GWsNS school, Aussois, June 2023

Neutron stars and gravitational waves

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Plan

Now:

- Basic physics of gravitational waves
- Neutron stars as sources of gravitational waves
- Gravitational wave detectors
- Analysis of gravitational wave data

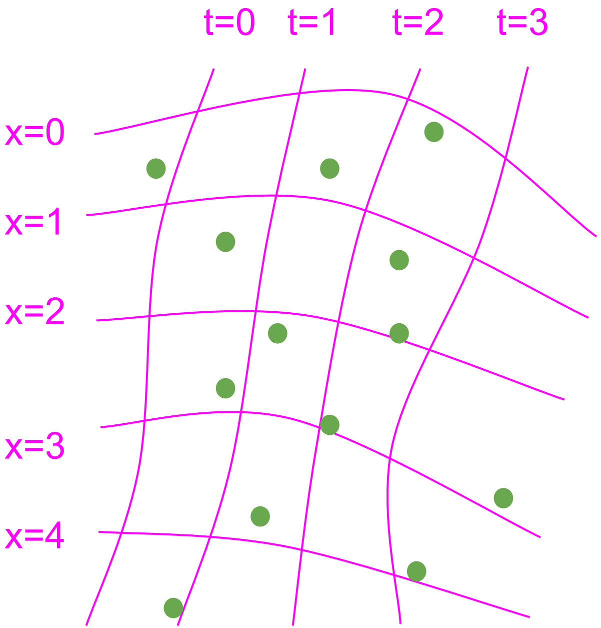
This afternoon: tutorials in form of Jupyter notebooks. Can be used:

- Online via Google Colab (hopefully this works!)
- Docker image where (almost) everything works offline
  (problem: this image is on my laptop, and you do not have it)

Basic physics of gravitational waves

General relativity review

4-dimensional spacetime
with coordinates and events



Metric tensor
(distance between events) $g_{\mu\nu}$

Spacetime interval
$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Curvature (gravity):
- Riemann tensor $R^\mu{}_{\nu\rho\sigma}$
- Ricci tensor $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$
- Ricci scalar $R = R_\mu{}^\mu$

Stress-energy tensor
(energy, matter, pressure) $T_{\mu\nu}$

Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

GR review: flat spacetime

Empty spacetime everywhere → $T_{\mu\nu} = 0$ → EFE satisfied by the Minkowski metric:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$ (Cartesian coordinates)

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$ (spherical coordinates)

All curvature quantities vanish everywhere, e.g. $R = 0$ → No gravity → Special relativity

Solution adequate for describing most (not all!) physics in the solar system.

GR review: Schwarzschild solution

Static spacetime (time invariance) + spherical symmetry → Schwarzschild metric

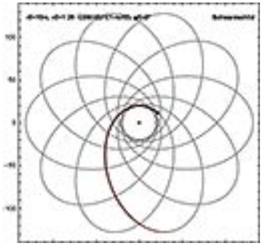
$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Characteristic scale set by the Schwarzschild radius $r_s = \frac{2GM}{c^2} \approx 3 \frac{M}{M_\odot}$ km

Solution adequate for describing physics in proximity of an object that is

- non-rotating
- electrically neutral

(non-spinning black hole, neutron star, main sequence star, planet…)



Gravitational waves in GR

Minkowski spacetime with a small perturbation $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

Introduce coordinates x^μ satisfying the harmonic gauge condition $$\Box x^\mu = 0$$

d'Alembert operator

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = -c^{-2}\partial_t^2 + \nabla^2$$

The EFE can be linearized and written as

$$\Box h_{\mu\nu} = -16\pi\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) \underset{\text{vacuum}}{=} 0$$

→ The perturbation satisfies the equation of waves traveling at the speed of light.

Gravitational waves in GR

We can decompose the perturbation field in plane waves, which look like

$$h_{\mu\nu} = A_{\mu\nu} \exp(ik_\rho x^\rho) + \text{compl. conj.}$$

Polarization tensor satisfying the transverse and traceless conditions

$$k^\mu A_{\mu\nu} = 0$$

$$A^\mu{}_\mu = 0$$

Null wave vector

$$k_\mu k^\mu = 0$$

$$k^\mu = (\omega, \vec{k})$$

With the appropriate choice of coordinates we can set

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Two degrees of freedom, "plus" and "cross" polarizations

Emission of gravitational waves

Inertia tensor of a source

$$I_{ij} = \int \varrho\, x^i x^j d^3x$$

Quadrupole moment

$$Q_{ij} = I_{ij} - \frac{1}{3} I \delta_{ij}$$

Quadrupole formula

$$h_{ij}^{TT} = \frac{2G}{c^4 r} P_{ij}{}^{kl} \ddot{Q}_{kl}$$

Distance to the source

Projection operator to extract the TT part orthogonal to the viewing direction

Energy loss (luminosity)

$$\frac{dE}{dt} = \frac{G}{5c^5} \left\langle \dddot{Q}_{ij} \dddot{Q}^{ij} \right\rangle$$

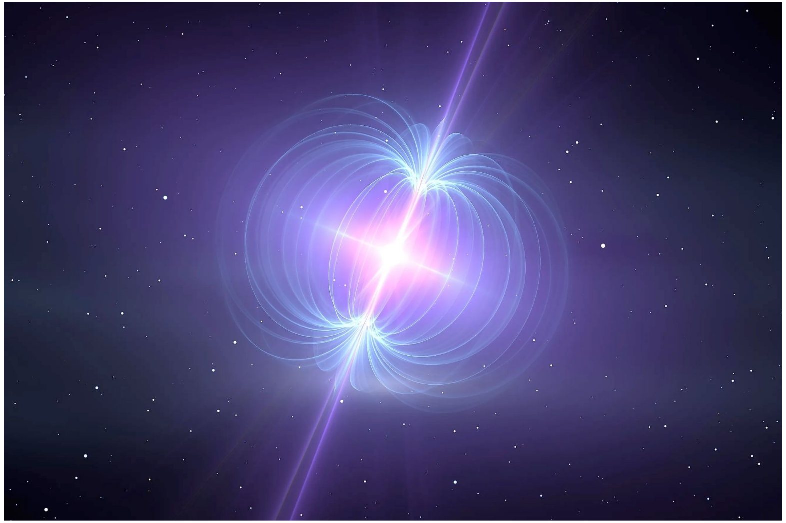
$$\frac{2G}{c^4} \approx 10^{-44}\ \mathrm{s}^2\ \mathrm{kg}^{-1}\ \mathrm{m}^{-1}$$

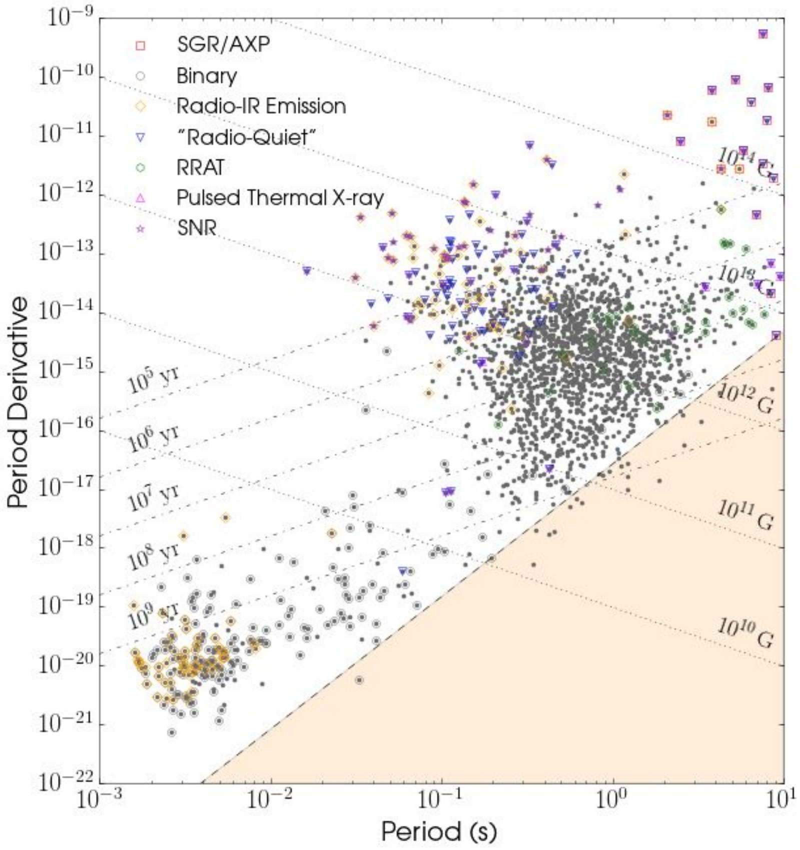
$$\frac{G}{5c^5} \approx 10^{-53}\ \mathrm{W}^{-1}$$

More terms in general: mass current quadrupole, mass octupole… (higher-order modes)

Neutron stars as sources of gravitational waves

Neutron stars



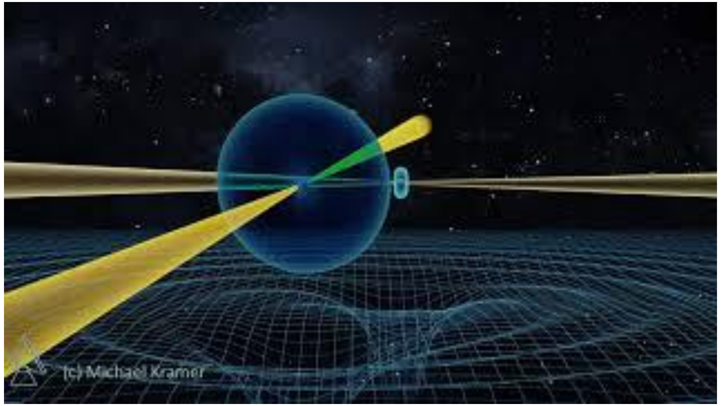


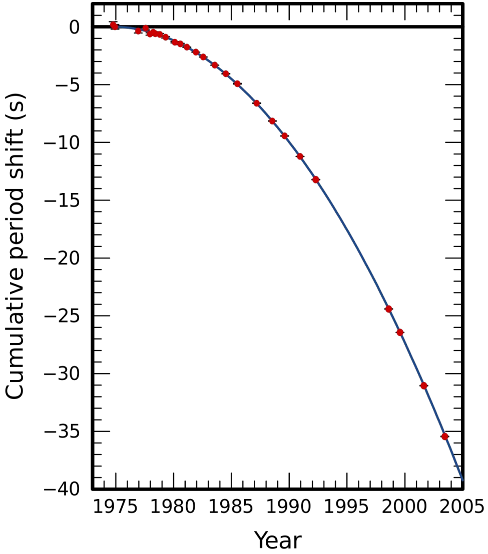
Observations:

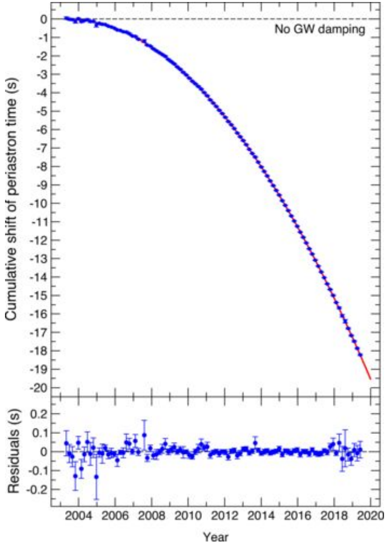
- Supernova remnant sites
- X-ray binaries
- Pulsars (radio, X-ray, γ-ray, optical)
- Gravitational waves

Neutron stars in binary systems

PSR 1913+16, the “Hulse-taylor pulsar”
PSR J0737−3039, the “double pulsar”
GW170817
…

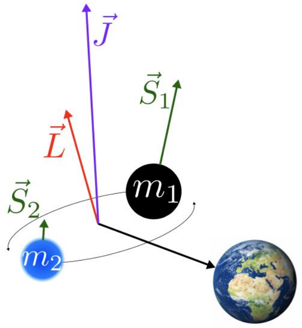






NS-NS (BNS), NS-BH

Inspiral of a binary system



Keplerian system of two point masses → Time-dependent quadrupole moment

$$E_{\text{orbit}}(v)$$

Quadrupole formula

Energy balance

$$\dot{E}_{\text{orbit}}(v) = -F(v)$$

→ Energy loss → Orbital decay

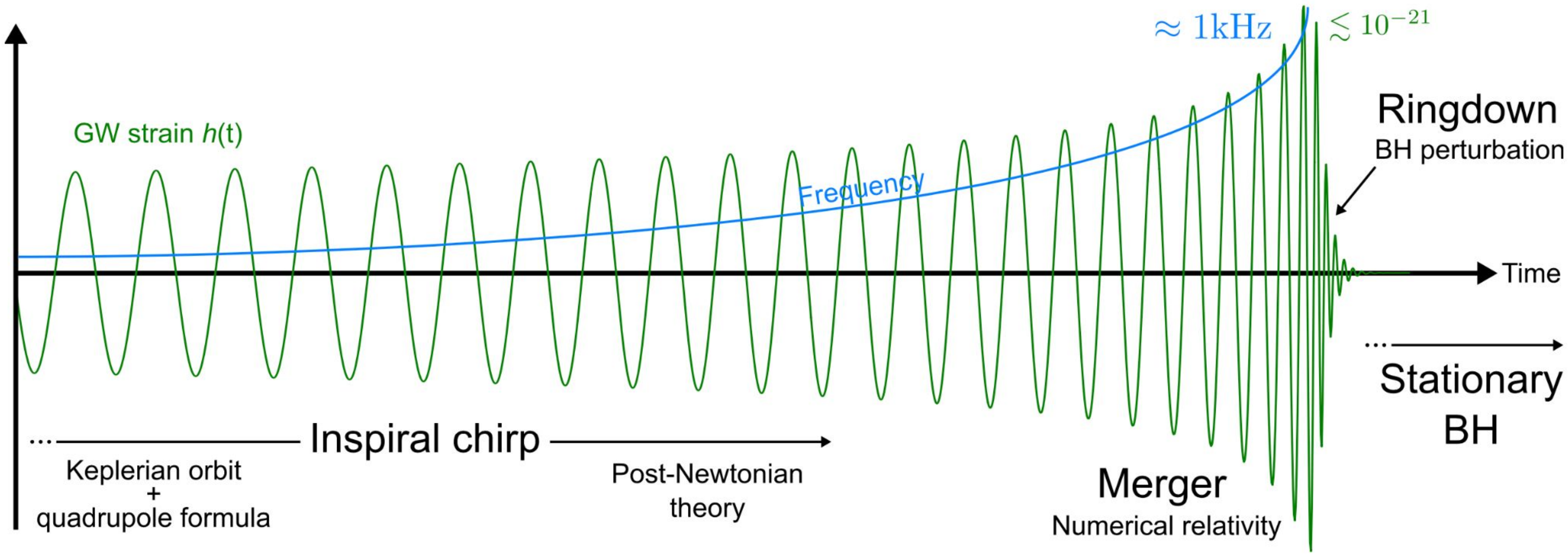
Expand both sides in power series of *v*/*c*
and truncate at some order:
(post-)Newtonian inspiral

Amplitude/phase vs time

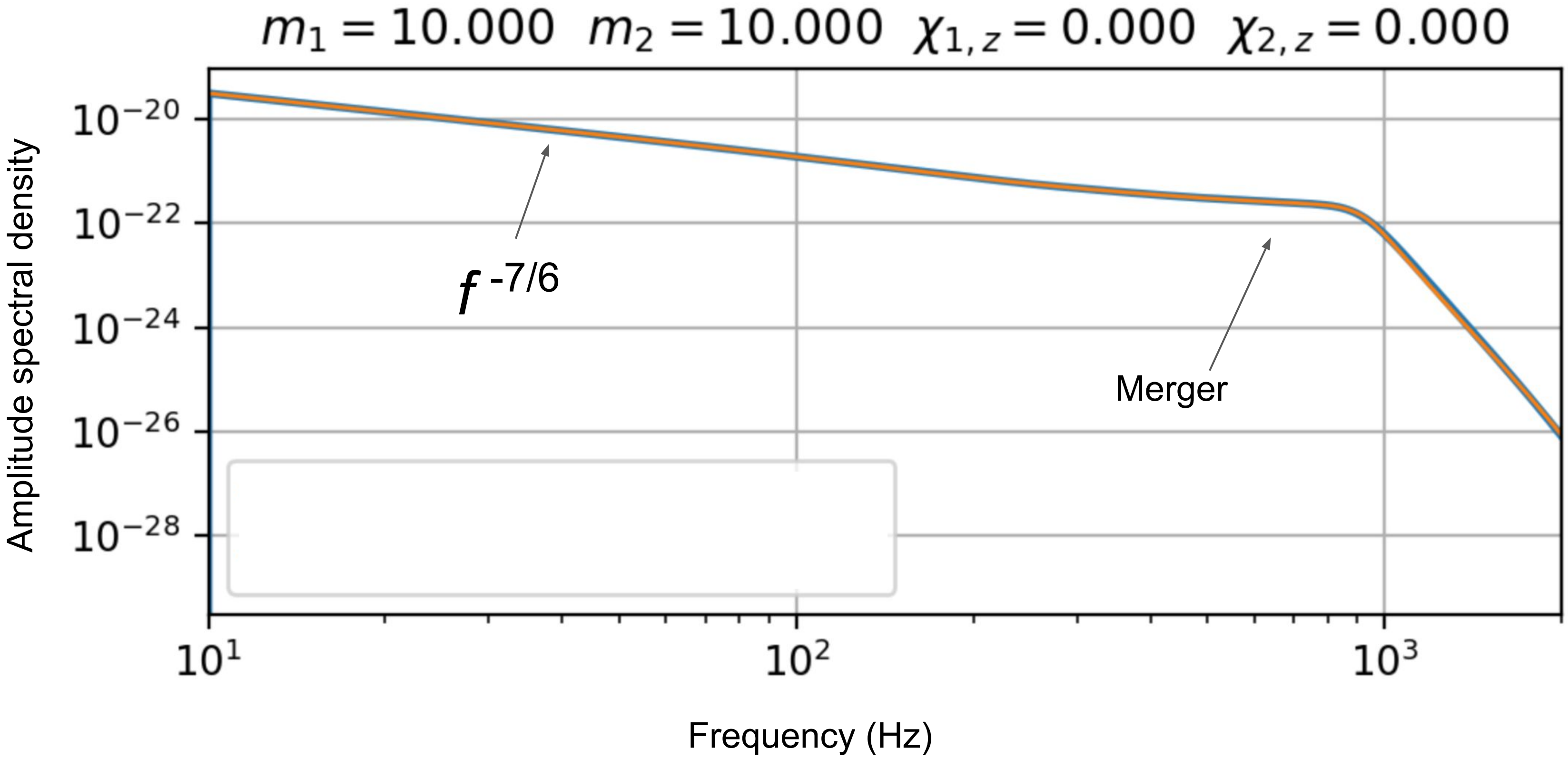
Approximate waveform
$h(t) \sim A(t) \exp\{i\psi(t)\}$

More details:
Peters and Mathews, *Gravitational Radiation from Point Masses in a Keplerian Orbit*, 1963
[Blanchet, *Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries*, LRR 2014]

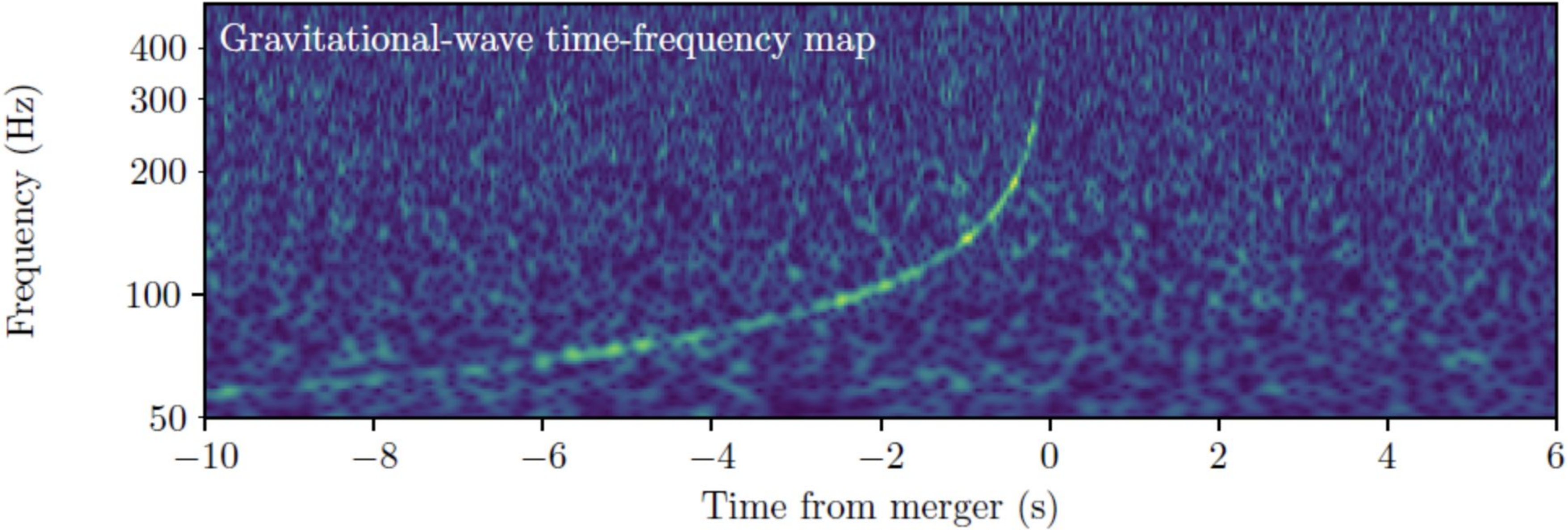
Complete compact binary coalescence waveform (BBH)



Complete compact binary coalescence waveform (BBH)



GW170817's inspiral



Post-Newtonian phase in the frequency domain

Correspondence between time and frequency (stationary phase signal) implies:

$$h(t) \propto A(t) e^{i\Phi(t)} \implies \tilde{h}(f) \propto B(f) e^{i\Psi(f)}$$

$$\Psi(f) = 2\pi f t_c + \phi_c - \frac{\pi}{4} + \frac{3}{128\eta u^5} \left\{ 1 + f(\eta)u^2 + (4\beta - 16\pi)u^3 + [g(\eta) + \sigma]\, u^4 + \cdots - \frac{39}{2}\tilde{\Lambda} u^{10} + \cdots \right\}.$$

$$\eta \equiv m_1 m_2 / M^2$$

$$u \equiv (\pi M f)^{1/3}$$

Spin effects

Newtonian ("zero pN") term

Leading finite-size (deformability) effect

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

Leading mass dependence: chirp mass

Effect of the spins of the inspiraling objects

Post-Newtonian evolution of the orbital angular momentum

(Apostolatos et al 1994)

$$\dot{\mathbf{L}} = \frac{1}{r^3}\left[\frac{4M_1+3M_2}{2M_1}\mathbf{S}_1 + \frac{4M_2+3M_1}{2M_2}\mathbf{S}_2\right]\times\mathbf{L}$$

Spin-orbit coupling

$$-\frac{3}{2}\frac{1}{r^3}\left[(\mathbf{S}_2\cdot\hat{\mathbf{L}})\mathbf{S}_1 + (\mathbf{S}_1\cdot\hat{\mathbf{L}})\mathbf{S}_2\right]\times\hat{\mathbf{L}}$$

Spin-spin coupling

$$-\frac{32}{5}\frac{\mu^2}{r}\left(\frac{M}{r}\right)^{5/2}\hat{\mathbf{L}}\,,$$

Radiation reaction

Components along the orbital angular momentum contribute only to higher-order terms in the pN series.

→ Affect the duration of the inspiral (orbital hangup effect)

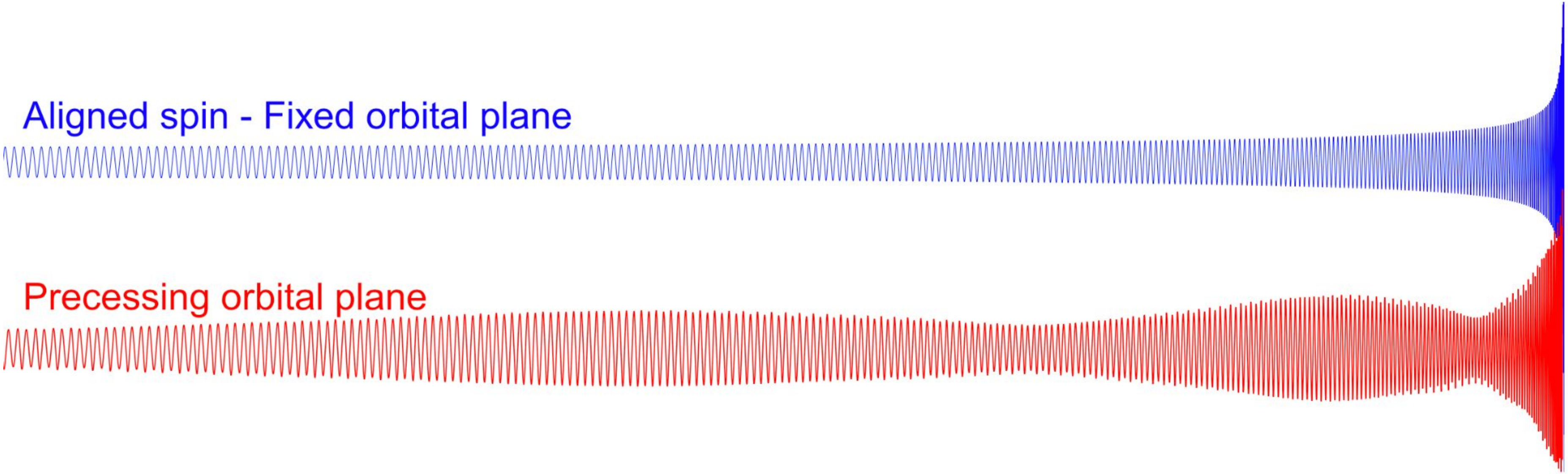
Components in the orbital plane alter the orientation of the momenta.

→ Orbital precession with time scale

$$\tau_{\text{inspiral}} \gg \tau_{\text{precession}} \gg \tau_{\text{orbit}}$$

→ Slow amplitude modulation of the inspiral

Effect of component spins



Matter effects in neutron star mergers

Put a neutron star in an external field

→ Quadrupole moment is induced

$$\lambda \equiv -\frac{Q_{ij}}{\mathcal{E}_{ij}}$$

Dimensionless tidal deformability

$$\Lambda_1 \equiv \lambda_1 / m_1^5$$

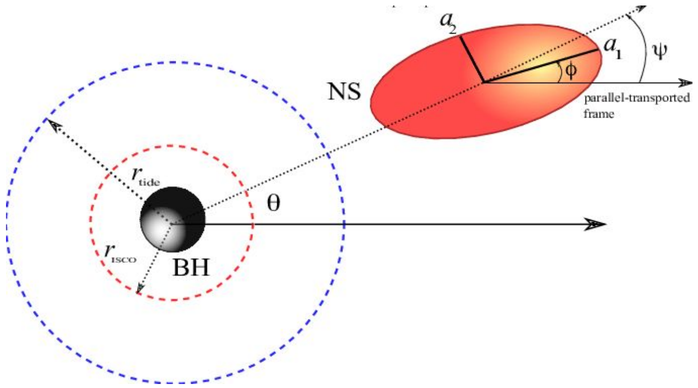
related to the structure of the neutron star

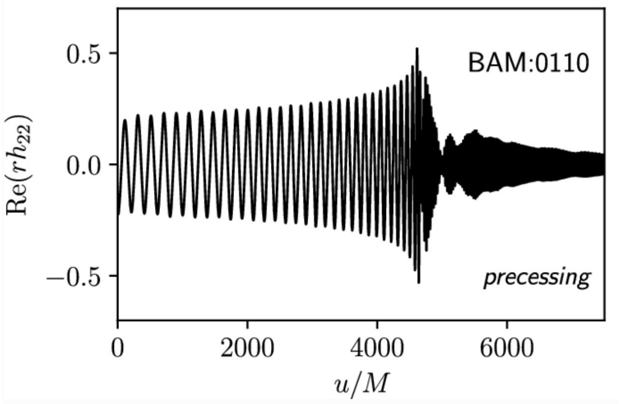
Leading order post-Newtonian tidal correction:

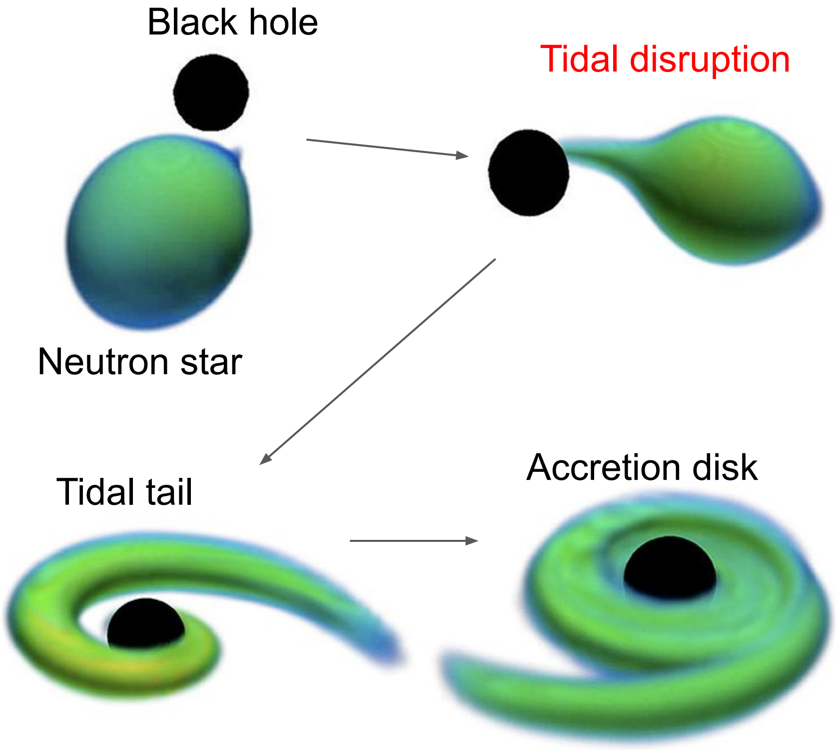
$$\tilde{\Lambda} \equiv \frac{16}{3} \frac{(m_1 + 12m_2)m_1^4 \Lambda_1 + (m_2 + 12m_1)m_2^4 \Lambda_2}{(m_1 + m_2)^5}$$

Deformation as the objects approach → Orbital energy loss → Faster inspiral

Matter effects in neutron star mergers







Practical calculation of a CBC waveform

LALSimulation: software library with "approximants", functions that return h_+, h_x given the source parameters.

Part of a larger library called LALSuite.

Used by most (all?) data analyses involving CBCs, both to search and to characterize signals.

Code: https://git.ligo.org/lscsoft/lalsuite

API documentation:

- https://lscsoft.docs.ligo.org/lalsuite/lalsimulation/index.html
- https://lscsoft.docs.ligo.org/lalsuite/lalsimulation/group__lalsimulation__inspiral.html

In practice I never use LALSimulation directly - I use it from PyCBC (more on this later)

Practical calculation of a CBC waveform

Typical approximants I am familiar with:

- TaylorF2 - Inspiral-only, frequency domain
- SpinTaylorT* - Inspiral-only, time domain
- SEOBNRv4_opt - Inspiral-merger-ringdown (IMR, good for BBH), time domain
- SEOBNRv4_ROM - IMR, frequency domain
- SEOBNRv4HM - IMR, time domain, with higher-order modes
- IMRPhenomPv2 - IMR, frequency domain, with orbital precession
- *_NRTidalv2 - Extension of simpler models to add tidal effects

Isolated, rotating and deformed neutron stars

Equatorial ellipticity ("mountain")

$$\epsilon \equiv \frac{|I_{xx} - I_{yy}|}{I_{zz}}$$

Results in GW amplitude at frequency $2f_{rot}$:

$$h_0 = \frac{4\pi^2 G \epsilon I_{zz} f_{GW}^2}{c^4 d} = (1.1 \times 10^{-24}) \left(\frac{\epsilon}{10^{-6}}\right) \left(\frac{I_{zz}}{I_0}\right) \left(\frac{f_{GW}}{1\,\mathrm{kHz}}\right)^2 \left(\frac{1\,\mathrm{kpc}}{d}\right)$$

Energy loss

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G}{c^5} I_{zz}^2 \epsilon^2 \omega_{rot}^6 = -(1.7 \times 10^{33}\,\mathrm{J/s}) \left(\frac{I_{zz}}{I_0}\right)^2 \left(\frac{\epsilon}{10^{-6}}\right)^2 \left(\frac{f_{GW}}{1\,\mathrm{kHz}}\right)^6$$

Results in a spindown

$$\dot{f}_{GW} = -\frac{32\pi^4}{5}\frac{G}{c^5} I_{zz} \epsilon^2 f_{GW}^5 = -(1.7 \times 10^{-9}\,\mathrm{Hz/s}) \left(\frac{\epsilon}{10^{-6}}\right)^2 \left(\frac{f_{GW}}{1\,\mathrm{kHz}}\right)^5$$

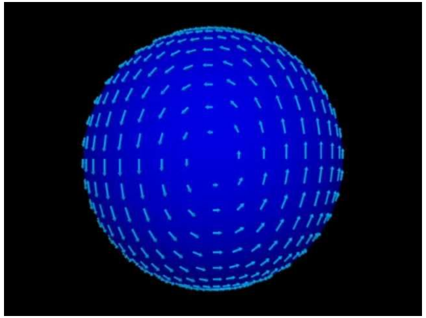
More details: [Riles, *Searches for continuous-wave gravitational radiation*, LRR 2023]

Radiation from mass-current quadrupole (r-) modes

Fluid modes supported by rotation; characteristic GW amplitude

$$h_0 = \sqrt{\frac{512\,\pi^7}{5}}\,\frac{G}{c^5}\,\frac{1}{d}\,f_{\rm GW}^3\,\alpha M R^3 \tilde{J}$$

$$= 3.6 \times 10^{-26} \left(\frac{1\,\mathrm{kpc}}{\mathrm{d}}\right) \left(\frac{f_{\rm GW}}{100\,\mathrm{Hz}}\right)^3 \left(\frac{\alpha}{10^{-3}}\right) \left(\frac{R}{11.7\,\mathrm{km}}\right)^3$$



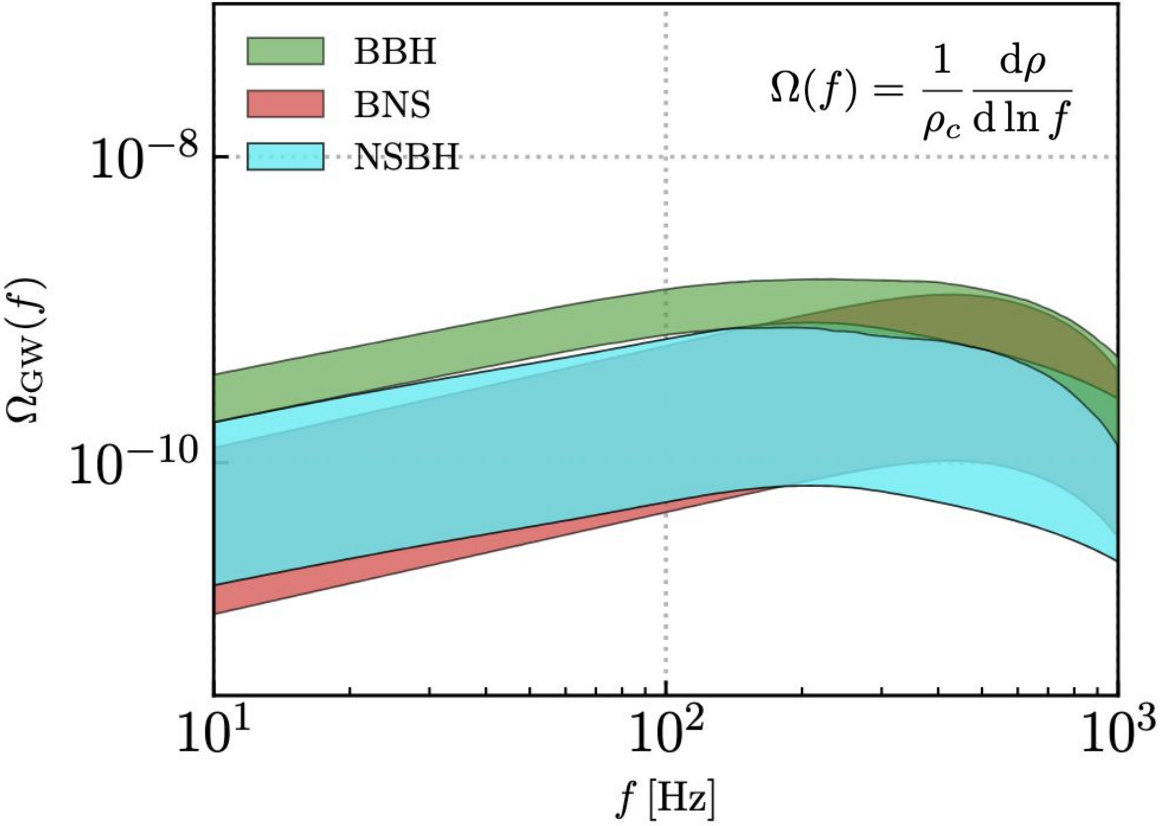
Associated spindown

$$\dot{f}_{\rm GW} = -\frac{4096\,\pi^7}{225}\,\frac{G}{c^7}\,\frac{M^2 R^6 \tilde{J}^2}{I_{zz}}\,\alpha^2 f_{\rm GW}^7$$

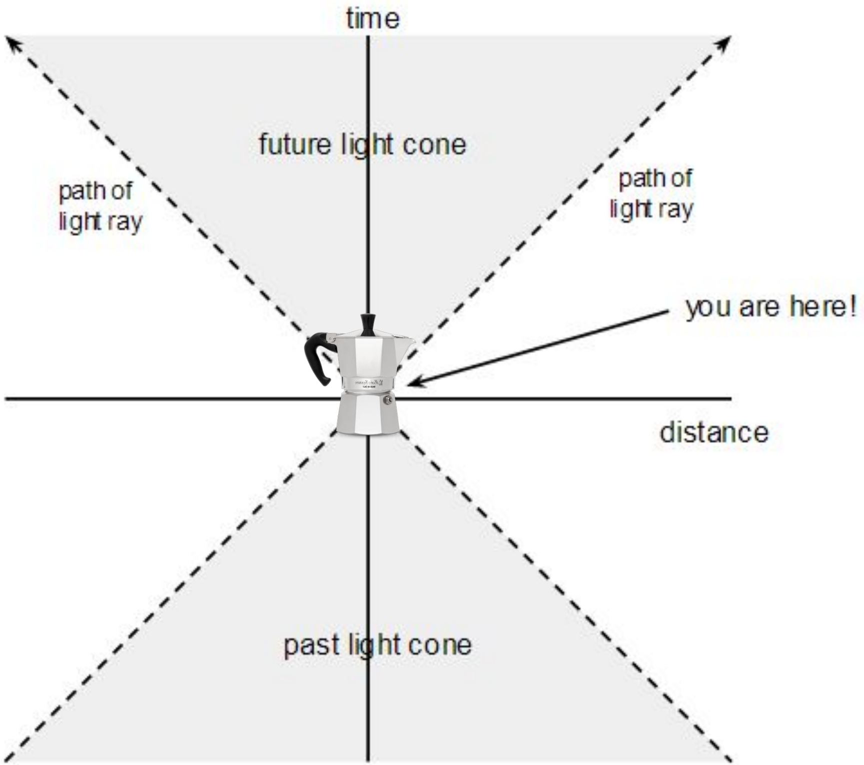
$$= -9.0 \times 10^{-14}\,\mathrm{Hz/s} \left(\frac{R}{11.7\,\mathrm{km}}\right)^6 \left(\frac{\alpha}{10^{-3}}\right)^2 \left(\frac{f_{\rm GW}}{100\,\mathrm{Hz}}\right)^7$$

More details: [Riles, *Searches for continuous-wave gravitational radiation*, LRR 2023]
Movies: https://research.physics.illinois.edu/CTA/movies/r-Mode/

Stochastic foregrounds

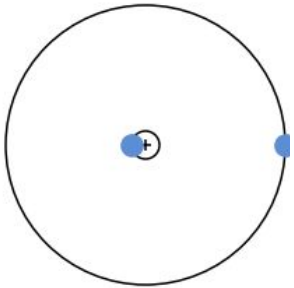


At this point, we may want to have a break



Gravitational-wave detectors

Why would we measure gravitational waves?

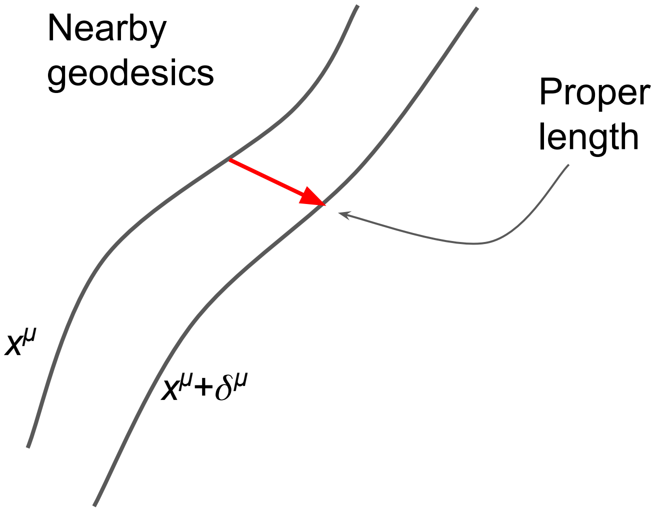


$r \gtrsim 100$ Mpc

Mass ~ 10 M_{Sun}
Velocity ~ c
Mass quadrupole Q

$$h_{ij} \sim \frac{G}{c^4}\frac{\ddot{Q}}{r} \lesssim 10^{-21}$$

GW detectors: general principle



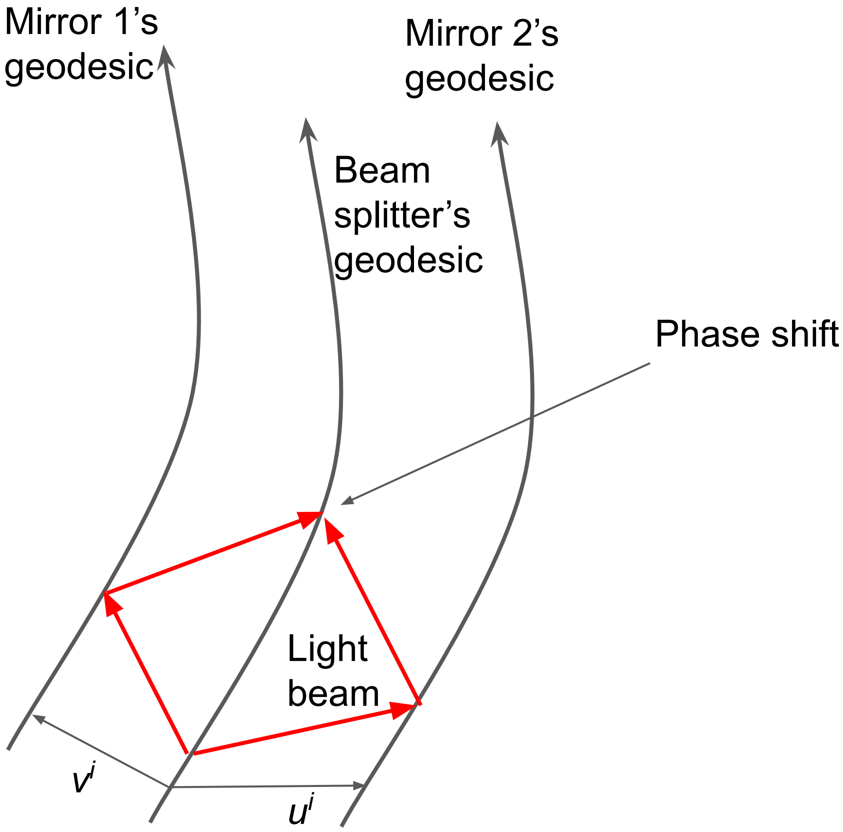
$$L(t) = \int_0^L \sqrt{g_{xx}}dx = \int_0^L \sqrt{1+h_{xx}(t,x)}dx$$

$$\approx L\left(1+\frac{1}{2}h_{xx}(t,x=0)\right)$$

Long wave / low frequency (“nearby”)
Weak perturbation

Time-dependent source → Change in proper length along direction u^i $$\frac{\delta L}{L} \approx \frac{1}{2}h_{ij}u^i u^j$$

GW detectors: general principle



$$\delta\phi = \frac{4\pi f_{\text{light}}}{c}(\delta L_1 - \delta L_2)$$

$$\propto L(u^i u^j - v^i v^j)h_{ij}$$

GW detectors: response to a plane GW

$$h_{ij} = \varepsilon^{+}_{ij} h_{+} + \varepsilon^{\times}_{ij} h_{\times}$$

$$h = (x^i x^j - y^i y^j) h_{ij} = D^{ij} h_{ij} = F_{+} h_{+} + F_{\times} h_{\times}$$

Antenna pattern functions,
beam pattern functions

Functions of instantaneous relative orientation of source and detector, and instantaneous signal frequency.

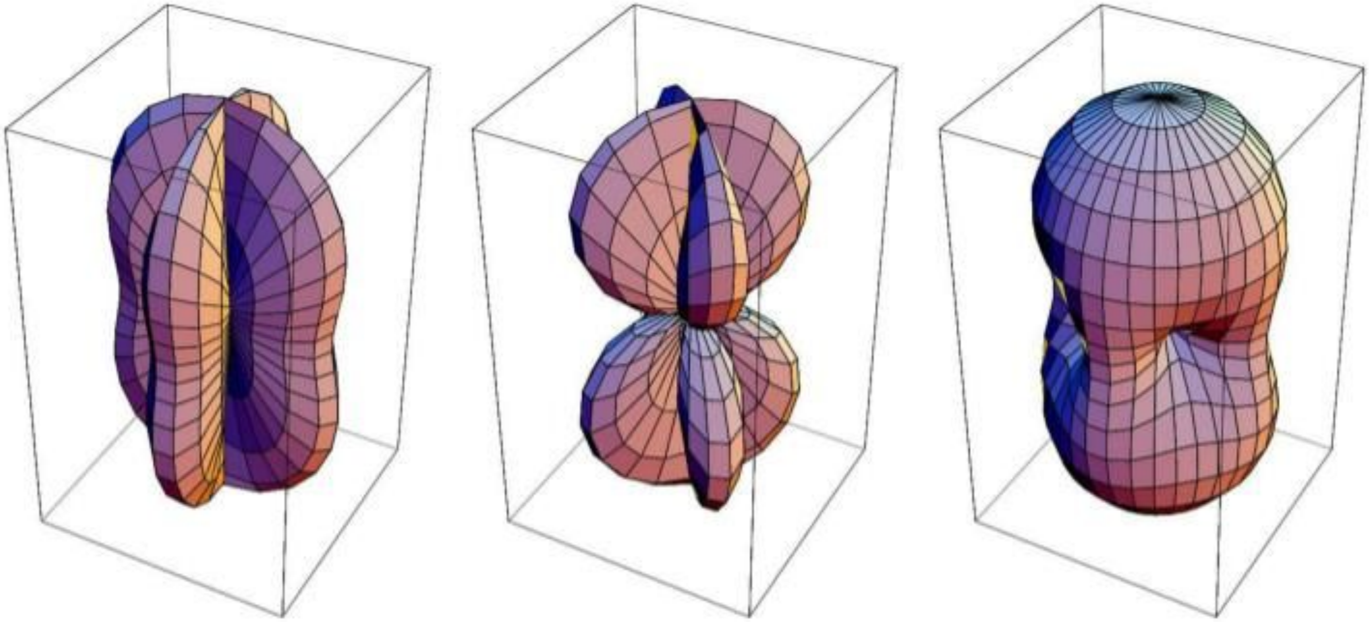
Time-independent if $T_{\text{signal}} \ll$ ~1 hr (nonrotating Earth).

Freq-independent if $\lambda_{\text{signal}} \gg L$ (long-wave/low-frequency approximation).

Various software libraries allow you to calculate these functions (e.g. LALSuite, PyCBC).

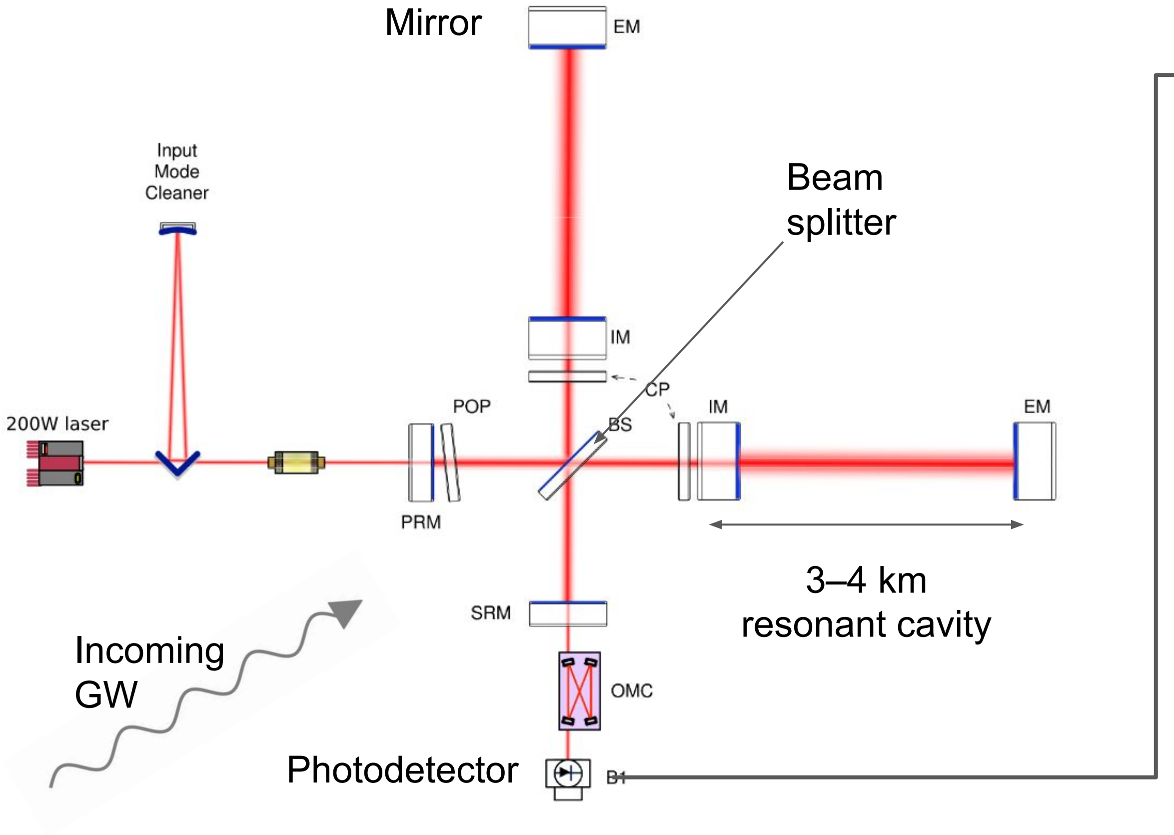
GW detectors: response to a plane GW

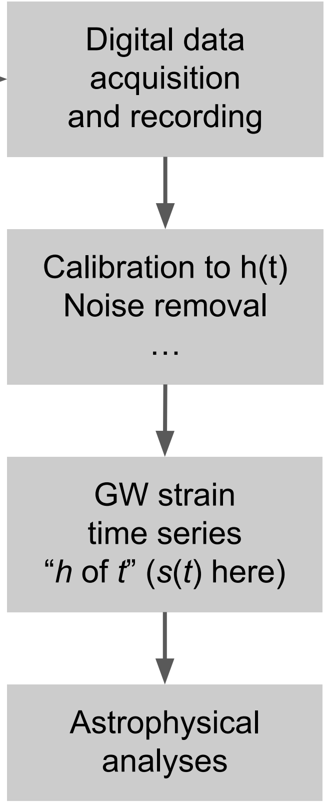
Antenna pattern in the low frequency limit



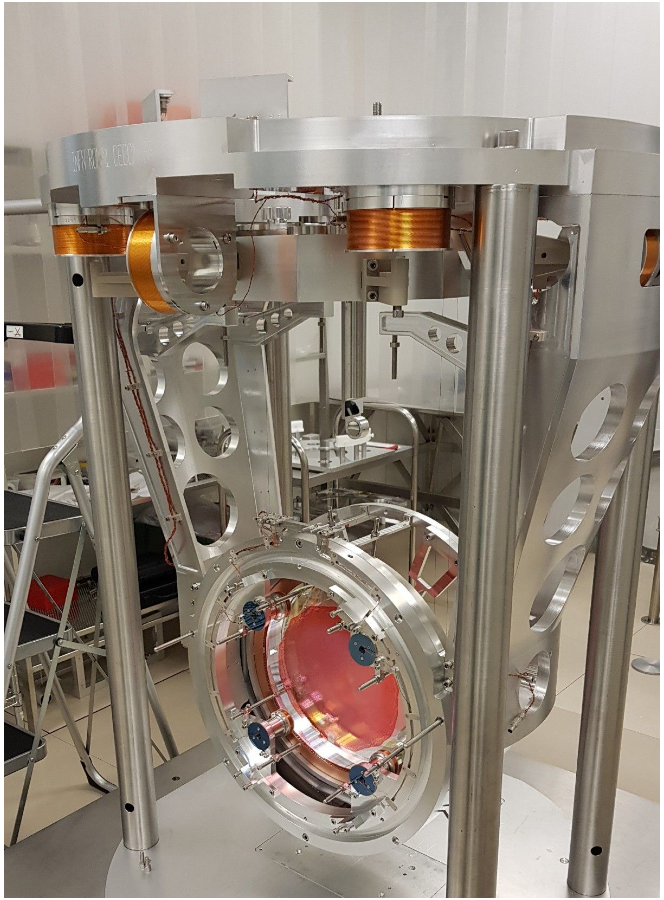
+ polarization × polarization averaged

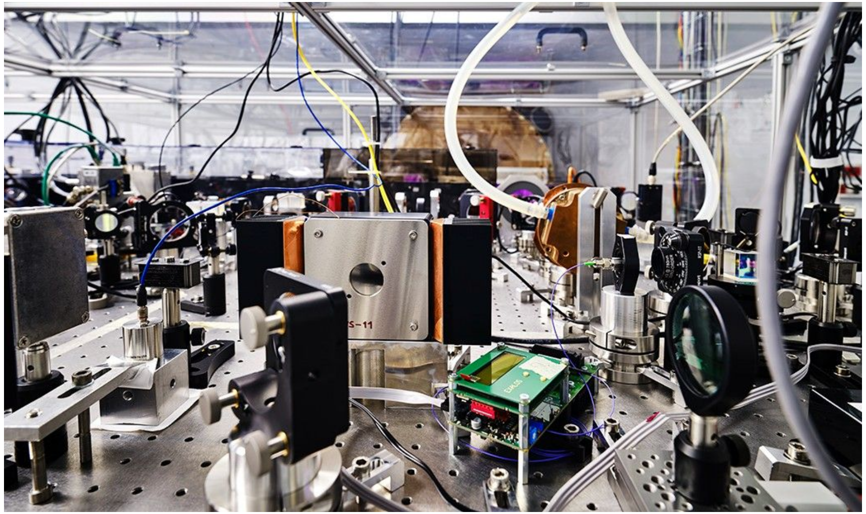
GW detectors: Michelson-Fabry-Perot interferometer



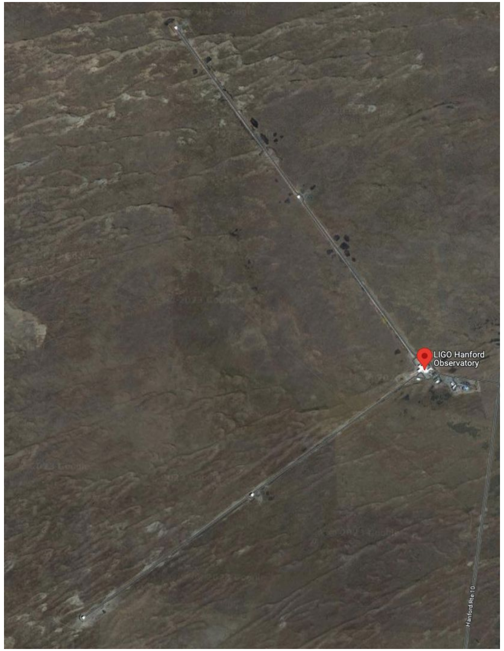


GW detectors: the present network

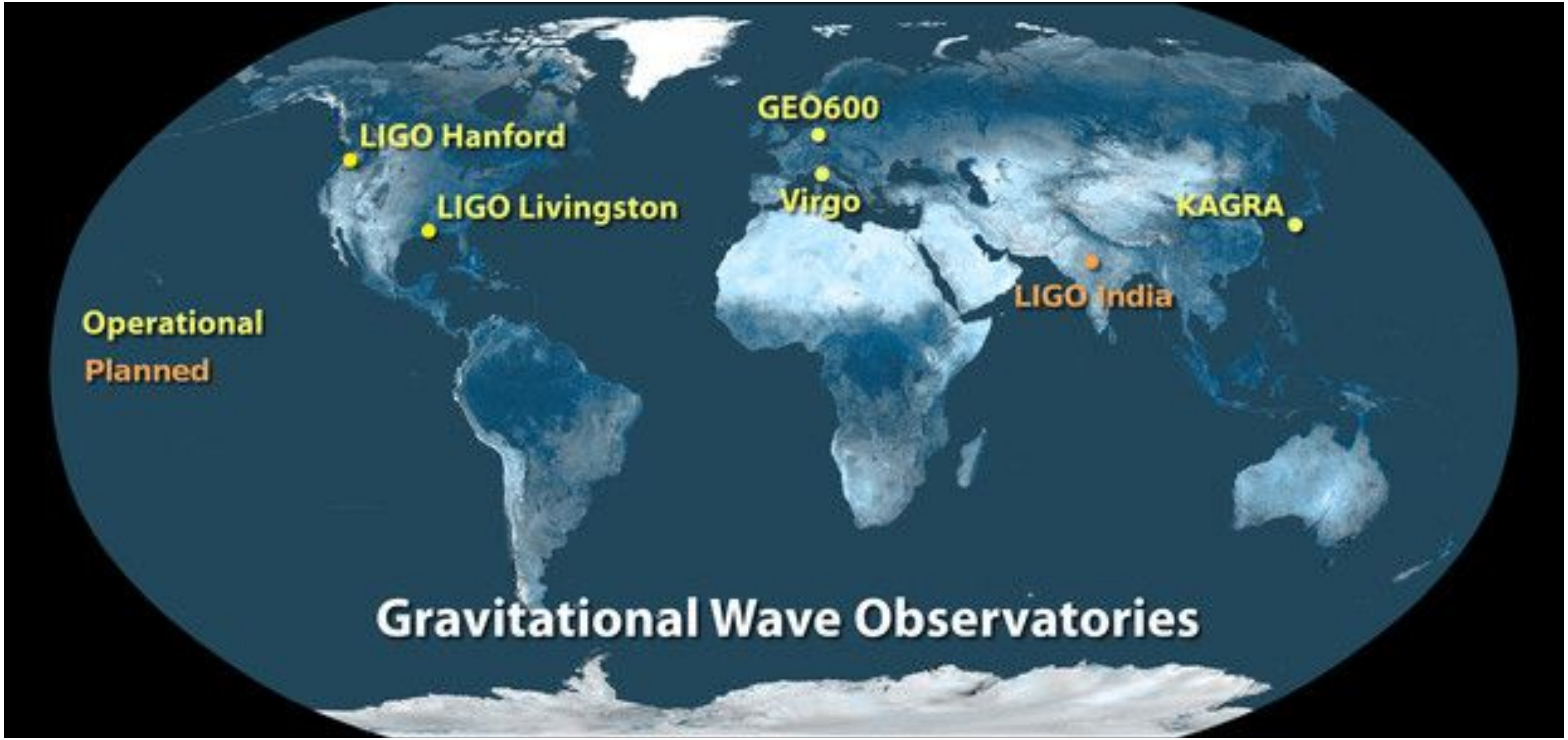




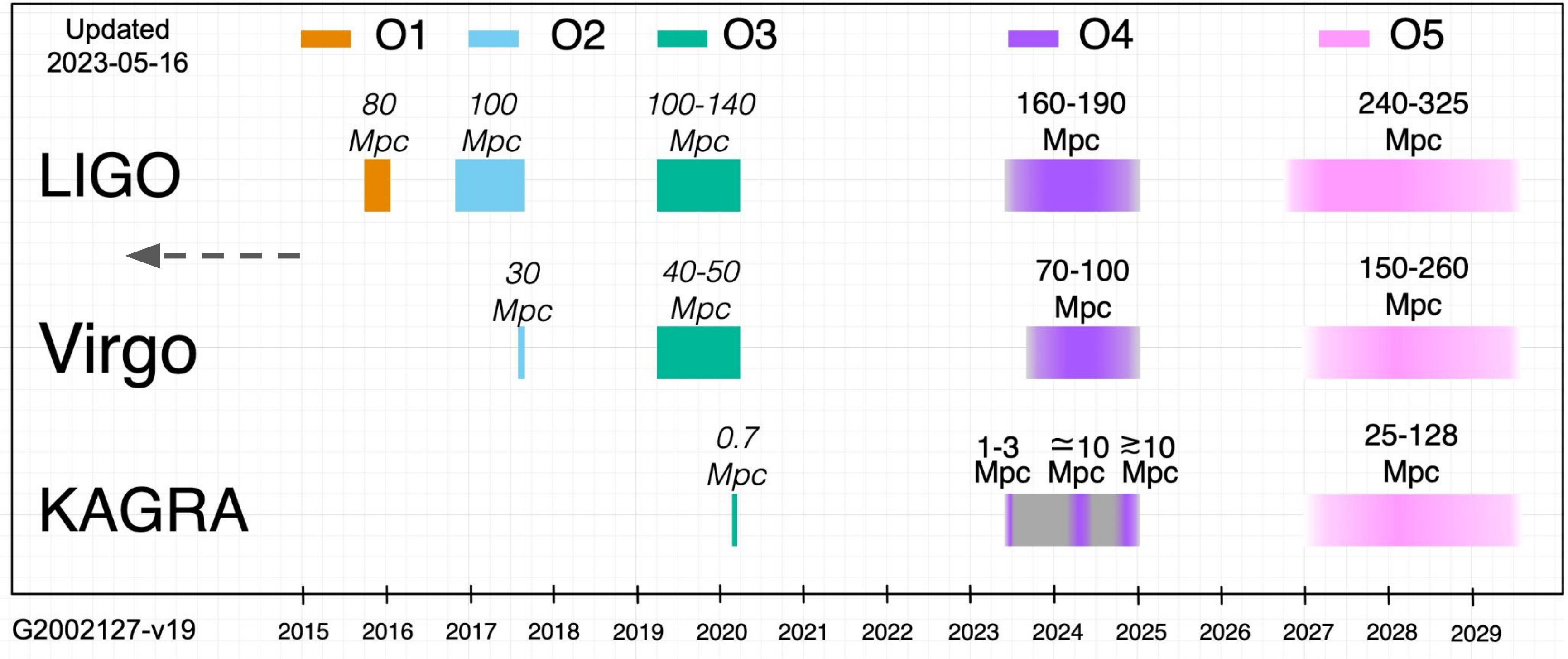
GW detectors: the present network



GW detectors: the present network



GW detectors: recent and near-future observing runs



GW detectors: data and results

- Rapid alerts:
  - General Coordinates Network: https://gcn.nasa.gov/
  - GraceDB: https://gracedb.ligo.org
- GW Transient Catalog (GWTC) papers
- GW Open Science Center (GWOSC) which includes $h(t)$: https://gwosc.org/

This afternoon we will use some data from GWOSC.

Analysis of gravitational-wave data

What do the data of a GW detector mean?

Continuous time series of spacetime strain measurement, sampled at ~10 kHz, contaminated with noise:

$$s(t) = \underbrace{n_{\mathrm{easy}}(t)} + \underbrace{n_{\mathrm{hard}}(t)} + \underbrace{\sum_i h_i(t; \vec{\lambda}_i)}$$

Detector noise that is easy to predict or model. Usually fundamental, and determines the sensitivity of the detector.

Detector/environment noise that is hard or impossible to predict or model. Usually technical, and determines the quality of the data.

Superposition of all gravitational-wave signals of the form given on slide 32, each with its own vector of parameters describing the source and possibly larger structures, or even the whole Universe.

What is “easy” noise?

Stochastic process: only its long-term statistical properties are well defined, i.e.

Probability distribution

$$P(n_{\mathrm{easy}})$$

Quasi-Gaussian process

Autocovariance or power spectral density

$$\langle n_{\mathrm{easy}}(t_1) n_{\mathrm{easy}}(t_2)\rangle = \frac{1}{2} W(t_2 - t_1) \quad \text{(time domain)}$$

$$\langle \tilde{n}_{\mathrm{easy}}(f_1) \tilde{n}^*_{\mathrm{easy}}(f_2)\rangle = \frac{1}{2} S(f_1)\delta(f_2 - f_1) \quad \text{(Fourier domain)}$$

Quasi-stationary process: $W(t)$ or $S(f)$ time-dependent
White process: $S(f)$ = const.

Typical sources: thermal noise and laser quantum (shot) noise.

Can only discover/study a signal h if it is “louder than noise” (to be defined soon…)
→ n_{easy} is the ultimate limit to the sensitivity of a GW detector.

What is “easy” noise?

Gaussianity → For N time-domain samples of n_{easy} (here n for simplicity) we can write

$$P(\boldsymbol{n}) = \frac{\exp\left(-\frac{1}{2}\boldsymbol{n}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{n}\right)}{\sqrt{(2\pi)^N \det \boldsymbol{\Sigma}}}$$

Stationarity → Covariance matrix $\boldsymbol{\Sigma}$ is constant across diagonals.

In the Fourier domain the covariance becomes (almost) diagonal

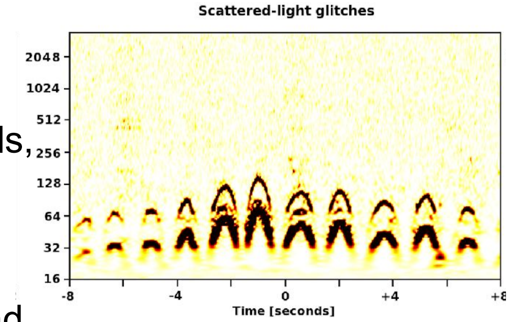
$$P(\tilde{\boldsymbol{n}}) \approx \frac{\exp\left(-\frac{1}{2}\sum_i \tilde{n}_i^2/\sigma_i^2\right)}{\sqrt{(2\pi)^N \prod_i \sigma_i^2}}$$

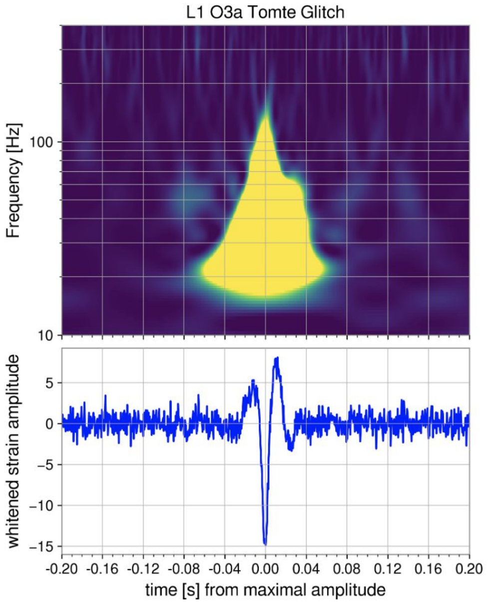
What is “hard” noise?

Potentially an arbitrary superposition of stationary, nonstationary, deterministic, nondeterministic, transient, continuous signals, whose origins may be known or unknown.

Any component that can be understood or modeled can be subtracted from the data, and hence drops out of this classification!

The rest can only be studied by running the detectors, observing how the noise behaves and trying to track down its origin in the detector (noise hunting).





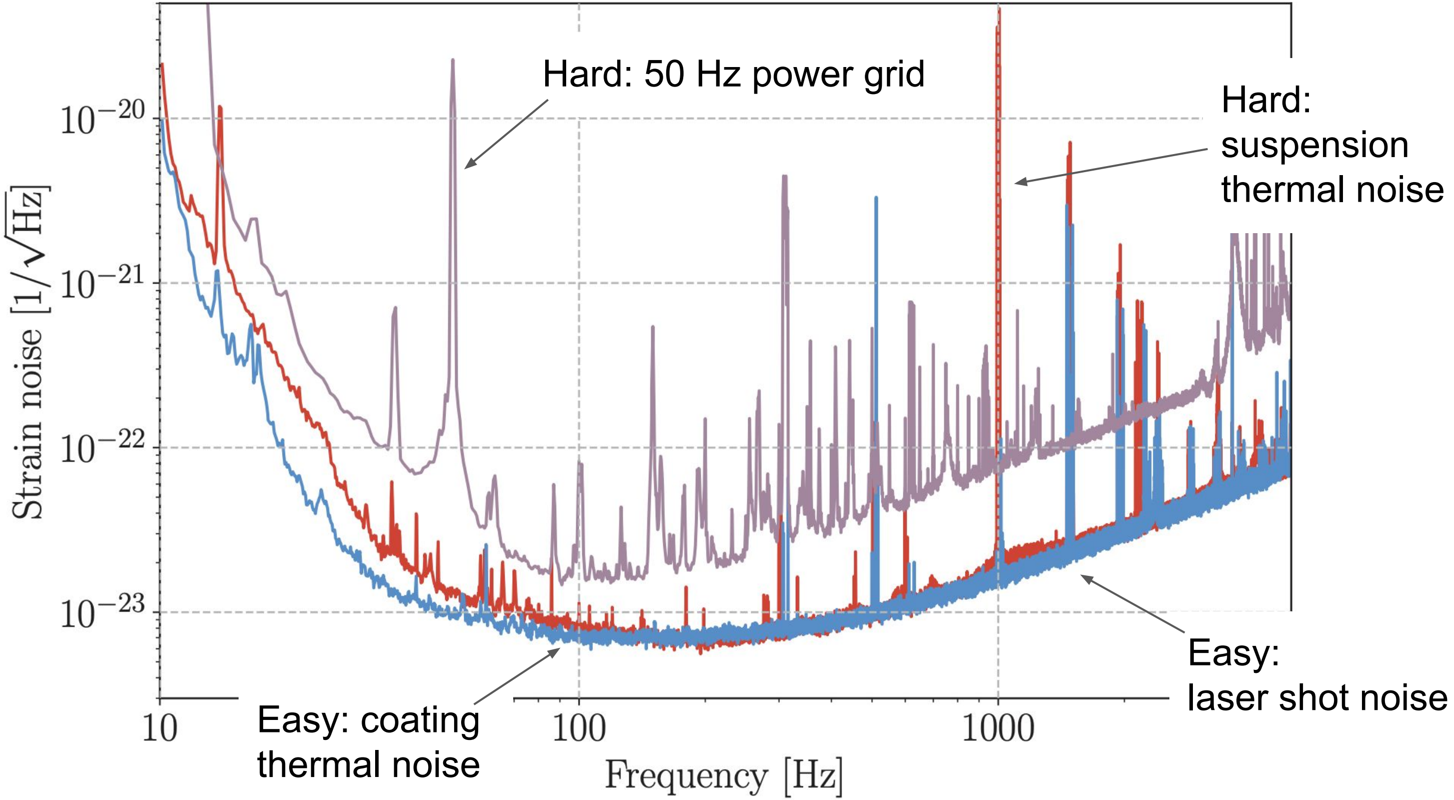
Different types of “hard” noise may only be a problem for specific analyses.

Spectral density of GW data

$$\sqrt{S(f)}$$

Amplitude spectral density

Not to be confused with the power spectral density

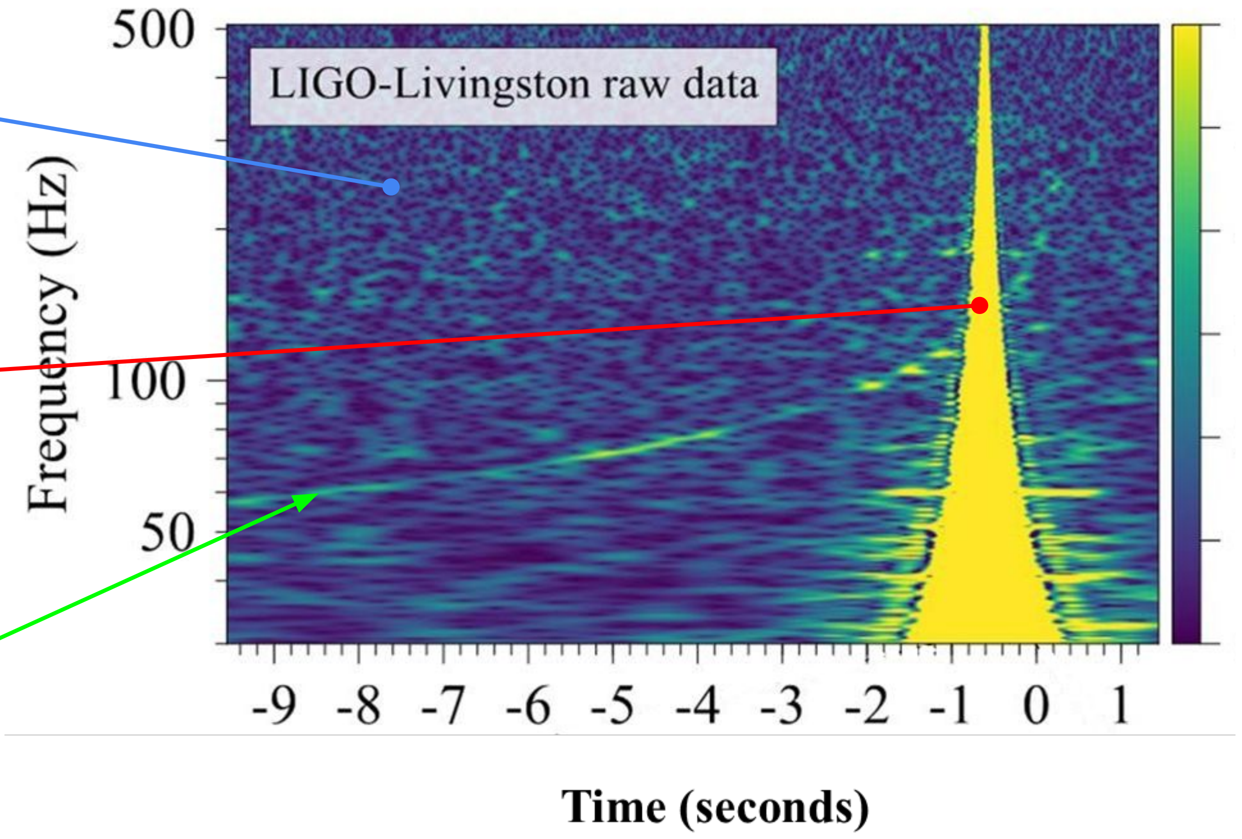


Time-frequency representation of GW data

“Easy” noise

“Hard” noise

Astrophysical signal



“Signal energy”

“Excess power”

“Local variance”

General workflow of gravitational-wave astronomy

Detector design and construction: build the machines that will get you $s(t)$

- Make n_{easy} as small as possible.
- Make n_{hard} much smaller than n_{easy} so we can forget about it.

Observing runs: obtain $s(t)$

- Run the detectors in order to obtain $s(t)$ for periods as long as possible.
- Understand how the detectors are behaving and learn how to improve them.

Competing activities!

Data analysis: derive scientific knowledge from $s(t)$

- Identify the presence of each signal h_i (without being fooled by n) and measure its parameters, i.e. characterize each individual source.
- Use the information for science: census of objects in the Universe, multimessenger astronomy, cosmology, fundamental physics…

Bayesian formalism

Probability of a statement *A*

$$0 \leq P(A) \leq 1$$

Bayes' theorem

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)}$$

Data *d*, parametric model *M*

Posterior distribution

Likelihood function

Prior distribution

$$p(\theta|d, M) = \frac{\mathcal{L}(d|\theta, M)\, \pi(\theta|M)}{\mathcal{Z}(d|M)}$$

Evidence (marginalized likelihood)

$$\mathcal{Z}(d|M) = \int \mathcal{L}(d|\theta, M)\, \pi(\theta|M)\, d\theta$$

$$\frac{P(M_A|d)}{P(M_B|d)} = \frac{\mathcal{Z}(d|M_A)}{\mathcal{Z}(d|M_B)} \frac{\pi(M_A)}{\pi(M_B)}$$

Posterior odds Bayes factor Prior odds

Data analysis tasks

- Model selection:
  estimate the posterior odds
- Parameter estimation:
  estimate the posterior distribution

Direct application of the Bayesian formalism - "Ideal" analysis

Calculate the evidences for the following models:

- Model 0: data contain only noise
- Model 1: data contain noise and one signal
- …
- Model *n*: data contain noise and n signals

We can then do model comparison and parameter estimation for the *n* signals.

Hard!!!*

- $n_{max} >> 1$
- *m*-dimensional parameter space for a single signal
- Must integrate over $nm >> 1$ dimensions
- Likelihood is an extremely complicated function

* But people are trying 🤫

Maximum likelihood formalism

For the first identification of an unknown GW signal, it is easier to solve a different problem:

$$\frac{P(M_A|d)}{P(M_B|d)} = \frac{\mathcal{Z}(d|M_A)}{\mathcal{Z}(d|M_B)} \frac{\pi(M_A)}{\pi(M_B)} \longrightarrow \mathcal{R} = \frac{\max_{\theta\in\Theta} \mathcal{L}}{\max_{\theta\in\Theta_0} \mathcal{L}}$$

Maximum likelihood ratio

A: noise + signal hypothesis
B: noise only (null) hypothesis

$$p(\theta|d, M) = \frac{\mathcal{L}(d|\theta, M)\, \pi(\theta|M)}{\mathcal{Z}(d|M)} \longrightarrow \begin{cases} \hat{\theta} = \arg\max \mathcal{L} \\ \boldsymbol{\Sigma}_\theta = \left. \dfrac{\partial^2 \mathcal{L}}{\partial\theta_i \partial\theta_j} \right|_{\hat{\theta}} \end{cases}$$

Maximum likelihood parameter estimate

Covariance matrix

Solving the ML problem gives only an approximate answer to the original problem.
In many cases not a very good one.

The signal + Gaussian noise likelihood function

Recall our data model $\tilde{\boldsymbol{s}} = \tilde{\boldsymbol{n}} + \tilde{\boldsymbol{h}}$ with quasi-Gaussian noise $P(\tilde{\boldsymbol{n}}) \approx \dfrac{\exp\left(-\frac{1}{2}\sum_i \tilde{n}_i^2/\sigma_i^2\right)}{\sqrt{(2\pi)^N \prod_i \sigma_i^2}}$

$$\mathcal{L}(\boldsymbol{s}|\boldsymbol{h}) \propto \exp\left(-\frac{1}{2}\sum_i (\tilde{s}_i - \tilde{h}_i)(\tilde{s}_i - \tilde{h}_i)^* \sigma_i^{-2}\right)$$

Whittle likelihood

Assume known noise PSD → The only free parameters are signal parameters

ML ratio becomes $\mathcal{R} = \dfrac{\max_h \mathcal{L}(\tilde{s}|h)}{\mathcal{L}(\tilde{s}|h=0)}$ simpler in log: $\ln\mathcal{R} = \max_h \left(\langle s|h\rangle - \frac{1}{2}\langle h|h\rangle\right)$

with the noise-weighted inner product between discrete-time signals *a* and *b*

$$\langle a|b\rangle = \sum_i \frac{\tilde{a}_i \tilde{b}_i^*}{\sigma_i^2}$$

(Note that I am not being very careful with constant factors in these expressions)

Matched filtering and signal-to-noise ratio

Want to maximize $\ln \mathcal{R} = \max_h \left(\langle s|h \rangle - \frac{1}{2} \langle h|h \rangle \right)$. Re-express the signal as $h = a h_{\mathrm{norm}}$

Then maximizing over *a* has a closed-form solution:

$$\ln \mathcal{R} = \max_{h_{\mathrm{norm}}} \left(\frac{1}{2} \frac{\langle s|h_{\mathrm{norm}} \rangle^2}{\langle h_{\mathrm{norm}}|h_{\mathrm{norm}} \rangle} \right) = \max_{h_{\mathrm{norm}}} \left(\frac{1}{2} \rho^2 \right)$$

with the (amplitude-maximized) signal-to-noise ratio (SNR)

$$\rho = \frac{\langle s|h_{\mathrm{norm}} \rangle}{\langle h_{\mathrm{norm}}|h_{\mathrm{norm}} \rangle^{1/2}} = \frac{\langle s|h \rangle}{\sqrt{\langle h|h \rangle}}$$

Template waveform

Matched filter

Maximizing over an overall phase shift is also possible if we use instead two templates that differ by a 90 deg phase rotation:

$$\rho = \frac{\sqrt{\langle s|h_{\mathrm{I}} \rangle^2 + \langle s|h_{\mathrm{Q}} \rangle^2}}{\sqrt{\langle h|h \rangle}}$$

Interpreting the signal-to-noise ratio

Remember that ultimately we are computing the LLR between “Gaussian noise + signal h” and “Gaussian noise only”.

How does the SNR behave if there is no signal?

Each <s|h> term is a unit-norm linear filter applied to Gaussian whitened data
→ Normal random variate

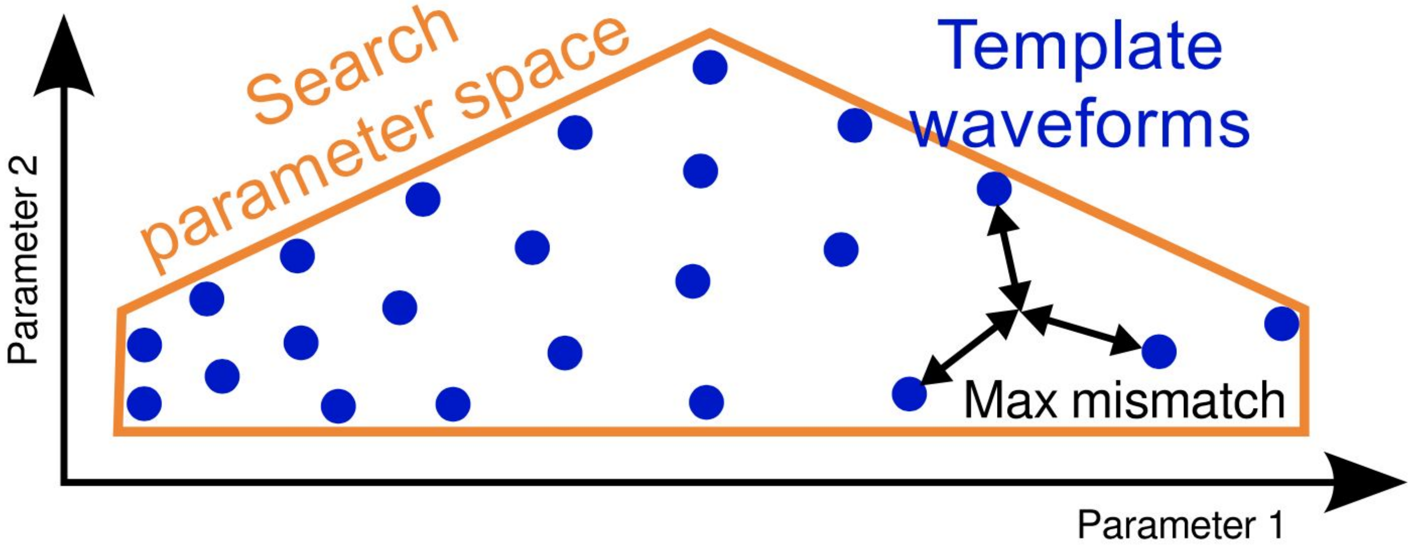
→ ϱ^2 distributed as a central χ^2 random variate → ϱ is on average ~1 far from the signals

Maximize the SNR ↔ Maximize the LLR for Gaussian noise

→ As SNR grows above some SNR_{min} (typically 4-8) the null hypothesis becomes more and more unlikely.

Assuming that the signals are well separated, finding the local maxima of the SNR over the remaining parameters will point out the (strongest) signals.
This generally requires a numerical search.

Needles in a haystack: the template bank



Discrete bank
Approximations
…

Loss of SNR

$$\varphi := \frac{\rho_{\mathrm{observed}}}{\rho_{\mathrm{optimal}}} < 1$$

Fitting factor

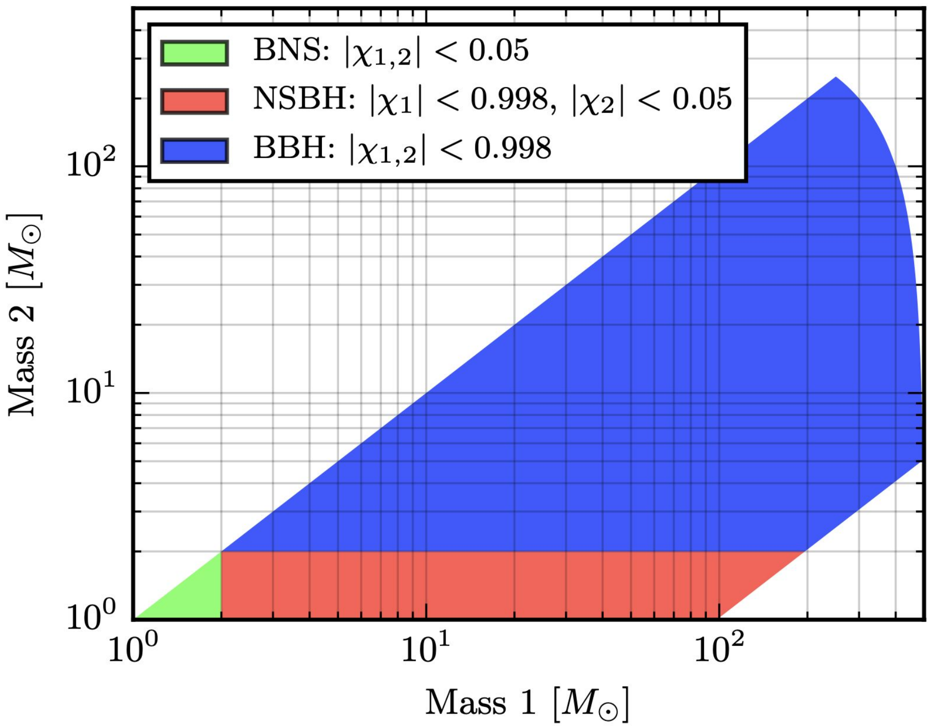
Loss of CBC detections

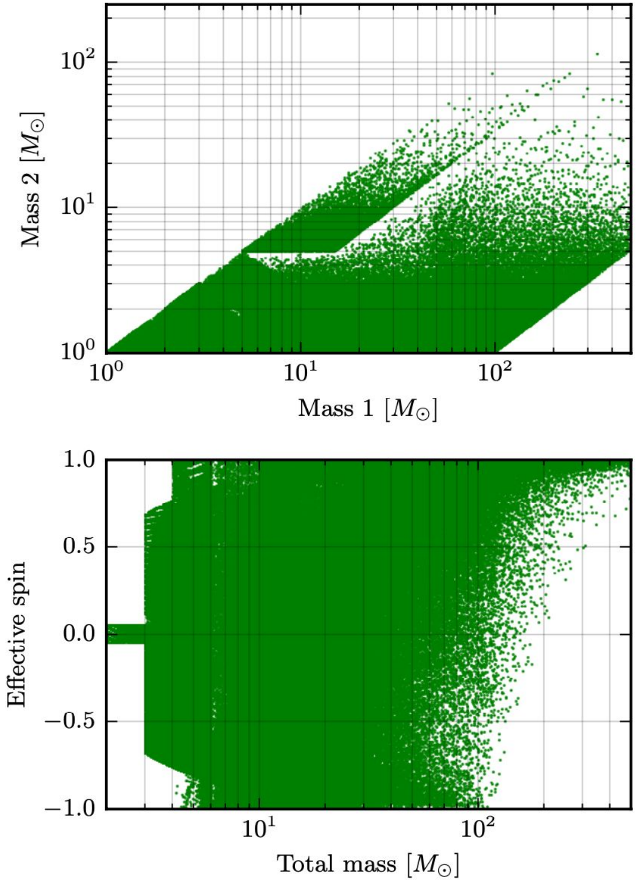
$$\frac{n}{n_{\mathrm{optimal}}} \propto \varphi^3$$

Bank must achieve

$$\varphi \approx 1$$

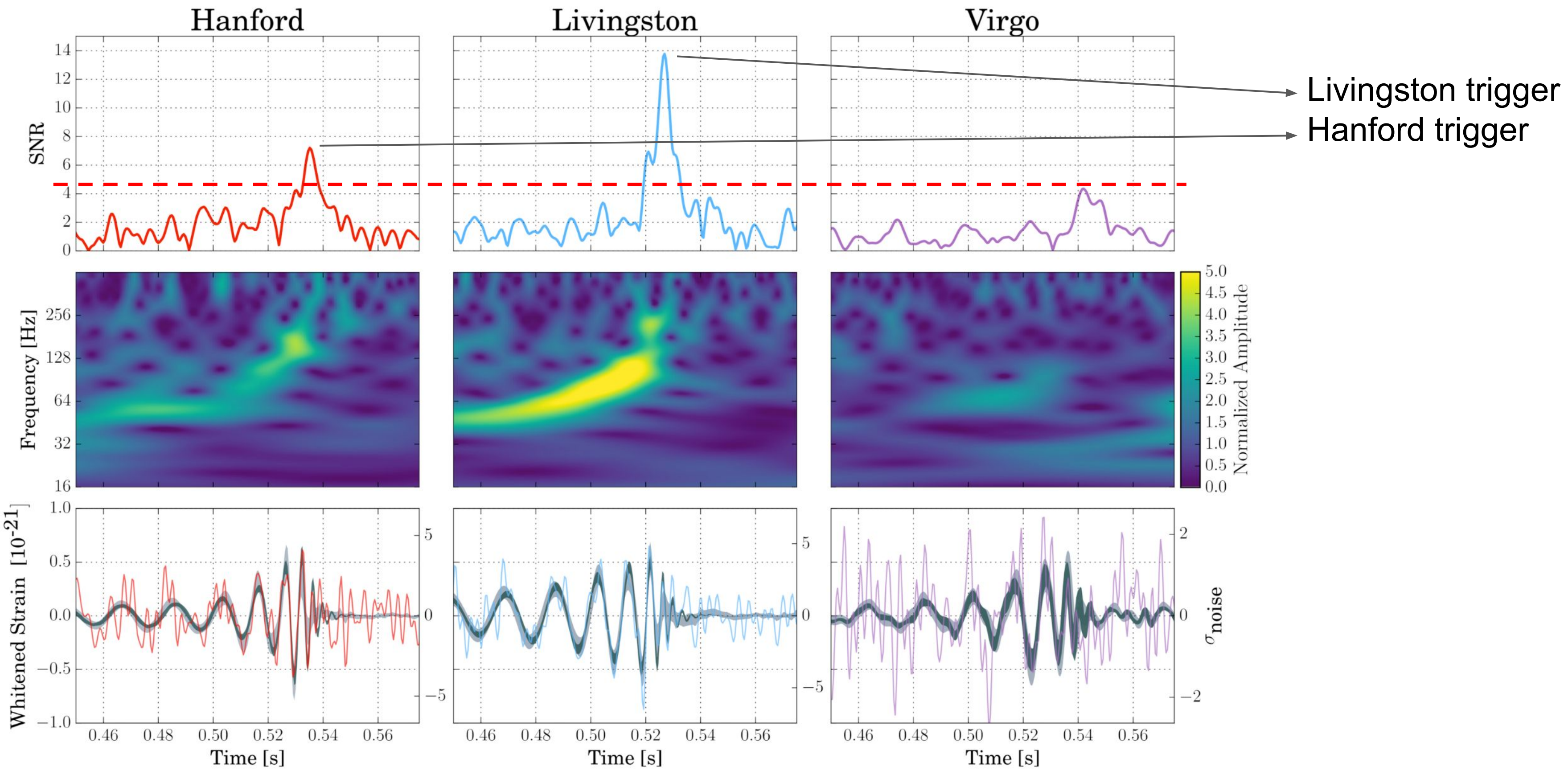
Needles in a haystack: the template bank





10^5-10^6 templates for CBC searches with LIGO/Virgo/KAGRA

Generation of candidate events



“Hard” noise and the SNR

Data

Template waveform

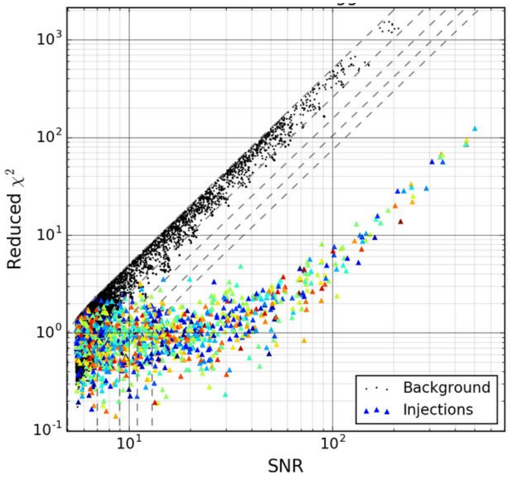
$$\rho \propto \int \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_\mathrm{n}(f)}\,\mathrm{d}f$$

Noise spectral density

SNR “proportional to the data”
→ Local fluctuations of the noise will reflect in the SNR
→ Large SNR no longer implies a trigger is astrophysical

Solution: signal-based discrimination statistics

- Time-frequency χ^2: check distribution of SNR over frequency
- Autocorrelation χ^2: check shape of SNR peak over merger time
- Bank χ^2: check shape of SNR peak over template bank parameters



Common statistical property:

- Distributed like a central χ^2 under Gaussian noise or Gaussian noise + matched signal
- Distributed like a noncentral χ^2 under Gaussian noise + mismatched signal

Combining data from different detectors

Incoherent methods

Solve the ML problem separately for each detector.

Identify triggers separately in each detector.

Time coincidence between detectors with a coincidence window accounting for the light travel time between detectors.

Rank each coincident candidate with an incoherent SNR-like quantity

$$\rho_{\text{net}}^2 = \sum_d \rho_d^2$$

Coherent methods

Solve the full ML problem simultaneously for all detectors with a common signal.

Requires exploring a larger search space (e.g. sky location).

More correct in principle, beneficial for many detectors.

Statistical significance of candidate events

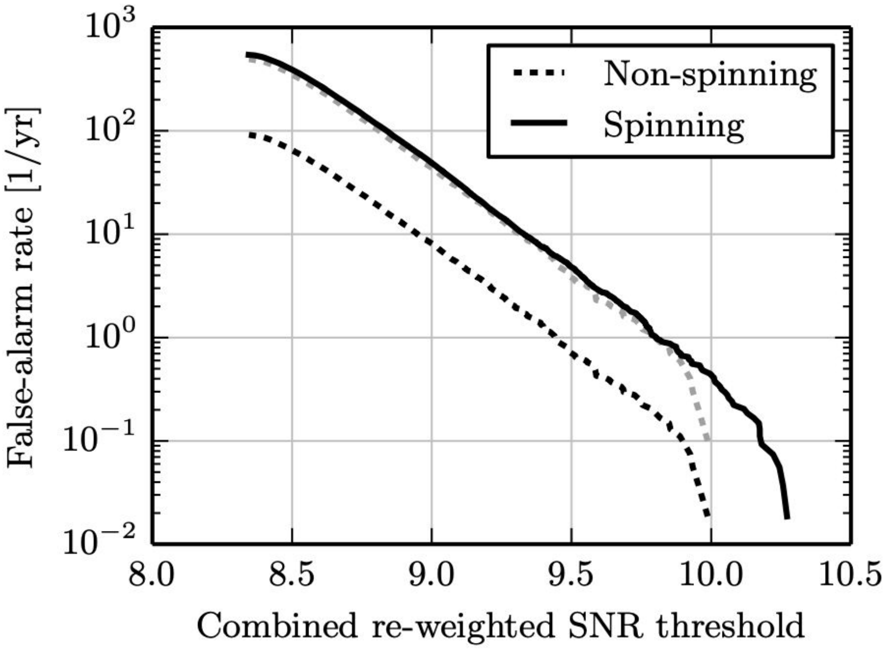
We have a list of candidate events each with a ranking value. But what does it mean?

How often does instrumental noise produce a candidate event ranked higher than what I got?
→ False-alarm rate (FAR)

Generate a “null” distribution of ranking statistic from a large sample of unphysical events:

- By time-sliding data from different GW detectors
- By extrapolating the bulk of the ranking statistic.

Obtain a map to “look up” the FAR associated with a given ranking statistic.



E.g. FAR $\lesssim$ 1/100 yr
→ Candidate is unlikely to come from noise

Statistical significance of candidate events

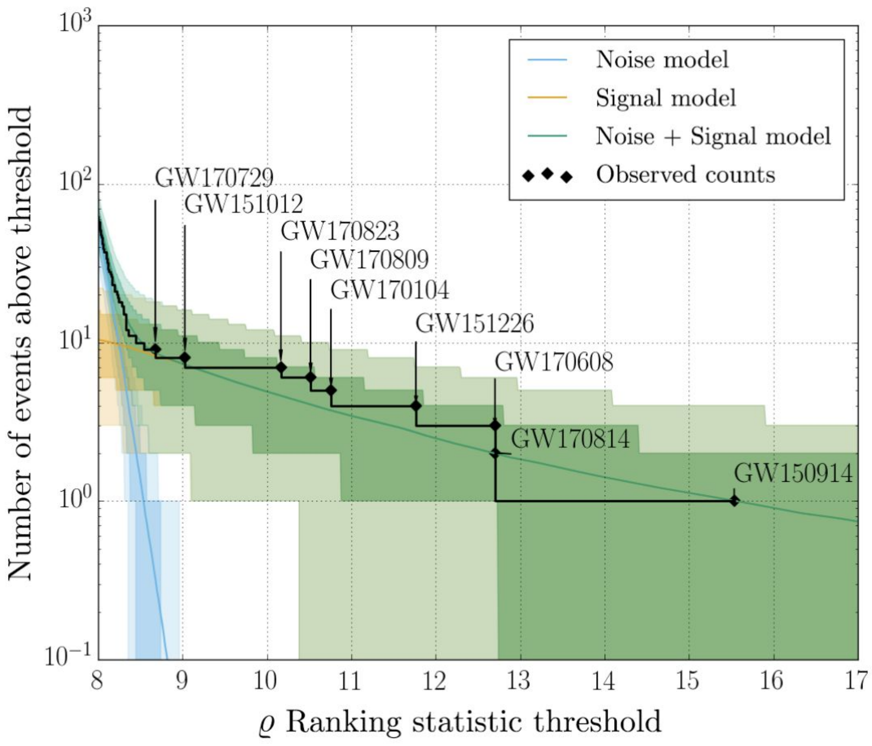
We have a list of candidate events each with a ranking value. But what does it mean?

How probable is a candidate event to be of astrophysical origin?
→ P(astro) or p_astro

Construct a model for the rate density of signal *f*(**λ**) and background *b*(**λ**) candidates over the space of candidate parameters **λ**

$$0 \leq P(\text{astro}) = \frac{f(\boldsymbol{\lambda})}{f(\boldsymbol{\lambda}) + b(\boldsymbol{\lambda})} \leq 1$$

0: candidate is certainly of terrestrial origin
0.5: ambiguous origin
1: candidate is certainly astrophysical



Modern implementations of matched-filter searches for CBCs

“Pipelines” developed by different teams

GstLAL

Time-domain matched filter

MBTA

Multiband matched filter

PyCBC

Frequency-domain matched filter

SPIIR

Time-domain fully coherent matched filter

Online (low latency)

Results available ~10 s after data acquisition.

Used to produce rapid alerts for electromagnetic followup observations.

Offline (archival)

Results available hours to weeks after data acquisition.

Used for “more careful” analyses, to compile ultimate event catalogs like GWTC.

See https://emfollow.docs.ligo.org/userguide/ for more info.

Estimation of source parameters of a candidate signal

Ok, we found a bunch of candidates from the ML search!
What can we infer about their properties?
Back to the original Bayesian problem:

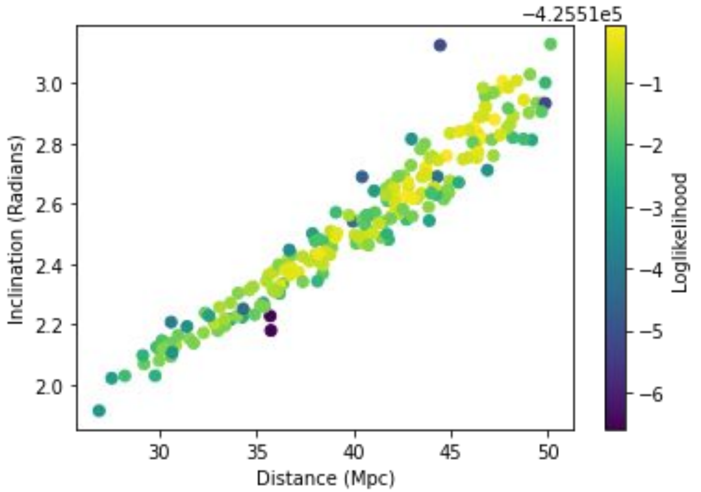
$$p(\theta|d,M) = \frac{\mathcal{L}(d|\theta,M)\,\pi(\theta|M)}{\mathcal{Z}(d|M)}$$

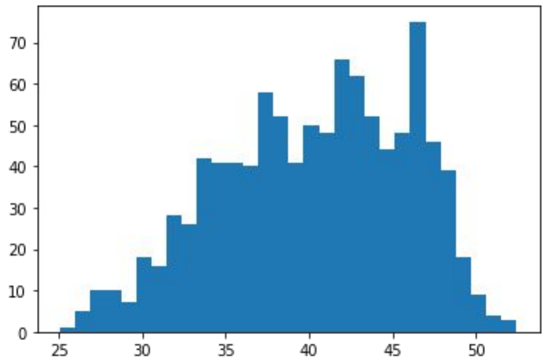
Stochastic sampling methods to "explore" the likelihood, e.g. Metropolis-Hastings algorithm

Produce a set of samples (points) in the parameter space of the model, drawn from the joint posterior density function.

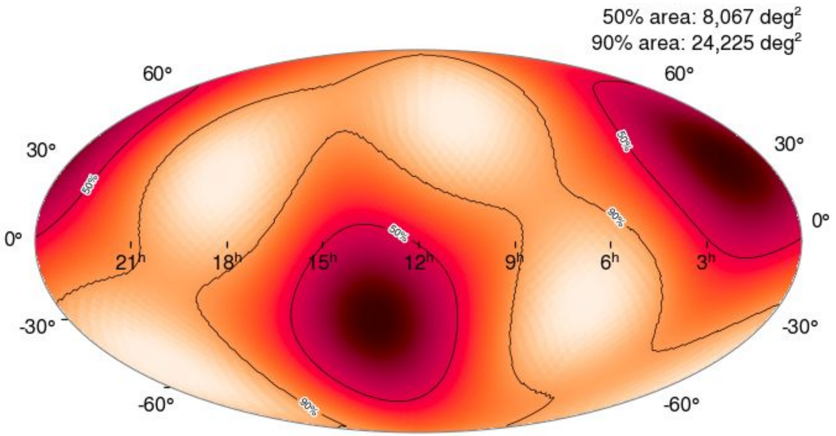
Marginal posteriors for a specific parameter can be obtained simply via histograms.

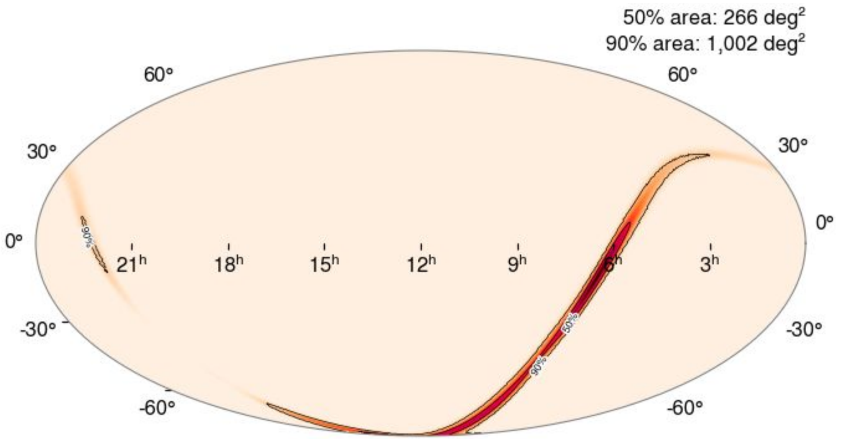
Challenges: complicated likelihoods (degeneracies), convergence (burn in), number of points.





Typical degeneracies for CBC signals: sky location



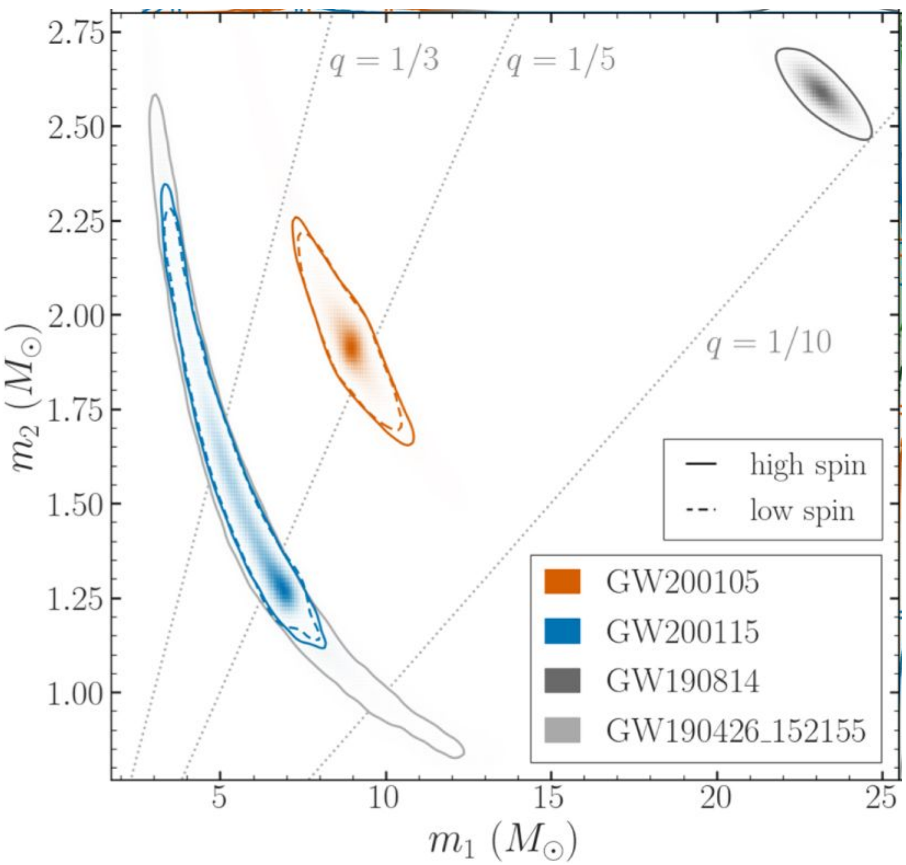


Single-interferometer observation:
the data contain no information about the parameters we are trying to infer, i.e. the problem is completely degenerate.

Then we just recover the prior distribution.

Two-interferometer observation:
the information about two parameters affects the data through a single effective parameter (the time of flight between the two interferometers in this case).

Typical degeneracies for CBC signals: component masses



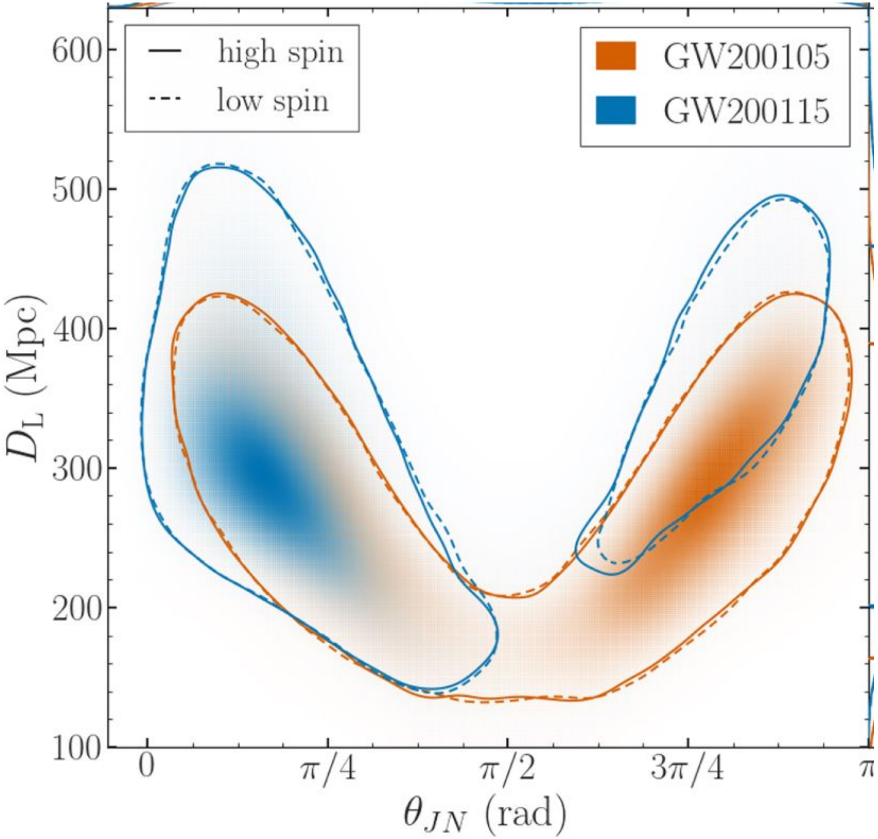
For a low-mass inspiral, the dominant effect is the chirp mass (leading parameter in the post-Newtonian expansion).

Trying to infer the component masses produces a posterior distribution that follows a line of constant chirp mass.

Solution:

- Add higher-order modes to the waveform model (add more physics to the model to break the degeneracy)
- Improve the detector sensitivity (make the higher-order modes more visible)

Typical degeneracies for CBC signals: distance/inclination



Another case of two parameters producing a very similar effect in the data:

$$D_{\text{eff}} = D\left[F_+^2\left(\frac{1+\cos^2\iota}{2}\right)^2 + F_\times^2\cos^2\iota\right]^{-1/2}$$

Solution:

- Add higher-order modes to the waveform model (add more physics to the model to break the degeneracy)
- Improve the detector sensitivity (make the higher-order modes more visible)

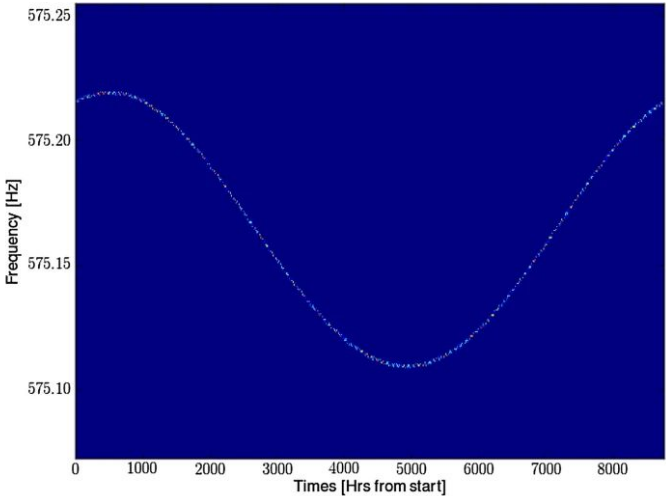
Continuous GW signal observed by an interferometer

$$\Phi(\tau) \approx \Phi_0 + 2\pi\left[f_s(\tau-\tau_0) + \frac{1}{2}\dot{f}_s(\tau-\tau_0)^2 + \frac{1}{6}\ddot{f}_s(\tau-\tau_0)^3\right]$$

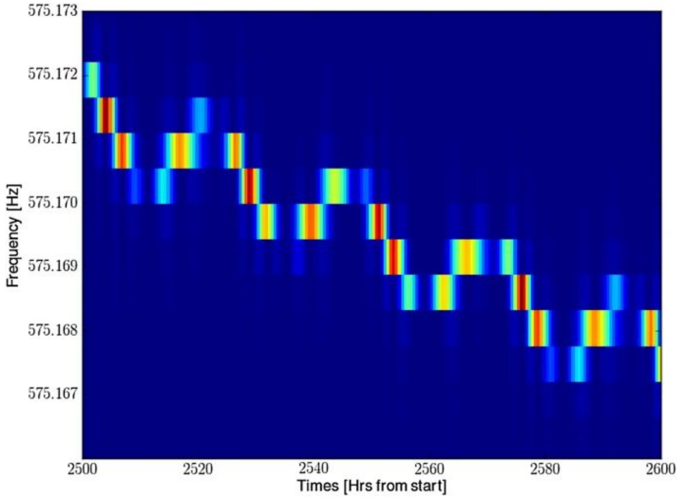
$$\tau(t) \equiv t + \delta t = t - \frac{\vec{r}_d \cdot \hat{k}}{c} + \Delta_{E\odot} + \Delta_{S\odot}$$

$$h(t) = F_+(t,\psi)\, h_0 \frac{1+\cos^2(\iota)}{2} \cos(\Phi(t)) + F_\times(t,\psi)\, h_0 \cos(\iota) \sin(\Phi(t))$$

Doppler modulation due to Earth revolution



Doppler modulation due to Earth rotation



More details: Riles, *Searches for continuous-wave gravitational radiation*, LRR 2023

Exercises

1. PyCBC tutorials 1-3 from “Gravitational-wave Data Analysis” and tutorial 0 from “Inference”.
2. What does the transverse property of GWs mean in terms of how a detector responds to a GW?
3. Explicitly calculate the antenna pattern functions in the low-frequency limit, assuming a coordinate system where the x and y axes are defined by the arms of the interferometer.
4. Calculate the response of an interferometer relaxing the low-frequency approximation, and assuming a monochromatic GW. For simplicity, only consider a source located directly above the interferometer. How does the response vary as a function of frequency?
5. Assume a single computer CPU core takes 10 ms to calculate the SNR for a single CBC template and 256 s of data. Assume we have a bank of 5×10^5 templates. What do we need to analyze one year of GW observations in one month?
6. Describe why a template bank to search for CBC signals including masses and spins will produce a larger FAR (at a fixed value of the candidate event’s ranking statistic) than a similar bank that only includes the masses.
7. Consider a CBC candidate event and its FAR and P(astro). Do you expect a small FAR to always be associated with a P(astro) value close to 1? Explain.

Backup

Multipole expansion of gravitational radiation

$$h_+ - \mathrm{i} h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m} \, Y^{\ell m}_{(-2)}(\theta, \phi)$$

Early-warning detection of a long inspiral