Causal Sets Theory

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Taketani's "Three-Stage Theory"

1. **Phenomenological stage**
   A substance presents itself, as it is, in a group of phenomena
   Ex : Resonance of hadrons
2. **Substantialistic stage**
   Investigation of the structure of the substance, distinction from the phenomena
   Ex : Discovery of quarks
3. **Essentialistic stage**
   Dynamics is understood : interactions and laws of motion are clarified
   Ex : Formulation of QCD

Quantum gravity needs to skip the first stage.

Causal Sets Theory

Space-time $(\mathcal{M}, g_{\mu\nu})$

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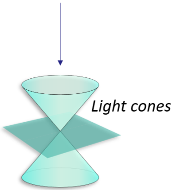
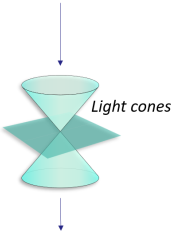


Partially ordered set $(\mathcal{M}, \prec)$

$$\forall x, y, z \in \mathcal{M}, \begin{cases} x \prec x \text{ (Reflexivity)} \\ x \prec y \text{ and } y \prec x \implies x = y \text{ (Acyclicity)} \\ x \prec y \text{ and } y \prec z \implies x \prec z \text{ (Transitivity)} \end{cases}$$

Causal Sets Theory

Space-time $(\mathcal{M}, g_{\mu\nu})$

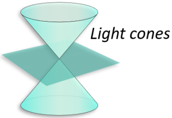


Partially ordered set

$(\mathcal{M}, \prec)$

Causal Sets Theory

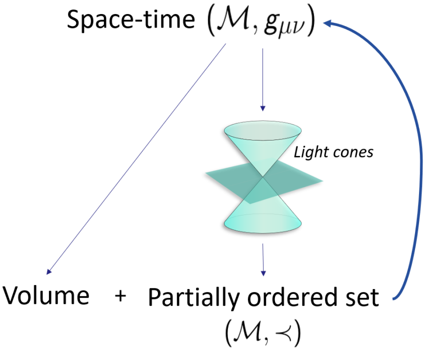
Space-time $(\mathcal{M}, g_{\mu\nu}) \neq \left(\mathcal{M}, \frac{g_{\mu\nu}}{|\det g|^{1/n}}\right)$



Partially ordered set

$(\mathcal{M}, \prec)$

Causal Sets Theory



Causal Sets Theory

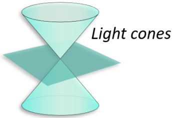
Causal set $(c, \prec)$

$(c, \prec)$ is a partially ordered set

$\forall x, y \in c, |\{z \in c / x \prec z \prec y\}| < \infty$

Causal Sets Theory

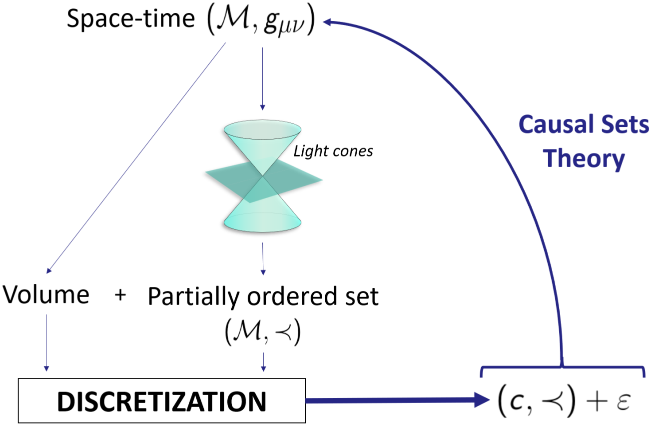
Space-time $(\mathcal{M}, g_{\mu\nu})$



Volume + Partially ordered set $(\mathcal{M}, \prec)$

DISCRETIZATION → $(c, \prec) + \varepsilon$

Causal Sets Theory



HKMML theorem :

Let $(\mathcal{M}_1, g_1)$ *and* $(\mathcal{M}_2, g_2)$ *be d-dimensional Lorentzian manifolds with* $d > 2$ *such that the chronological (or timelike) past and future of each point in space-time is unique.*

If there exists a causal bijection between $(\mathcal{M}_1, g_1)$ *and* $(\mathcal{M}_2, g_2)$.

(If $\exists f : (\mathcal{M}_1, \prec_1) \to (\mathcal{M}_2, \prec_2) | \, \forall x, y \in \mathcal{M}_1, \; x \prec_1 y \Leftrightarrow f(x) \prec_2 f(y)$)

Then $(\mathcal{M}_1, g_1)$ *and* $(\mathcal{M}_2, g_2)$ *are conformally isometric.*

This means that the causal structure determine not just one space-time, but the full conformal equivalence class of it.

Volume element information

If we know only $(c_N, \prec)$, with c_N a causal set of N elements, we cannot recover a volume of space-time out of it.

We need the information contained in ρ (or ε) such that :

$$\frac{N}{\rho} = \int_V \sqrt{\mathrm{det}\, g}\, \mathrm{d}^d x.$$

Sorkin's slogan

$$\underbrace{\textit{Causal order}}_{\text{Proto}-\text{causality}} + \underbrace{\textit{Spacetime Volume}}_{\text{Number}} = \textit{Geometry}$$

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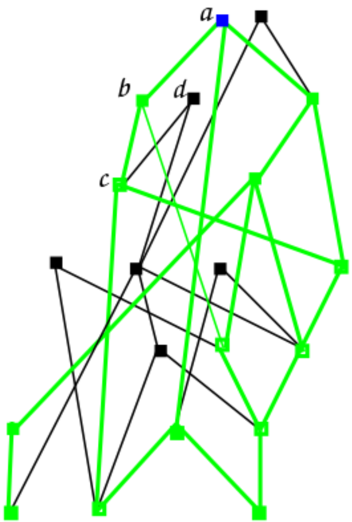
Formalism

A causal set could be represented by a Hasse diagram.



- $c \prec b \prec a$
- $\{b, a\}$ is a link or a 2-chain
- $\{c, b, a\}$ is a 3-chain
- $\{d, a\}$ is a 2-antichain
- $\mathrm{Past}(a)$ is the set of all elements e such that $e \prec a$ (the green set)
- This is a **Past-finite** causal set c : $\forall e \in c, |\mathrm{Past}(e)| < \infty$.

Faithful embedding map $\Phi : c \to (\mathcal{M}, g)$

- $x \prec_c y \Leftrightarrow \Phi(x) \prec_M \Phi(y)$
- Embedded points are distributed uniformly with unit density.
- The characteristic length over which the continuous geometry varies $\gg$ mean spacing between embedded points

Poissonian selection of random positions $(\mathcal{M}, g) \to c$

- The probability of finding n elements in a spacetime region of volume V is given by :

$$P_V(n) = \frac{(\rho V)^n e^{-\rho V}}{n!}.$$

- $\langle N \rangle = \rho V$
- $\Delta N = \sqrt{\rho V}$

Fundamental conjecture

If a causal set c could have arisen from a sprinkling process into $(\mathcal{M}, g)$ "with relatively high probability" :

$$(c, \prec) + \varepsilon \;\simeq\; (\mathcal{M}, g) \quad \left(\Leftrightarrow \;\; (c, \prec) + \varepsilon \; \underset{\textit{Embedding}}{\overset{\textit{Sprinkling}}{\leftrightarrows}} \; (\mathcal{M}, g) \right)$$

<u>The Hauptvermutung of CST :</u>

$$(c, \prec) \; + \; \varepsilon \;\; \simeq \;\; (\mathcal{M}, g) \;\; \& \;\; (c, \prec) \; + \; \varepsilon \;\; \simeq \;\; (\mathcal{M}', g')$$
$$\Longrightarrow \; (\mathcal{M}, g) \;\; \simeq \;\; (\mathcal{M}', g')$$

then $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ differ only at scale smaller than ρ.

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Quantum partition function

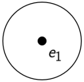
We interpret $Z = \int \mathcal{D}g e^{iS[g]}$ as $Z_\Omega \equiv \sum_{c \in \Omega} e^{i\frac{S(c)}{\hbar}}$ where Ω is the space of all past-finite causal set that we need to construct.

The standard action for causal set is the Benincasa-Dowker action defined as :

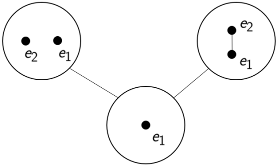
$$S(c_n^i) = \frac{4}{\sqrt{6}}\left[n - N_0^{(n,i)} + 9N_1^{(n,i)} - 16N_2^{(n,i)} + 8N_3^{(n,i)}\right].$$

Where $N_k^{(n,i)}$ is the total number of k-element order interval in the causal set c_n^i. It gives the Einstein-Hilbert action in the continuum limit.

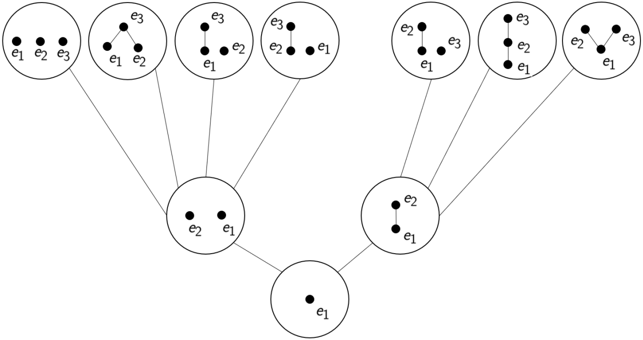
Sequential growth



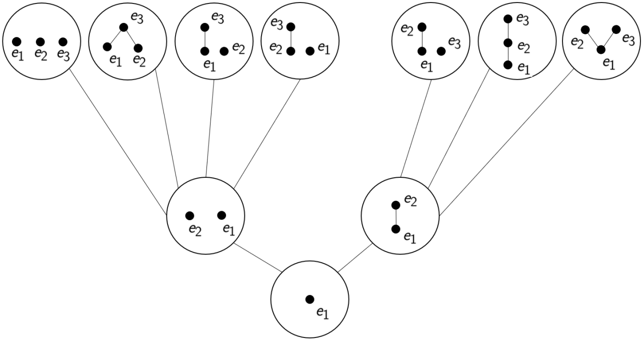
Sequential growth



Sequential growth

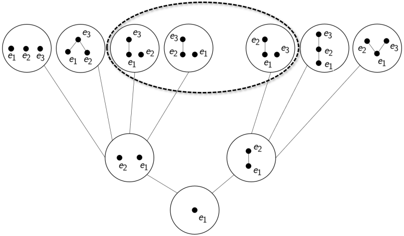


Sequential growth



As $n \to \infty$, this growth process generates the sample space Ω of countable *labelled* past finite causal sets.

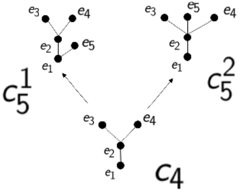
Sequential growth

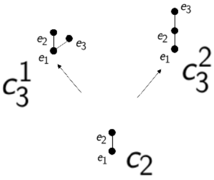


Physical requirements of the dynamics :

- Markov sum rule
- Internal temporality
- Discrete general covariance
- Bell causality

Bell Causality





$$\frac{\alpha(c_4 \to c_5^1)}{\alpha(c_4 \to c_5^2)} = \frac{\alpha(c_2 \to c_3^1)}{\alpha(c_2 \to c_3^2)}$$

Causal Sets Theory

The continuum may be a mathematical construct which approximates an underlying physical discreteness.

$$(\mathcal{M}, g) \underset{\textit{Sprinkling}}{\overset{\textit{Embedding}}{\leftrightharpoons}} (c, \prec) + \varepsilon$$

Causal set approach :

- ✓ cures divergences in QFT.
- ✓ cures curvature singularity in GR.
- ✓ cures infinite entanglement entropy of black holes.
- ✓ measures metric at sub-Planckian scale.
- ✓ is compatible with Lorentz invariance.
- ✓ predicts the right magnitude of Λ.
- ? could give fruitful formulation of quantum fields dynamics.
- ? could solve the Hard Problem of Consciousness as a birth process happening in the brain.

Thank you for your attention

Additional slides

General measure space

Let Ω be a non-empty set. $\mathfrak{A} \subseteq \mathcal{P}(\Omega)$ is an *algebra* if :

(i) $\Omega \in \mathfrak{A}$
(ii) $A \in \mathfrak{A} \implies A^c \in \mathfrak{A}$
(iii) $A_1, A_2, ..., A_n \in \mathfrak{A} \implies \bigcup_{k=1}^{n} A_k \in \mathfrak{A}$
$\mathfrak{A}$ is closed under **finite** unions

$\mathfrak{A}$ is a σ-*algebra* if is also closed under **countable** unions.
$A \in \mathfrak{A}$ is a *measurable set*.
$(\Omega, \mathfrak{A})$ is a *measurable space*.
A *measure* on a measurable space is a map satisfying :

(i) $\mu(\varnothing) = 0$
(ii) $A_1, ..., A_n \in \mathfrak{A}$ pairwise disjoint $\implies \mu(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} \mu(A_k)$

$(\Omega, \mathfrak{A}, \mu)$ is a *measure space* $\rightarrow$ gives the CST dynamics.

CHK theorem

Caratheodary-Hahn-Kluvnek (CHK) theorem

If μ is bounded weakly countably additive vector measure over $\mathfrak{A}$:
μ is strongly additive $\Leftrightarrow \exists!$ countably additive extension of μ to $\mathfrak{S}_{\mathfrak{A}}$

- *bounded*

$$\sup_{x^*}\{\sup_\pi \sum_{\alpha_i \in \pi} ||x^*(\mu(\alpha_i))||; x^* \in \mathcal{H}^*, ||x^*|| \leq 1\} < \infty$$

  where the second supremum is taken over all partitions π of Ω

- *weakly countably additive*
  For every $x^* \in \mathcal{H}^*$, $x^*(\mu)$ countably additive.

$$\Leftrightarrow x^*\left(\mu\left(\bigcup_i \alpha_i\right)\right) = \sum_i x^*(\mu(\alpha_i))$$

  for infinite sequence $\{\alpha_n\}$ of pairwise disjoint element of $\mathfrak{A}$

CHK theorem

Caratheodary-Hahn-Kluvnek (CHK) theorem

If μ is bounded weakly countably additive vector measure over $\mathfrak{A}$:
μ is strongly additive $\Leftrightarrow \exists!$ countably additive extension of μ to $\mathfrak{S}_{\mathfrak{A}}$

- *vector measure*

DEFINITION 1. A function F from a field $\mathscr{F}$ of subsets of a set Ω to a Banach space X is called a *finitely additive vector measure*, or simply a *vector measure*, if whenever E_1 and E_2 are disjoint members of $\mathscr{F}$ then $F(E_1 \cup E_2) = F(E_1) + F(E_2)$.

- *strongly additive*

$$||\sum_{n=1}^{\infty} \mu(\alpha_n)|| < \infty$$

for every sequence $\{\alpha_n\}$ of pairwise disjoint element of $\mathfrak{A}$

CHK theorem simplified

Caratheodary-Hahn-Kluvnek (CHK) theorem

If μ is bounded weakly countably additive vector measure over $\mathfrak{A}$:
μ is strongly additive $\Leftrightarrow \exists!$ countably additive extension of μ to $\mathfrak{S}_{\mathfrak{A}}$

Variation of μ, $\forall \alpha \in \mathfrak{A}$, where π is a finite partition of α :

$$|\mu|(\alpha) \equiv \sup_\pi \sum_{\alpha_i \in \pi} ||\mu(\alpha_i)||$$

Measure μ is *of bounded variation* if $|\mu|(\Omega) < \infty$

Theorem : μ of bounded variation $\implies$ μ strongly additive
Theorem : μ strongly additive $\implies$ μ bounded

Simplified CHK theorem

If μ is a weakly countably additive vector measure over $\mathfrak{A}$:
μ of bounded variation $\Rightarrow \exists!$ countably additive extension to $\mathfrak{S}_{\mathfrak{A}}$

Need for extension

cylinder set : $\mathrm{cyl}(c_n^i) \equiv \{c \in \Omega | c|_n = c_n^i\} = \bigcup_{j(i)} \mathrm{cyl}(c_{n+1}^{j(i)}) \subset \Omega$

Nesting property for $m > n$:

$$\mathrm{cyl}(c_m^i) \cap \mathrm{cyl}(c_n^j) \neq 0 \implies \mathrm{cyl}(c_m^i) \subset \mathrm{cyl}(c_n^j)$$

$\mathrm{cyl}(c_n^i) \subset \Omega$. $\mathfrak{A}$ is generated from the cylinder sets via finite unions, intersections and set differences.

$$\mu(c_n^i) \equiv \mu(\mathrm{cyl}(c_n^i))$$

The event algebra $\mathfrak{A}$ does not suffice to be able to define covariant observables like the originary event : $\alpha_{\mathrm{orig}} = \left(\bigcup_{n>1} \bigcup_{i \in \mathcal{I}_n} \mathrm{cyl}(c_n^i)\right)^c$.
$\implies$ One needs to include countable set operations on $\mathfrak{A}$.

$$\text{Covariant events} \in \mathfrak{S}_{\mathfrak{A}} / \sim$$

$c \sim c' \Leftrightarrow c, c'$ are *order-isomorphic* to each other

Transitive percolation

Generate a random causal set by the following algorithm :

1. Start with n elements labeled $0, 1, 2, \cdots, n-1$
   ($n = \infty$ not excluded.)
2. With a fixed probability p, introduce a relation between every pair of points labeled i and j, where $i < j$.
3. Form the transitive closure of these relations (e.g. if $2 \prec 5$ and $5 \prec 8$ then enforce that $2 \prec 8$.)

The transition probability α_n from c_n^i to a specified child $c_{n+1}^{j(i)}$:

$$\alpha_n^{(S)} = p^m (1-p)^{n-\varpi}$$

m = number of maximal elements in the past S of the new element
ϖ = size of the past S of the new element

PHYSICAL REQUIREMENTS OF THE DYNAMICS : ✓

Physical requirement for transitive percolation

- ✓ Internal temporality

  Build into our definition of the growth process

- ✓ Discrete general covariance

  Net probability of a given c_n^i in "manifestly covariant form" is $P(c_n^i) = Wp^L q^{\binom{n}{2}-R}$ where L is the number of links in c_n^i, R the number of relations, and W the number of (natural) labelings of c_n^i.

- ✓ Bell causality

  Consider two different children, one with $(m, \varpi) = (m_1, \varpi_1)$ and the other with $(m, \varpi) = (m_2, \varpi_2)$

$$\frac{\alpha_n^{(m_1,\varpi_1)}}{\alpha_n^{(m_2,\varpi_2)}} = \frac{\alpha_{n'}^{(m_1,\varpi_1)}}{\alpha_{n'}^{(m_2,\varpi_2)}} \Leftrightarrow \frac{p^{m_1} q^{n-\varpi_1}}{p^{m_2} q^{n-\varpi_2}} = \frac{p^{m_1} q^{n'-\varpi_1}}{p^{m_2} q^{n'-\varpi_2}}$$

  where $n' \leq n$ is the cardinality of the union of the precursor sets of the two transitions.

- ✓ Markov sum rule

  Trivial in a well-defined probabilistic procedure.

General transition probability

Parameters of the growth = q_n = probabilities to add a completely disconnected element at stage n.

$$\alpha_n^{(S)} = \alpha_n^{(m,\varpi)} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{q_n}{q_{\varpi-k}} = \frac{\sum_{l=m}^{\varpi} \binom{\varpi-m}{\varpi-l} t_l}{\sum_{j=0}^{n} \binom{n}{j} t_j}$$

m = number of maximal elements in the past S of the new element
ϖ = size of the past S of the new element

Alternative parameters : $t_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{q_k}$.

PHYSICAL REQUIREMENTS OF THE DYNAMICS : ✓

Physical requirement for general transition probability

✓ Internal temporality

Build into our definition of the growth process

✓ Discrete general covariance

Probability of a labeled causal set $\tilde{c}_n^i$: $P(\tilde{c}_n^i) = \prod_{i=0}^{N-1} \alpha(i, \varpi_i, m_i)$

This is a product over all elements $x \in c_n^i$ of poset invariant quantities that depends only on the structure of $past(x)$.

✓ Bell causality

$$\frac{\alpha_n^{(m_1,\varpi_1)}}{\alpha_n^{(m_2,\varpi_2)}} = \frac{\alpha_{n'}^{(m_1,\varpi_1)}}{\alpha_{n'}^{(m_2,\varpi_2)}} \Leftrightarrow \frac{\sum_{k=0}^{m_1}(-1)^k \binom{m_1}{k} \frac{q_n}{q_{\varpi_1 - k}}}{\sum_{k=0}^{m_2}(-1)^k \binom{m_2}{k} \frac{q_n}{q_{\varpi_2 - k}}} = \frac{\sum_{k=0}^{m_1}(-1)^k \binom{m_1}{k} \frac{q_{n'}}{q_{\varpi_1 - k}}}{\sum_{k=0}^{m_2}(-1)^k \binom{m_2}{k} \frac{q_{n'}}{q_{\varpi_2 - k}}}$$

The ratios depends only on precursor set structure.

✓ Markov sum rule

Impose a constraint :

$$\sum_{i=0}^{N-1} \alpha(i, \varpi_i, m_i) = 1 \Leftrightarrow \sum_S \sum_l t_l \binom{|S| - m(S)}{l - m(S)} = \sum_j t_j \binom{n}{j}$$

$$\Leftrightarrow \forall l, \sum_S \binom{|S| - m(S)}{l - m(S)} = \binom{n}{l}$$

Corresponding measure space

The dynamics is a specification of the measure over $\mathfrak{A}$.

$$\mu(\mathrm{cyl}(c_n^i)) \equiv P(c_n^i) = \prod_{i=1}^{n} \alpha_i$$

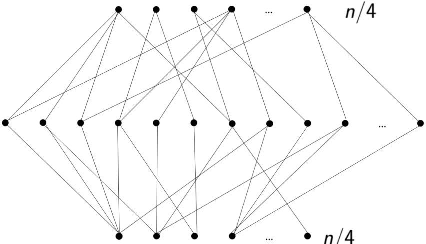
$$\mu : \mathfrak{A} \to [0,1], \;\; \mu(\Omega) = \mu(\mathrm{cyl}(c_1^1)) = 1$$

$\forall \alpha \in \mathfrak{A}$, there exists a smallest $n < \infty$ and a subset $S \subset \{1, 2, ..., |\Omega_n|\}$ such that $\alpha = \bigcup_{k \in S} \mathrm{cyl}(c_n^k)$.

$$\mu(\alpha) = \sum_{k \in S} P(c_n^k)$$

μ scalar real measure $\implies$ μ extends to $\mathfrak{S}_{\mathfrak{A}}$.

Entropy catastrophe : KR posets in CSG



Lemma (Brightwell, Dowker, Garcia, Henson, Sorkin) :
In the CSG dynamics with $t_k \neq 0$ for some $k > 1$, a causet containing an infinite level almost surely does not occur.

$$\sum_{n=|S|+1}^{\infty} \alpha_n^{(S)} = \sum_{l=m}^{\varpi} \binom{\varpi - m}{\varpi - l} t_l \sum_{n=|S|+1}^{\infty} \frac{1}{\sum_{j=0}^{n} \binom{n}{j} t_j} < \infty$$

Straightforward generalisation

Straightforward generalisation of classical sequential growth, the transition amplitudes are :

$$A_n^{(S)} = A_n^{(m,\varpi)} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{q_n}{q_{\varpi-k}} = \frac{\sum_{l=m}^{\varpi} \binom{\varpi-m}{\varpi-l} t_l}{\sum_{j=0}^{n} \binom{n}{j} t_j}$$

with now $q_n, t_n \in \mathbb{C}$.

The measure of $\mathrm{cyl}(c_n^i) \in \mathfrak{A}$ yields :

$$|c_n^i\rangle \equiv \mu(\mathrm{cyl}(c_n^i)) \propto \prod_{m \text{ in branch}} A(c_m \to c_{m+1}) \in \mathbb{C}$$

The product is over transition along the nodes from c_1^1 to c_n^i.

PHYSICAL REQUIREMENTS OF THE DYNAMICS : ✓

Physical requirement on $\mathbb{CSG}$

✓ Internal temporality

Build into our definition of the growth process

✓ Discrete general covariance

$$\Longrightarrow \; |c_n^i >= |c_n^j > \text{ whenever } c_n^i \sim c_n^j$$

✓ Bell causality :

$$\frac{A_n^{(m_1,\varpi_1)}}{A_n^{(m_2,\varpi_2)}} = \frac{A_{n'}^{(m_1,\varpi_1)}}{A_{n'}^{(m_2,\varpi_2)}} \Leftrightarrow \frac{\sum_{k=0}^{m_1}(-1)^k\binom{m_1}{k}\frac{1}{q_{\varpi_1-k}}}{\sum_{k=0}^{m_2}(-1)^k\binom{m_2}{k}\frac{1}{q_{\varpi_2-k}}} = \frac{\sum_{k=0}^{m_1}(-1)^k\binom{m_1}{k}\frac{1}{q_{\varpi_1-k}}}{\sum_{k=0}^{m_2}(-1)^k\binom{m_2}{k}\frac{1}{q_{\varpi_2-k}}}$$

✓ Markov sum rule

$$|c_{n+1}^{j(i)} >= \hat{O}(c_n^j \to c_{n+1}^{j(i)})|c_n^i > \;\; \& \; \mathrm{cyl}(c_n^i) = \bigcup_{j(i)} \mathrm{cyl}(c_{n+1}^{j(i)})$$

$$\mu(\mathrm{cyl}(c_n^i)) = \mu\Big(\bigcup_{j(i)} \mathrm{cyl}(c_{n+1}^{j(i)})\Big) = \sum_{j(i)} \mu(\mathrm{cyl}(c_{n+1}^{j(i)})) = \sum_{j(i)} \hat{O}(c_n^j \to c_{n+1}^{j(i)})|c_n^i >$$

$$\Longrightarrow \sum_{j(i)} \hat{O}(c_n^j \to c_{n+1}^{j(i)}) = \mathbb{1}$$

CHK extension theorem for complex measure space

Simplified CHK theorem

If μ is a weakly countably additive vector measure over $\mathfrak{A}$:
μ of bounded variation $\Rightarrow \exists !$ countably additive extension to $\mathfrak{S}_{\mathfrak{A}}$

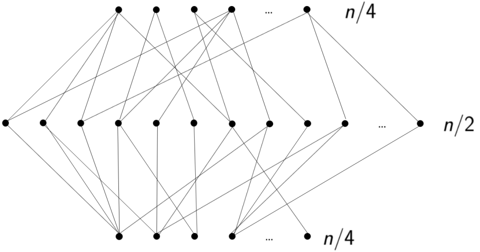
Define $\zeta_n^i \geq 0$ such as $\sum_{j(i)} |A(c_n^i \to c_{n+1}^{j(i)})| = 1 + \zeta_n^i \geq 1$ we have :

$$\zeta_n^{\mathrm{max}} \equiv \max_{c_n^i \in \Omega_n} \zeta_n^i \overset{!}{=} \frac{\sum_{k=0}^{n} \binom{n}{k} |t_k|}{|\sum_{k=0}^{n} \binom{n}{k} t_k|} - 1 \overset{!}{=} \zeta_n^{\mathrm{a}}$$

$$\zeta_n^{\mathrm{min}} \equiv \min_{c_n^i \in \Omega_n} \zeta_n^i \overset{!}{=} \sum_{\varpi=1}^{n} \frac{|\sum_{k=1}^{\varpi-1} \binom{\varpi-1}{k-1} t_k|}{|\sum_{k=0}^{n} \binom{n}{k} t_k|} + \frac{|t_0|}{|\sum_{k=0}^{n} \binom{n}{k} t_k|} - 1 \overset{!}{=} \zeta_n^{\mathrm{c}}$$

Theorem (Surya, Zalel) :
μ is of bounded variation if $\sum_{n=1}^{\infty} \zeta_n^{\mathrm{max}}$ converges.
μ is not of bounded variation if $\sum_{n=1}^{\infty} \zeta_n^{\mathrm{min}}$ diverges.

Entropy catastrophe : KR posets in $\mathbb{C}$SG



In complex sequential growth :

$$\sum_{n=1}^{\infty} \zeta_n^{\max} < \infty \implies \sum_{n=|S|+1}^{\infty} \alpha_n^{(S)} < \infty$$

Personal work in progress.

Quantum sequential growth

Event $=$ set of histories : $E = \{\gamma_1, \gamma_2, ...\}$

$$\mu(E) = \mu(\gamma_1) + \mu(\gamma_2) + ... + I(\gamma_1, \gamma_2) + I(\gamma_1, \gamma_3) + I(\gamma_2, \gamma_3) + ... = D(E, E)$$

$I(x, y) = D(x, y) + D(y, x)$ are *interferences terms*.

$D : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ is the *decoherence functional* defined with :

- *Hermiticity* : $\forall \alpha, \beta \in \mathfrak{A}, D(\alpha, \beta) = D(\beta, \alpha)^*$
- *Linearity* :
  $\forall \alpha, \beta, \delta \in \mathfrak{A} / \beta \cap \delta = \varnothing, D(\alpha, \beta \cup \delta) = D(\alpha, \beta) + D(\alpha, \delta)$
- *Normalisation* : $D(\Omega, \Omega) = 1$
- *Strong positivity* : for any $\{\alpha_i\}$ finite collection in $\mathfrak{A}$:
  $M_{ij} = D(\alpha_i, \alpha_j)$ has non-negative eigenvalues

Extension of the quantum measure

$$\exists \alpha, \beta \in \mathfrak{A}, \alpha \cap \beta = \varnothing / \mu(\alpha \cup \beta) \neq \mu(\alpha) + \mu(\beta)$$

$$\mu(\alpha \cup \beta \cup \delta) = \mu(\alpha \cup \beta) + \mu(\alpha \cup \delta) + \mu(\beta \cup \delta) - \mu(\alpha) - \mu(\beta) - \mu(\delta)$$

GNS Construction $\implies$ *Quantum vector measure* $\mu_v : \mathfrak{A} \to \mathcal{H}$.

$$\forall \alpha, \beta \in \mathfrak{A}, \alpha \cap \beta = \varnothing / \mu(\alpha \cup \beta) = \mu(\alpha) + \mu(\beta)$$

PHYSICAL REQUIREMENTS OF THE DYNAMICS : ✘ Bell Causality ?

Physical requirement on QSG

- ✓ Internal temporality
  Build into our definition of the growth process
- ✓ Discrete general covariance
  $\implies |c_n^i>=|c_n^j>$ whenever $c_n^i \sim c_n^j$
- Bell causality : ???
- ✓ Markov sum rule

$$|c_{n+1}^{j(i)}>=\hat{O}(c_n^j \to c_{n+1}^{j(i)})|c_n^i> \ \& \ cyl(c_n^i) = \bigcup_{j(i)} \mathrm{cyl}(c_{n+1}^{j(i)})$$

$$\mu(\mathrm{cyl}(c_n^i)) = \mu\Big(\bigcup_{j(i)} \mathrm{cyl}(c_{n+1}^{j(i)})\Big) = \sum_{j(i)} \mu(\mathrm{cyl}(c_{n+1}^{j(i)})) = \sum_{j(i)} \hat{O}(c_n^j \to c_{n+1}^{j(i)})|c_n^i>$$

$$\implies \sum_{j(i)} \hat{O}(c_n^j \to c_{n+1}^{j(i)}) = \mathbb{1}$$

Extension of complex percolation $\mathbb{CP}$

Complex percolation is a natural quantum generalisation of transitive percolation in which real probabilities are replaced by complex amplitudes.

$$P(c_n^i) = Wp^L q^{\binom{n}{2}-R} \to A(c_n^i)$$

Lemma : The quantum vector measure of complex percolation is not of bounded variation when the parameter p is not real.

Quantum sequential growth

The construction of a Hilbert space from the event algebra $\mathfrak{A}$ and the decoherence functional D implies that the quantum measure is equivalent to a Hilbert space valued measure which is additive, unlike the quantum measure :

Vector pre-measure $\eta_v : \mathfrak{A} \to \mathcal{B}/$

$$\eta_v\Big(\bigcup_{n=1}^{N} \alpha_n\Big) = \sum_{n=1}^{N} \eta_v(\alpha_v)$$

Vector measure $\eta_v : \mathfrak{S} \to \mathcal{B}/$

$$\bar{\eta}_v\Big(\bigcup_{n=1}^{\infty} \alpha_n\Big) = \sum_{n=1}^{\infty} \bar{\eta}_v(\alpha_v)$$

Quantum sequential growth

Quantum Measure Theory is a formulation of quantum theory based on the path integral.

Systems described by a *quantum measure space* $(\Omega, \mathfrak{A}, \mu)$.

Ω : *sample space* of histories γ or spacetime configurations.
$\mathfrak{A}$: *event algebra* or set of proposition about the system.
μ : *quantum pre-measure* given by the path integral, $\mu : \mathfrak{A} \to \mathbb{R}^+$.

$$\exists \alpha, \beta \in \mathfrak{A}, \alpha \cap \beta = \varnothing / \mu(\alpha \cup \beta) \neq \mu(\alpha) + \mu(\beta)$$

$$\mu(\alpha \cup \beta \cup \delta) = \mu(\alpha \cup \beta) + \mu(\alpha \cup \delta) + \mu(\beta \cup \delta) - \mu(\alpha) - \mu(\beta) - \mu(\delta)$$

To make predictions about infinite-time events
$\implies$ extension of the quantum pre-measure to a σ-algebra.

Construction of the inner product vector space $(\mathcal{H}_1, +, \cdot, \langle \cdot, \cdot \rangle_1)$

$\mathcal{H}_1 \equiv$ set of all complex-valued functions on $\mathfrak{A}$ which are non-zero only on a finite number of events.

$$\forall u, v \in \mathcal{H}_1, \alpha \in \mathfrak{A}, \quad (u+v)(\alpha) \equiv u(\alpha) + v(\alpha)$$

$$\forall u \in \mathcal{H}_1, \lambda \in \mathbb{C}, \quad (\lambda \cdot u)(\alpha) \equiv \lambda u(\alpha)$$

$$\forall u, v \in \mathcal{H}_1, \quad \langle u, v \rangle_1 \equiv \sum_{\alpha \in \mathfrak{A}} \sum_{\beta \in \mathfrak{A}} u^*((\alpha) v(\beta) D(\alpha, \beta)$$

Problem : The inner product is degenerate

Construction of the Hilbert space of histories $(\mathcal{H}_2, +, \cdot, \langle \cdot, \cdot \rangle_2)$

$$\{u_n\} \sim \{v_n\} \Leftrightarrow \lim_{n \to +\infty} ||u_n - v_n||_1 = 0$$

$$\mathcal{H}_2 \equiv \mathcal{H}_1 / \sim$$

$\sim$ equivalence class of a Cauchy sequence $\{u_n\}$ is denoted by $[u_n]$

$$\forall [u_n], [v_n] \in \mathcal{H}_1, \alpha \in \mathfrak{A}, \quad [u_n] + [v_n] \equiv [u_n + v_n]$$

$$\forall [u_n] \in \mathcal{H}_1, \lambda \in \mathbb{C}, \quad \lambda \cdot [u_n] \equiv [\lambda u_n]$$

$$\forall [u_n], [v_n] \in \mathcal{H}_1, \quad \langle [u_n], [v_n] \rangle_2 \equiv \lim_{n \to +\infty} \langle u_n, v_n \rangle_1$$

Quantum vector measure

$$\mu_v(\alpha) \equiv [\chi_\alpha] \in \mathcal{H} \text{ with the indicator } \chi_\alpha(\beta) = \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{if } \beta \neq \alpha. \end{cases}$$

If $\mathcal{H} = \mathbb{C}^n$, and $\mu_v^{(i)} : \mathfrak{A} \to \mathbb{C}$, for $i = 1, \dots, n$ are the components of μ_v in an orthonormal basis :

$$\mu_v \text{ is of bounded variation } \Leftrightarrow \mu_v^{(i)} \text{ is of bounded variation}$$

$$\langle \mu_v(\alpha), \mu_v(\beta) \rangle = D(\alpha, \beta) \qquad \langle \bar{\mu}_v(\alpha), \bar{\mu}_v(\beta) \rangle = \bar{D}(\alpha, \beta)$$

$$\bar{D} : \mathfrak{S}_{\mathfrak{A}} \times \mathfrak{S}_{\mathfrak{A}} \to \mathbb{C} \qquad \bar{D}|_{\mathfrak{A}} = D$$

Conclusion

Sorkin's slogan

$$ORDER + NUMBER = GEOMETRY$$

CST dynamics is given by $(\Omega, \mathfrak{A}, \mu)$.

- ■ Extension of measure for classical sequential growth.
- ■ Condition on extension of measure for complex sequential growth.
- ■ Definition of quantum vector measure.
- □ How to get Bell causality in quantum sequential growth ?
- □ What are conditions for extension in quantum sequential growth ?