compton scattering see e.g. [Hatta, Xiao, Yuan]

Rq: x is strictly 0 in the uGPD

Summary

Interpolating scheme for exclusive Compton scattering

Overarching scheme

$$\int \mathrm{d}x \int \mathrm{d}^d k \mathcal{G}^{ij}(x, \xi, k, \Delta) H^{ij}(x, \xi, k, \Delta)$$

Bjorken limit

$$\int \mathrm{d}x H^{ij}(x, \xi, 0, \Delta) \times [\int \mathrm{d}^d k \mathcal{G}^{ij}(x, \xi, k, \Delta)]$$

Regge limit

$$\int \mathrm{d}^d k \mathcal{G}^{ij}(0, \xi, k, \Delta) \times [\int \mathrm{d}x H^{ij}(x, \xi, k, \Delta)]$$

We found an interpolating scheme

Double limit

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2\to\infty} \mathcal{A}_{\mathrm{Regge}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d\boldsymbol{\ell}}{(2\pi)^d}$$
$$\times (-i\pi) G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\boldsymbol{\ell})(\partial^j\Phi^*)(z,\boldsymbol{\ell})$$

Eikonal limit of the Bjorken limit

$$\lim_{x_{\mathrm{Bj}},\xi\to 0} \mathcal{A}_{\mathrm{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d\boldsymbol{\ell}}{(2\pi)^d}$$
$$\times \lim_{x_{\mathrm{Bj}},\xi\to 0} \int \mathrm{d}x \frac{G^{ij}(x,\xi,t)(\partial^i\Phi)(z,\boldsymbol{\ell})(\partial^j\Phi^*)(z,\boldsymbol{\ell})}{x - x_{\mathrm{Bj}} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+P^-} + i0}$$

Double limit

Do the two limits commute?

If $G^{ij}(x,\xi,t)$ is a constant at $x = 0$:

$$\int \mathrm{d}x \, \frac{G^{ij}(x,\xi,t)(\partial^i \Phi)(z,\ell)(\partial^j \Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$\simeq G^{ij}(0,\xi,t) \int \mathrm{d}x \, \frac{(\partial^i \Phi)(z,\ell)(\partial^j \Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$= G^{ij}(0,\xi,t)(\partial^i \Phi)(z,\ell)(\partial^j \Phi^*)(z,\ell)$$

$$\times \ln \left(\frac{1 - x_{\mathrm{Bj}} - \frac{\ell^2}{z\bar{z}\frac{Q^2+Q'^2}{2}}\xi + i0}{-1 - x_{\mathrm{Bj}} - \frac{\ell^2}{z\bar{z}\frac{Q^2+Q'^2}{2}}\xi + i0} \right)$$

and thus

$$\lim_{x_{\mathrm{Bj}},\xi \to 0} \int \mathrm{d}x \, \frac{G^{ij}(x,\xi,t)(\partial^i \Phi)(z,\ell)(\partial^j \Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$\simeq -i\pi G^{ij}(0,\xi,t)(\partial^i \Phi)(z,\ell)(\partial^j \Phi^*)(z,\ell)$$

Double limit

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2\to\infty} \mathcal{A}_{\mathrm{Regge}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \times (-i\pi) G^{ij}(0,\xi,t)(\partial^i \Phi)(z,\boldsymbol{\ell})(\partial^j \Phi^*)(z,\boldsymbol{\ell})$$

Eikonal limit of the Bjorken limit **provided the GPDs are constant at $x = 0$**

$$\lim_{x_{\mathrm{Bj}},\xi\to 0} \mathcal{A}_{\mathrm{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \times (-i\pi) G^{ij}(0,\xi,t)(\partial^i \Phi)(z,\boldsymbol{\ell})(\partial^j \Phi^*)(z,\boldsymbol{\ell})$$

Checked with explicit final expressions for both double limits

Conclusion

Where do we stand?

Bad news

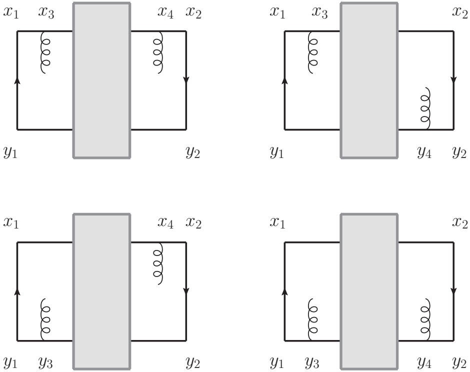
- Semi-classical small x physics has, at its core, issues with collinear logarithms
- The problem can be traced down to the very starting point

Good news

- We now have a minimal correction of semi-classical small x which solves the problem from first principles
- Wave functions, and thus hard parts, are not modified by the scheme
- All we need is the right evolution equation...

BACKUP

The energy denominators



$$\mathrm{tr}\, G^R_{\mathrm{scal}}(x_2, x_1)\, G^A_{\mathrm{scal}}(y_1, y_2)$$

$$= 16g^2 \int \mathrm{d}^D x_3 \int \mathrm{d}^D x_4 \int \mathrm{d}^D y_3 \int \mathrm{d}^D y_4 \delta(y_3^+ - x_3^+)\delta(x_4^+ - y_4^+)$$

$$\times (\partial^+_{x_3} G^R_0)(x_3, x_1)(\partial^+_{x_4} G^R_0)(x_2, x_4)(\partial^+_{y_3} G^A_0)(y_1, y_3)(\partial^+_{y_4} G^A_0)(y_4, y_2)$$

$$\times\, \mathrm{tr} \left\{ \left[A^-(y_3) - A^-(x_3)\right] G^A_{\mathrm{scal}}(y_3, y_4) \left[A^-(y_4) - A^-(x_4)\right] G^R_{\mathrm{scal}}(x_4, x_3) \right\}$$

$$
\begin{aligned}
&\mathrm{tr}\, G^R_{\mathrm{scal}}(x_2,x_1)\, G^A_{\mathrm{scal}}(y_1,y_2) \\
&= 16g^2 \int \mathrm{d}^D x_3 \int \mathrm{d}^D x_4 \int \mathrm{d}^D y_3 \int \mathrm{d}^D y_4 \delta(y_3^+ - x_3^+)\delta(x_4^+ - y_4^+) \\
&\times (\partial^+_{x_3} G_0^R)(x_3,x_1)(\partial^+_{x_4} G_0^R)(x_2,x_4)(\partial^+_{y_3} G_0^A)(y_1,y_3)(\partial^+_{y_4} G_0^A)(y_4,y_2) \\
&\times \mathrm{tr}\left\{ \left[A^-(y_3) - A^-(x_3)\right] G^A_{\mathrm{scal}}(y_3,y_4) \left[A^-(y_4) - A^-(x_4)\right] G^R_{\mathrm{scal}}(x_4,x_3) \right\}
\end{aligned}
$$

Can be proven via the repeated use of Klein-Gordon in a potential, or by proving the generalization to G_{scal} of the relation

$$
\frac{\partial}{\partial x^+}[y^+,x^+]_{\boldsymbol{x}_1}[x^+,z^+]_{\boldsymbol{x}_2} = -ig[y^+,x^+]_{\boldsymbol{x}_1}\left[A^-(x^+,\boldsymbol{x}_1) - A^-(x^+,\boldsymbol{x}_2)\right][x^+,z^+]_{\boldsymbol{x}_2}
$$

Structurally ready for a so-called dilute (perturbative) expansion

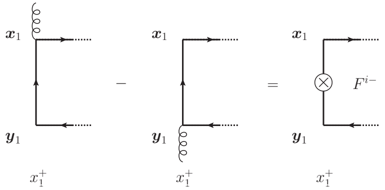
The free propagators G_0 provide the energy denominators.

Two useful technical details

- The classical field does not depend on x^- so $G_{\rm scal}(x, x_0)$ only depends on $(x^- - x_0^-)$, not on each separately: we can define

$$G_{\rm scal}(x, x_0) \equiv \int \frac{\mathrm{d}p^+}{2\pi} \frac{\mathrm{e}^{-ip^+(x^- - x_0^-)}}{2ip^+} (\boldsymbol{x}|\mathcal{G}_{p^+}(x^+, x_0^+)|\boldsymbol{x}_0)$$

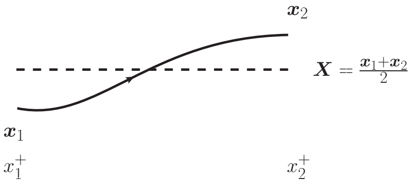
$\mathcal{G}$ satisfies the Schrödinger equation instead of Klein-Gordon



- Since $A^i = 0$, we have

$$A^-(x^+, \boldsymbol{x}) - A^-(x^+, \boldsymbol{y}) = -(\boldsymbol{x}^i - \boldsymbol{y}^i) \int_0^1 \mathrm{d}s\, F^{i-}(x^+, s\boldsymbol{x} + (1-s)\boldsymbol{y})$$

Partial twist expansion



$$(\boldsymbol{x}_1|\mathcal{G}^R_{p^+}(x_1^+,x_2^+)|\boldsymbol{x}_2) \simeq \theta(p^+)\,(\boldsymbol{x}_1|\mathcal{G}^{(0)R}_{p^+}(x_1^+,x_2^+)|\boldsymbol{x}_2)\,\left[x_1^+,x_2^+\right]_{\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}}$$

$$(\boldsymbol{y}_2|\mathcal{G}^A_{p^+}(x_2^+,x_1^+)|\boldsymbol{y}_1) \simeq \theta(-p^+)\,(\boldsymbol{y}_2|\mathcal{G}^{(0)A}_{p^+}(x_2^+,x_1^+)|\boldsymbol{y}_1)\,\left[x_2^+,x_1^+\right]_{\frac{\boldsymbol{y}_1+\boldsymbol{y}_2}{2}}$$

[Altinoluk, Armesto, Beuf, Martinez, Salgado]

$$F^{i-}(x_1^+, s\boldsymbol{x}_1+\bar{s}\boldsymbol{y}_1) \simeq F^{i-}\left(x_1^+, s\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2}+\bar{s}\frac{\boldsymbol{y}_1+\boldsymbol{y}_2}{2}\right)$$

What about the overlapping limit?

Illustration: collinearly divergent contributions

- Leading twist limit of the Regge limit

$$\lim_{Q^2\to\infty} F_T(x_{\rm Bj}\to 0, Q^2)\Big|_{\rm div} \propto \frac{1}{\epsilon}\left[xg(x)\right]_{x=0}\int_0^1 {\rm d}y\,\mathcal{P}^{qg}(y)$$

- Eikonal limit of the Bjorken limit

$$\lim_{x_{\rm Bj}\to 0} F_T(x_{\rm Bj}, Q^2\to\infty)\Big|_{\rm div} \propto \frac{1}{\epsilon}\lim_{x_{\rm Bj}\to 0}\int_{x_{\rm Bj}}^1 {\rm d}y\left[xg(x)\right]_{x=x_{\rm Bj}/y}\mathcal{P}^{qg}(y)$$

What about the overlapping limit?

$$\lim_{Q^2\to\infty} F_T(x_{\rm Bj}\to 0, Q^2)\Big|_{\rm div} =_? \lim_{x_{\rm Bj}\to 0} F_T(x_{\rm Bj}, Q^2\to\infty)\Big|_{\rm div}$$

The two limits commute, and **the collinear divergence of the Regge limit can be canceled using DGLAP** if and only if

$$\lim_{x_{\rm Bj}\to 0}\int_{x_{\rm Bj}}^{1} {\rm d}y\,[xg(x)]_{x=x_{\rm Bj}/y}\,\mathcal{P}^{qg}(y) = [xg(x)]_{x=0}\int_0^1 {\rm d}y\,\mathcal{P}^{qg}(y)$$

What about the overlapping limit?

$$\lim_{x_{\mathrm{Bj}}\to 0}\int_{x_{\mathrm{Bj}}}^{1}\mathrm{d}y\,[xg(x)]_{x=x_{\mathrm{Bj}}/y}\,\mathcal{P}^{qg}(y) =_? [xg(x)]_{x=0}\int_0^1 \mathrm{d}y\mathcal{P}^{qg}(y)$$

This equation is only correct provided that:

- The PDF is constant at $x = 0$ (arguably false)
- The splitting function is integrable on $\{0, 1\}$
  $\hookrightarrow$ True for $\mathcal{P}^{qg}$, false for $\mathcal{P}^{gg}$

In the Regge limit, $\mathcal{P}^{qg}$ appears at LL and $\mathcal{P}^{gg}$ at NLL.
This is why the issue was only noticed at NLL.
However, **the issue was here at LL all along** and its origin is the fact that x is 0, which traces back to the shock wave approximation