Numerical simulations of coalescing neutron star binaries Part I – Numerical Relativity

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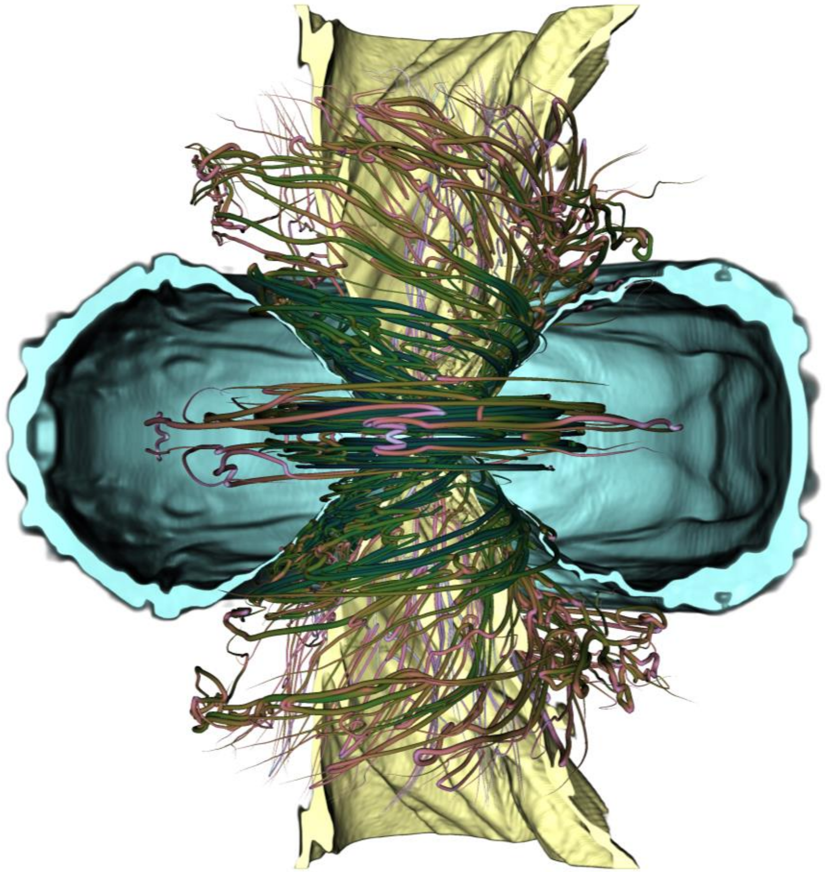
General Relativity and Astrophysics

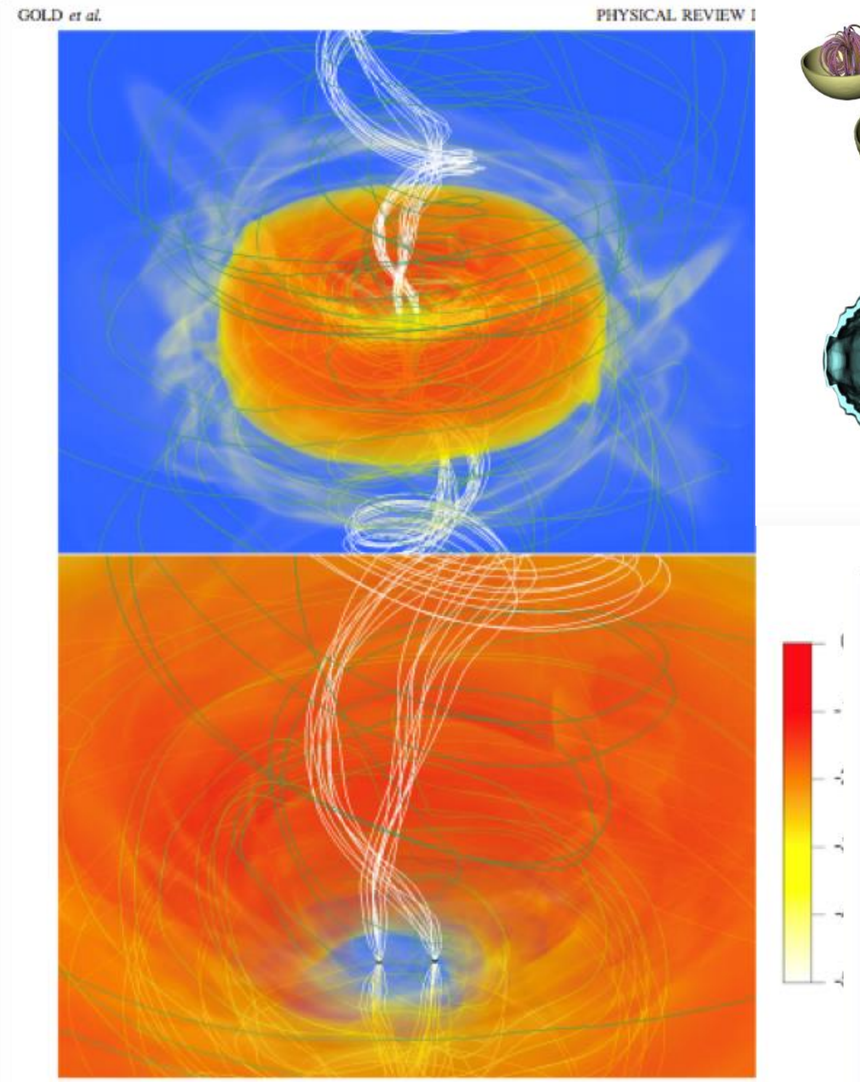
- Binary Black Hole Mergers
- Binary Neutron Star Mergers
- Neutron Star – Black Hole Mergers
- Supernovae
- Accretion Disks
- Cosmology



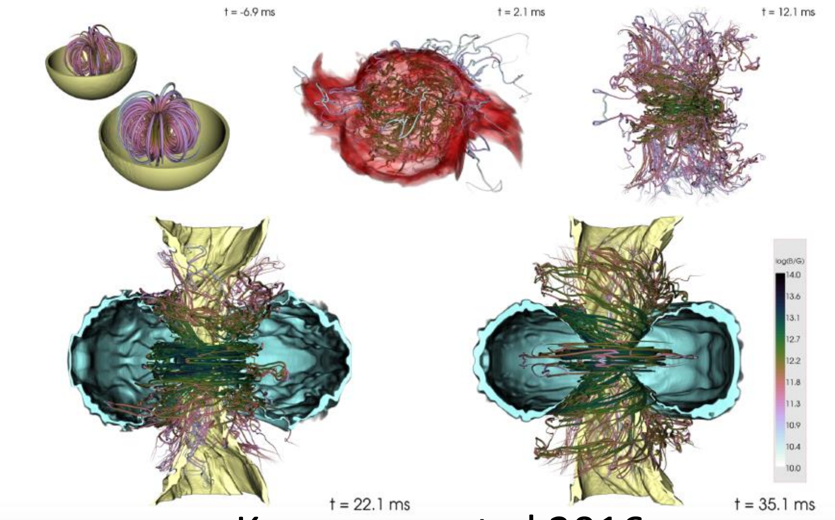
Kawamura et al 2016

In all these scenarios general relativity plays a fundamental role.

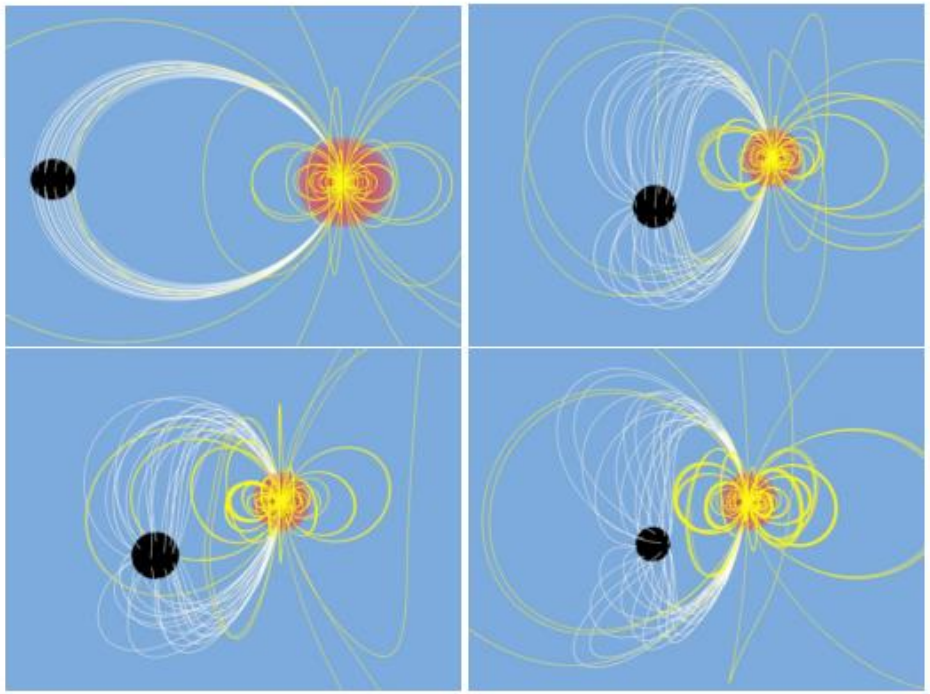
APPLICATIONS



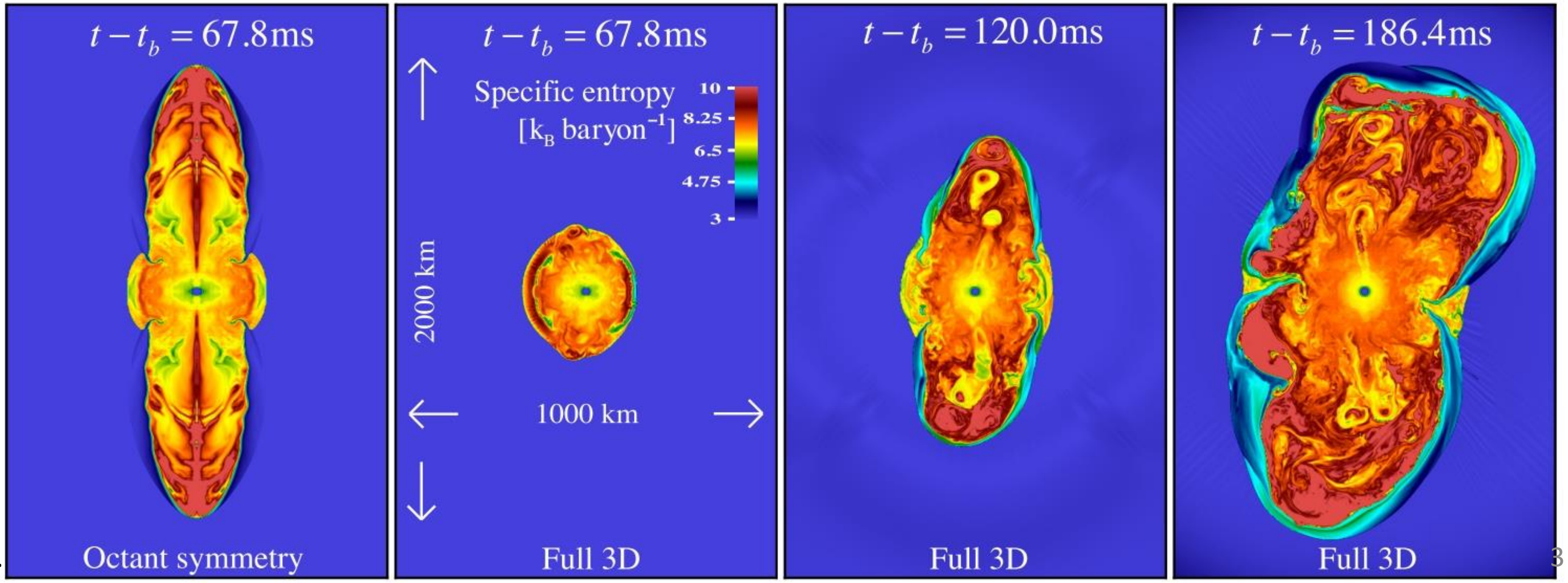
Gold et al 2014



Kawamura et al 2016



Paschalidis et al 2013



Moesta et al 2014

History of Numerical Relativity
(see https://link.springer.com/article/10.1007/lrr-2015-1)

- 1962 Arnowitt, Deser and Misner (ADM) 3+1 formulation
- 1966 May and White first 1D GR simulation of collapse to BH
- 1976 Smarr et al first simulation of head-on collision of two BHs
- 1985 Stark and Piran extract GWs from a simulation of rotating collapse to a BH in NR.
- 1994 “Binary Black Hole Grand Challenge Project” is launched
- 1995-1998 BSSN formulation
- 1996 Brügmann mesh refinement simulation of BHs
- 1997 Cactus 1.0 is released
- 1997 Brandt & Brügmann “puncture” initial data

History of Numerical Relativity
(see https://link.springer.com/article/10.1007/lrr-2015-1)

- 2000 Brandt et al. simulate the first grazing collisions of BHs using a revised version of the Grand Challenge Alliance code
- 2000 Shibata and Uryū first NS-NS merger simulation in GR
- 2003 Schnetter et al “Carpet” driver for Cactus
- 2005 Pretorius first simulation of BH-BH inspiral and merger
- 2006 Shibata and Uryū first NS-BH merger simulation
- 2008 Anderson et al first GRMHD simulation of an NS-NS merger
- 2010 Chawla et al first GRMHD simulation of an NS-BH merger

Numerical Relativity:
3+1 Formulation

Einstein Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}$$

$R \equiv R^{\mu}_{\mu}$ Ricci scalar

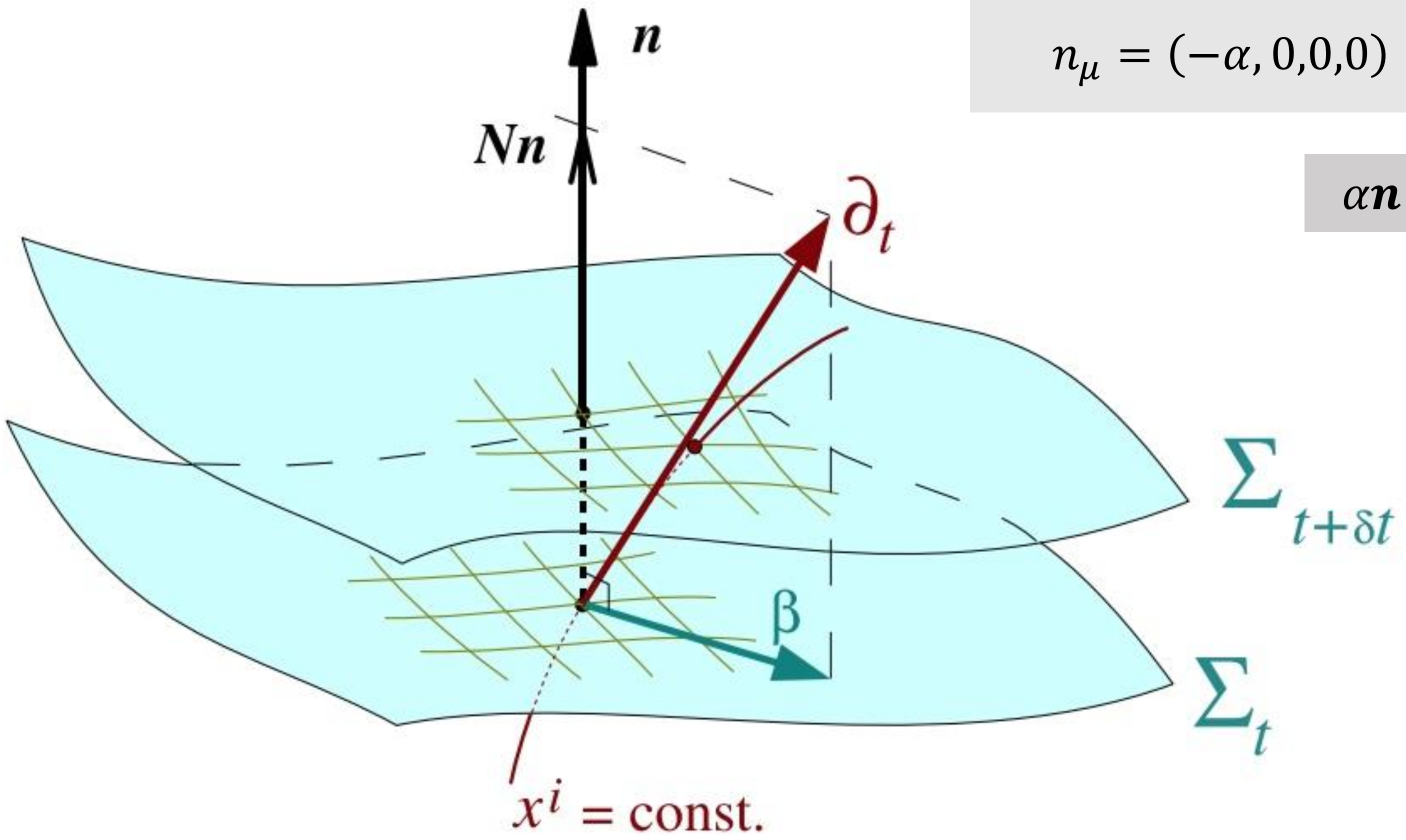
$R_{\mu\nu} \equiv R^{\rho}_{\mu\rho\nu}$ Ricci tensor

$R^{\sigma}_{\mu\rho\nu} \equiv \partial_\rho \Gamma^{\sigma}_{\mu\nu} - \partial_\nu \Gamma^{\sigma}_{\mu\rho} + \Gamma^{\sigma}_{\tau\rho}\Gamma^{\tau}_{\mu\nu} - \Gamma^{\sigma}_{\tau\nu}\Gamma^{\tau}_{\mu\rho}$ Riemann tensor

$$\Gamma^{\sigma}_{\mu\rho} \equiv \frac{1}{2} g^{\sigma\tau} \left(\partial_\mu g_{\rho\tau} + \partial_\rho g_{\mu\tau} - \partial_\tau g_{\mu\rho} \right)$$

The metric in the 3+1 form

$G = c = 1$



$n_\mu = (-\alpha, 0,0,0) \quad n^\mu = \frac{1}{\alpha}(1, -\beta^i)$

$\alpha \boldsymbol{n} = \boldsymbol{t} - \boldsymbol{\beta}$

https://arxiv.org/abs/gr-qc/0703035

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

ADM Equations (1962)

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\gamma_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \left(\beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k\right) +$$
$$\alpha\left({}^{(3)}R_{ij} + K K_{ij} - 2K_{ik}K_j^k\right) + 4\pi\alpha\left[\gamma_{ij}(S - E) - 2S_{ij}\right]$$

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi E$$

$$D_j\left(K^{ij} - \gamma^{ij}K\right) = 8\pi S^i$$

$$S_{\mu\nu} \equiv \gamma_\mu^\sigma \gamma_\nu^\tau T_{\sigma\tau} \qquad S_\mu \equiv -\gamma_\mu^\sigma n^\tau T_{\sigma\tau} \qquad S \equiv S_\mu^\mu \qquad E \equiv n^\sigma n^\tau T_{\sigma\tau}$$

plus a (free) choice for the lapse function α and the shift vector $\boldsymbol{\beta}$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

BSSN Equations (1987-1995-1998)

$$K_{ij} = e^{4\phi}\tilde{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$

$${}^{(3)}R_{ij} = {}^{(3)}\bar{R}_{ij} + {}^{(3)}R^{\phi}_{ij}$$

$$\gamma_{ij} \equiv e^{4\phi}\tilde{\gamma}_{ij}$$

$\partial_t \gamma_{ij}$

$$\partial_t \phi = -\frac{1}{6}\alpha K + \frac{1}{6}\partial_i \beta^i + \beta^i \partial_i \phi$$

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \beta^k \partial_k \tilde{\gamma}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k + \tilde{\gamma}_{kj}\partial_i \beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k$$

$\partial_t K_{ij}$

$$\partial_t K = -D^i D_i \alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(E+S) + \beta^i D_i K$$

$$\partial_t \tilde{A}_{ij} = e^{-4\phi}\left[-\left(D_i D_j \alpha\right)^{TF} + \alpha\left({}^{(3)}R^{TF}_{ij} - 8\pi S^{TF}_{ij}\right)\right]$$

$$+\alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j\right) + \beta^k \partial_k \tilde{A}_{ij} + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{kj}\partial_i \beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k \beta^k$$

$$\partial_t \tilde{\Gamma}^i = -2\tilde{A}^{ij}\partial_j \alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{kj} - \frac{2}{3}\tilde{\gamma}^{ij}\partial_j K - 8\pi\tilde{\gamma}^{ij}S_j + 6\tilde{A}^{ij}\partial_j \phi\right)$$

$$+\beta^j \partial_j \tilde{\Gamma}^i - \tilde{\Gamma}^j \partial_j \beta^i + \frac{2}{3}\tilde{\Gamma}^i \partial_j \beta^j + \frac{1}{3}\tilde{\gamma}^{li}\partial_l \partial_j \beta^j + \tilde{\gamma}^{lj}\partial_j \partial_l \beta^i$$

Numerical Relativity: Introduction to GRHD

Equations

Einstein Equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$

Hydro Equations

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\nabla_\mu J^\mu = 0$$

$$P = P(\rho, \epsilon)$$

$$J^\mu = \rho u^\mu$$

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$$

$$h \equiv 1 + \epsilon + P/\rho$$

GRHD equations

The system of equations can be written in a flux-conservative form (**Valencia formulation**, Banyuls et al 1997, Anton et al 2006):

$$\nabla_\mu T^{\mu\nu} = 0$$
$$\nabla_\mu J^{\mu} = 0$$

$$\longrightarrow \quad \partial_t \boldsymbol{U} + \partial_i \boldsymbol{F}^i = \boldsymbol{S}$$

where $\boldsymbol{U}$ is the vector of conserved variables, $\boldsymbol{F}^i$ the fluxes, and $\boldsymbol{S}$ the source terms.

The importance of flux-conservative Form

- **Lax-Wendroff Theorem** (1960): If a consistent numerical method written in a flux conservative form converges to a function u(x,t) for dx that goes to zero, then u(x,t) is a solution of the conservation law*.

- **Hou-LeFlock Theorem** (1994): non-conservative schemes do not converge to the correct solution if a shock wave is present in the flow.

*note that the proper formulation of the Lax-Wendroff theorem is slightly different from what reported here (but for our purposes it is OK).

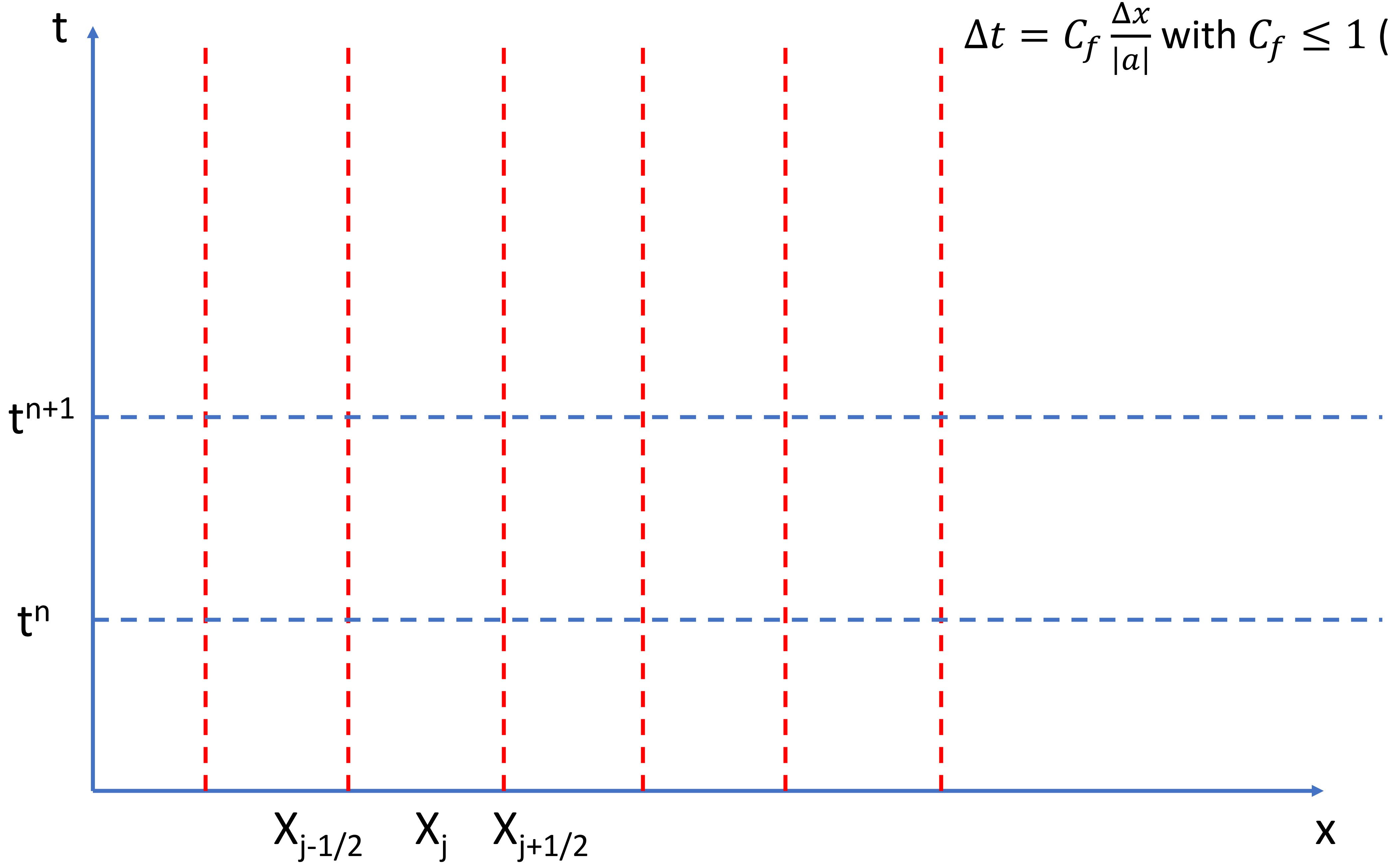
NUMERICAL SCHEME IN A FLUX-CONSERVATIVE FORM

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

Let's solve it on a numerical grid

$$x_j = j \times \Delta x \ , j = 0, \cdots , J-1$$
$$t^n = n \times \Delta t \ , n = 0, \cdots , N-1$$

$$\Delta t = C_f \frac{\Delta x}{|a|} \text{ with } C_f \leq 1 \text{ (Courant Factor)}$$



$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

We now take the integral in t and x within a numerical cell

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \int_{t^n}^{t^{n+1}} \frac{\partial u}{\partial t} dx\, dt + \int_{t^n}^{t^{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\partial f(u)}{\partial x} dx\, dt = 0$$

$$\int_{x_{j-1/2}}^{x_{j+1/2}} \left[u(x, t^{n+1}) - u(x, t^n)\right] dx + \int_{t^n}^{t^{n+1}} \left[f\left(u(x_{j+1/2}, t)\right) - f\left(u(x_{j-1/2}, t)\right)\right] dt = 0$$

We then divide by Δx

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n+1})\,dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n})\,dx - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2},t)\right)dt - \int_{t^n}^{t^{n+1}} f\left(u(x_{j-1/2},t)\right)dt\right]$$

$$\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n+1})\,dx = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^{n})\,dx - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2},t)\right)dt - \int_{t^n}^{t^{n+1}} f\left(u(x_{j-1/2},t)\right)dt\right]$$

We now define

$$\tilde{u}_j^n \equiv \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t^n)\,dx$$

$$\tilde{u}_j^{n+1} = \tilde{u}_j^n - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2},t)\right)dt - \int_{t^n}^{t^{n+1}} f\left(u(x_{j-1/2},t)\right)dt\right]$$

$$\tilde{u}_j^{n+1} = \tilde{u}_j^n - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt - \int_{t^n}^{t^{n+1}} f\left(u(x_{j-1/2}, t)\right) dt\right]$$

Let's also define

$$f_{j+1/2}^n \equiv \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt$$

And our equation reduces to:

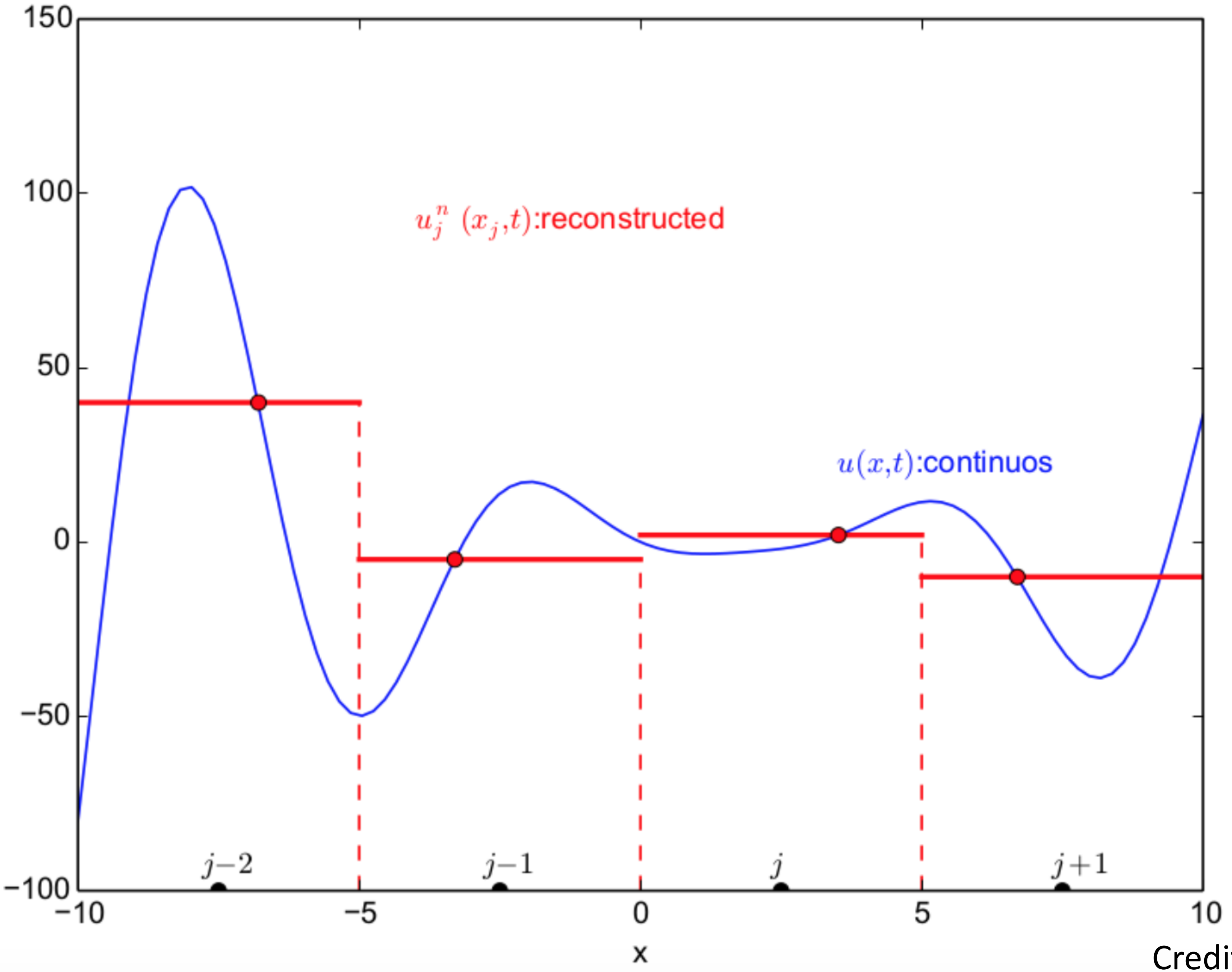
$$\tilde{u}_j^{n+1} = \tilde{u}_j^n - \frac{\Delta t}{\Delta x}\left(f_{j+1/2}^n - f_{j-1/2}^n\right)$$

$$\tilde{u}_j^n \equiv \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x, t^n)\, dx$$

$$f_{j+1/2}^n \equiv \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt$$

a numerical method written in this way is said to be in **flux conservative form**.

Godunov Method



Credit: A. Endrizzi

RIEMANN PROBLEM

- By solving the Riemann problem one can compute

$$f^n_{j+1/2} \equiv \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f\left(u(x_{j+1/2}, t)\right) dt$$

- More computationally convenient to use approximate Riemann solvers, e.g., HLLE:

$$\boldsymbol{F}^i = \frac{c_{\min} \boldsymbol{F}^i_{\mathrm{r}} + c_{\max} \boldsymbol{F}^i_{\mathrm{l}} - c_{\max} c_{\min} \left(\boldsymbol{F}^0_{\mathrm{r}} - \boldsymbol{F}^0_{\mathrm{l}} \right)}{c_{\max} + c_{\min}}$$

$$c_{max} \equiv \max(0, c_{+,r}, c_{+,l}),$$

$$c_{min} \equiv -\min(0, c_{-,r}, c_{-,l})$$

HIGH RESOLUTION SHOCK-CAPTURING METHODS

- To increase the order, instead of assuming a step function one could use a piecewise linear function with a TVD slope-limiter:

$$\tilde{u}(x,t^n) = \tilde{u}_j^n + \sigma_j^n (x - x_j) \quad \text{for} \quad x_{j-1/2} < x < x_{j+1/2}$$

$$\sigma_j^n = \text{minmod}\left(\frac{\tilde{u}_j^n - \tilde{u}_{j-1}^n}{\Delta x}, \frac{\tilde{u}_{j+1}^n - \tilde{u}_j^n}{\Delta x} \right)$$

$$\text{minmod}(a,b) \equiv \begin{cases} a \text{ if } |a|<|b| \text{ and } ab>0 \\ b \text{ if } |b|<|a| \text{ and } ab>0 \\ 0 \text{ if } ab<0 \end{cases}$$

or higher orders functions (e.g., PPM, WENO-Z, etc…).

einstein toolkit

Einstein Toolkit
einsteintoolkit.org

- Set of publicly available tools for relativistic astrophysics
- Latest release on May 24 2023 (codename “Schwarzschild ”)
- More than 150 users on 6 continents
- Tested on several HPC infrastructures around the world
- Includes over 100 Cactus thorns, including:
  - McLachlan and Baikal (space-time evolution)
  - GRHydro and IllinoisGRMHD (GRMHD equations)
  - Several initial data and analysis routines
- Data can be read and visualized by open source codes:
  - Visit https://visit.llnl.gov/
  - Kuibit https://sbozzolo.github.io/kuibit/

- Open source framework
  - Decentralized code development
  - Active and friendly user community
  - Module based approach
- Infrastructure modules
  - Parameter file handling
  - Parallelization (MPI + OpenMP)
  - Adaptive Mesh refinement (Carpet)
  - IO (ASCII and HDF5) + checkpointing

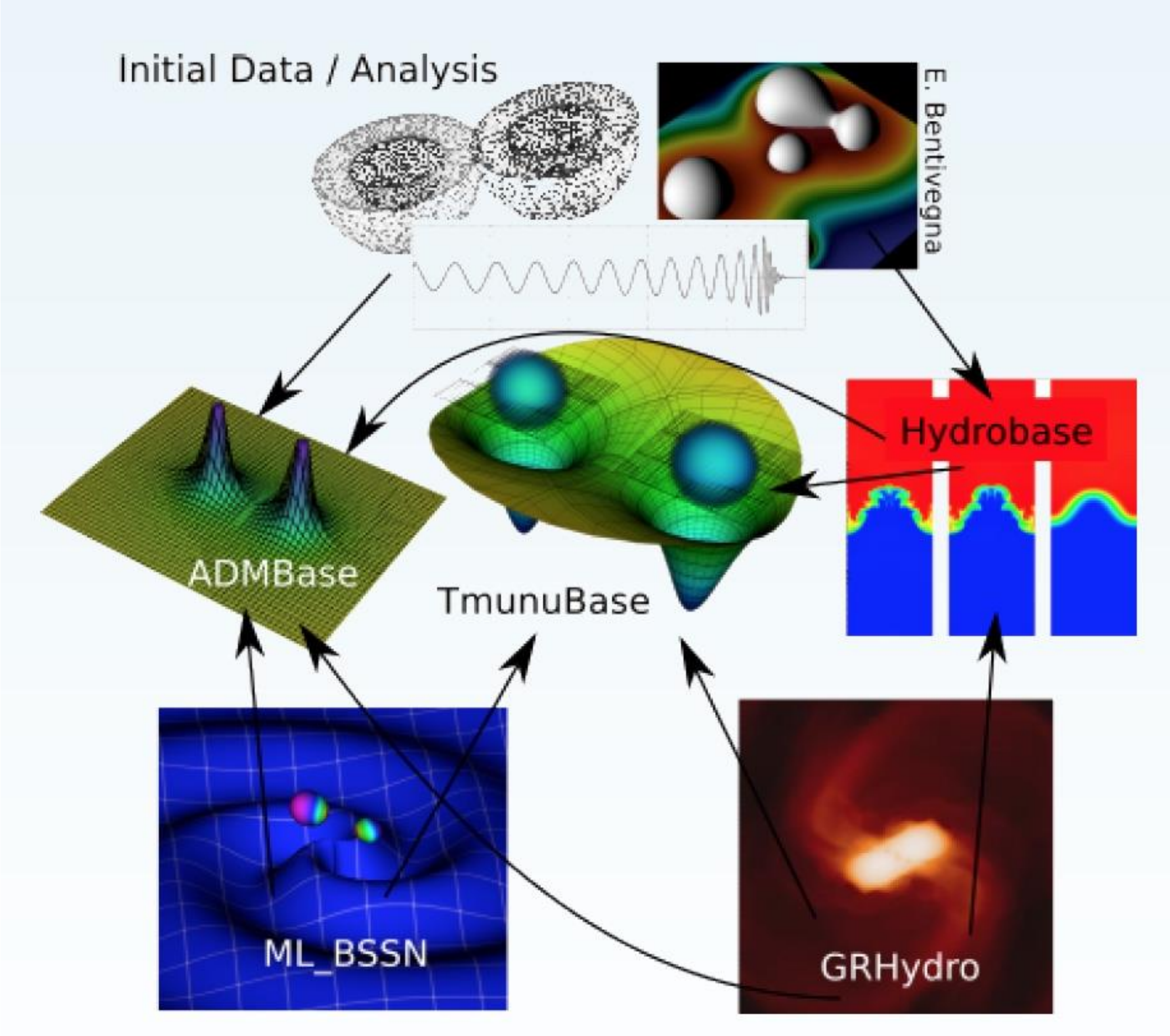
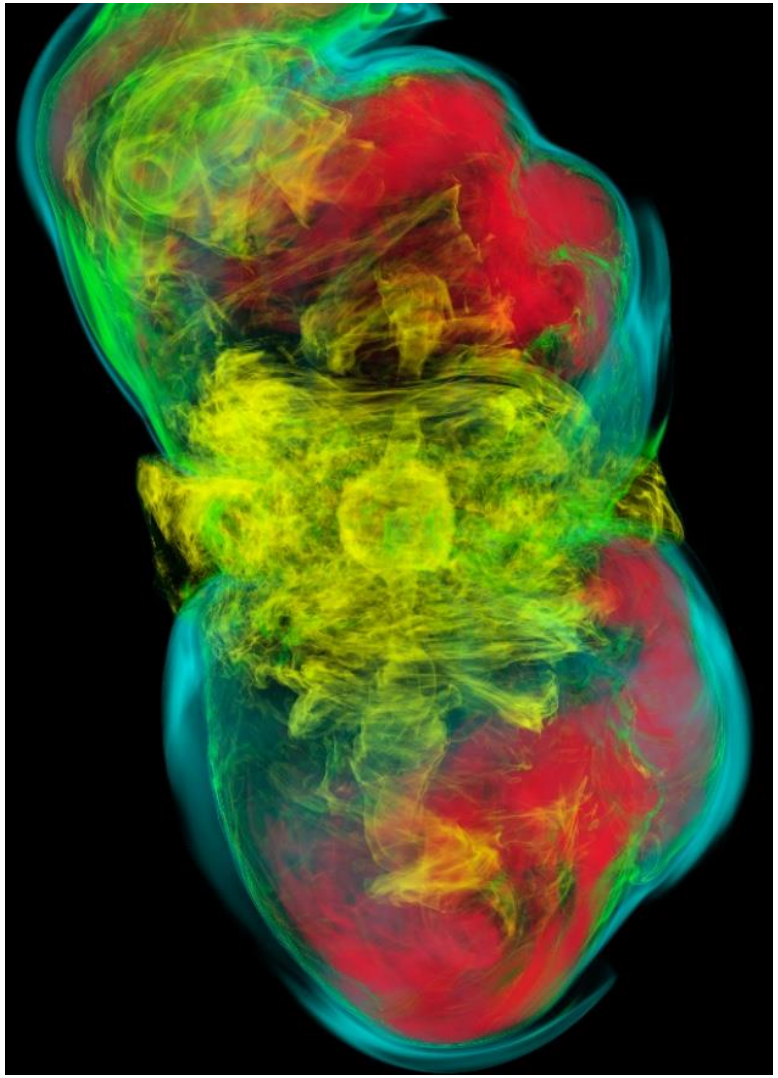


Figure by R. Haas

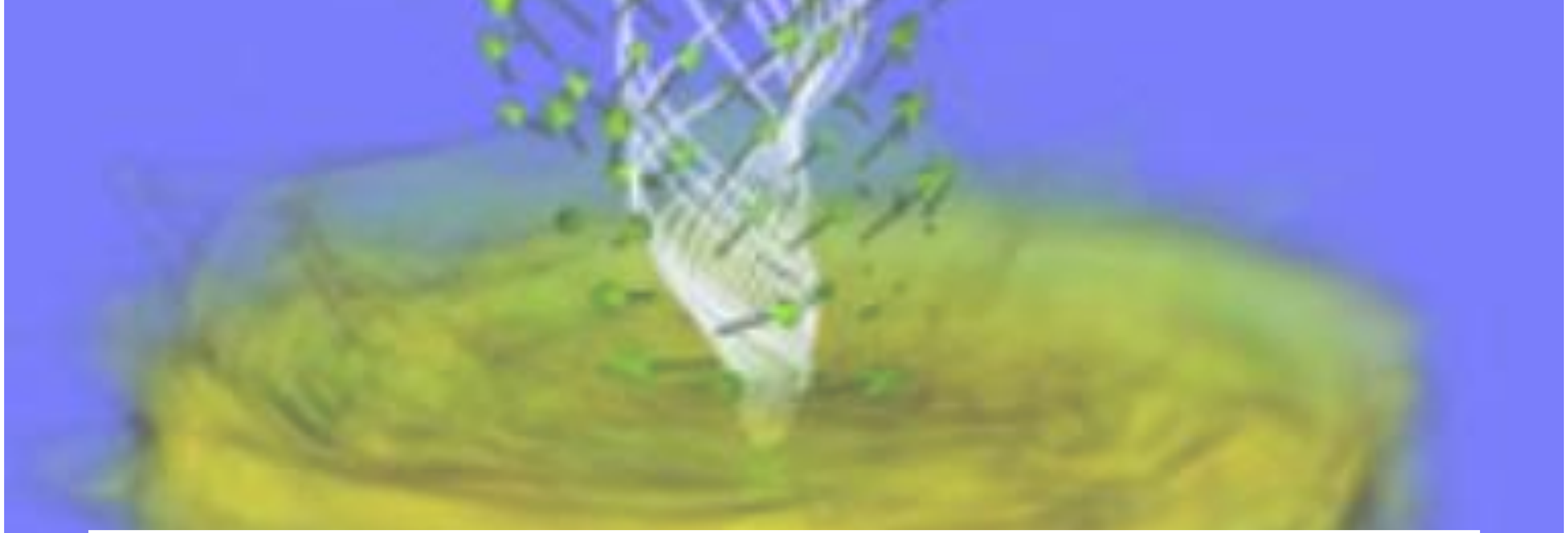
GRHYDRO



Moesta et al 2013: http://arxiv.org/abs/1304.5544

- First publicly available fully GRMHD code
- Based on the public version of the Whisky code
- Fully embedded in the Einstein Toolkit
- Uses Valencia formulation
- Implements up to 5th order reconstruction
- Divergence cleaning and Constrained Transport

IllinoisGRMHD



- First publicly available fully GRMHD code using the vector potential as evolution variable
- Included in the Einstein Toolkit
- Very robust code for GRMHD in AMR
- Implements PPM+HLLE
- It now includes support for tabulated EOS and neutrinos

http://math.wvu.edu/~zetienne/ILGRMHD/index.html

The Spritz Code

https://zenodo.org/record/4350072

Cipolletta, Kalinani, **Giacomazzo***, Ciolfi 2020, CQG 37, 135010
Cipolletta, Kalinani, Giangrandi, **Giacomazzo***, Ciolfi, Sala, Giudici 2021, CQG 38, 085021
Kalinani, Ciolfi, Kastaun, **Giacomazzo**, Cipolletta, Ennoggi 2022, PRD 105, 103031

BNS sims require to account for magnetic fields, but also for EOS and neutrino emission. No public code was available that included all these effects.

We therefore developed a new General Relativistic MHD code named Spritz:

- Publicly available on Zenodo
- Based on the Einstein Toolkit Infrastructure (http://einsteintoolkit.org)
- GRMHD Valencia formulation
- Staggered vector potential formulation to evolve the magnetic field
- Support for finite-temperature tabulated Equations Of State
- Neutrino transport via a leakage scheme with a grey approximation (https://stellarcollapse.org/Zelmani) and 3 neutrino species ($\nu_e, \bar{\nu}_e, \nu_x$)
- 5-th order WENO-Z scheme for hydro
- Currently used for NS-NS simulations

EINSTEIN TOOLKIT SCHOOLS AND WORKSHOPS

www.einsteintoolkit.org

EU – **19-23 June 2023** at Universidade de Aveiro, Portugal
https://euet2023.web.ua.pt/

USA – **17-21 July 2023** at Rochester Institute of Technology, New York, USA
https://compact-binaries.org/content/events/2023-07-17/north-american-einstein-toolkit-school-and-workshop
(hybrid format)

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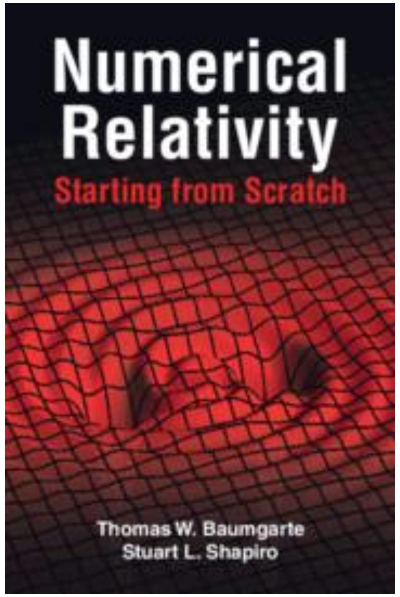

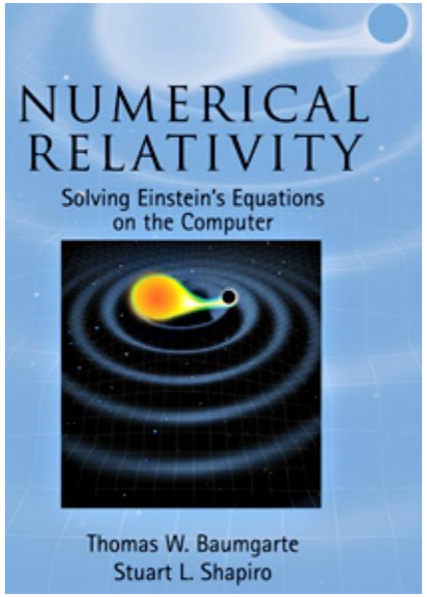

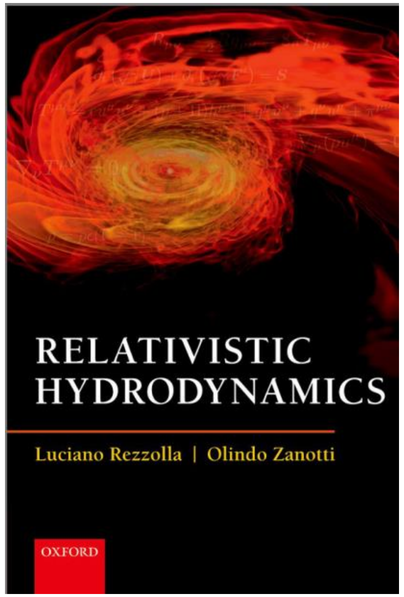

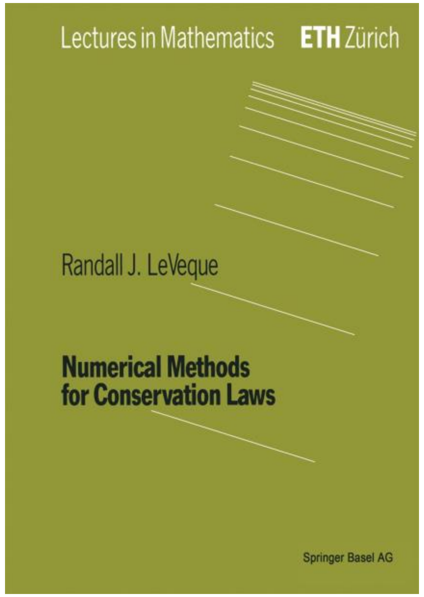

USEFUL TEXTBOOKS

Numerical Relativity master course at Milano-Bicocca (42 hours, passwd GW170817): https://elearning.unimib.it/course/view.php?id=45924