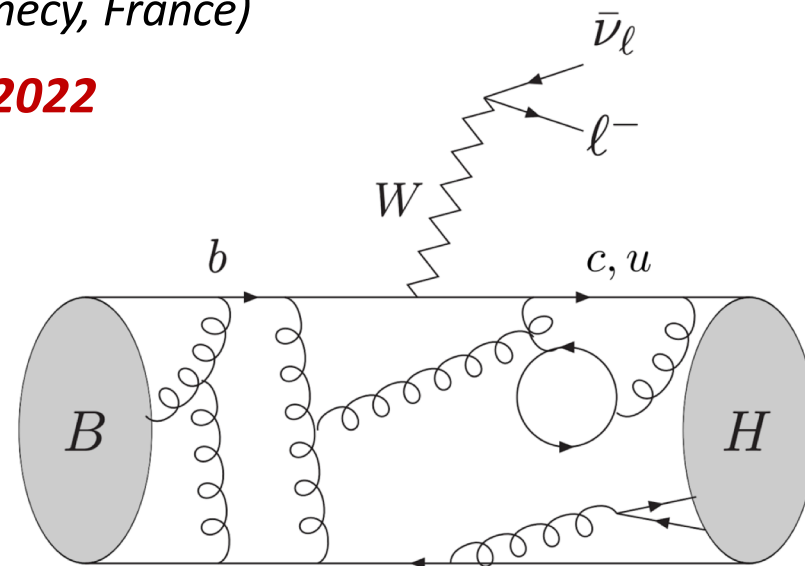# *The Dispersive Method approach to V$_{cb}$*

Work in collaboration with G. Martinelli, M. Naviglio and S. Simula

[PRD '21 (2105.02497), PRD '21 (2105.07851), PRD '22 (2105.08674),

2109.15248, PRD '22 (2204.05925), …]

Ludovico Vittorio (LAPTh & CNRS, Annecy, France)

GDR-InF Annual Workshop 2022

(from J.Phys.G 46 (2019) 2, 023001)

www.cnrs.fr

State-of-the-art of the semileptonic B → D(*) decays

Two critical issues:

- V_{cb} puzzle:

$$\boxed{2.8\,\sigma \text{ discrepancy!}}$$

EXCLUSIVE

FLAG Review 2021 [EPJC '22 (2111.09849)]

$$|V_{cb}| \times 10^3 = 39.36(68)$$

Fit only to the B → (D, D) channels*

VS

INCLUSIVE

$$|V_{cb}| \times 10^3 = 42.00(65)$$

$$|V_{cb}|_{\text{incl}} \times 10^3 = 42.16 \pm 0.50$$

Bordone et al., PLB '21 [arXiv:2107.00604]

$$|V_{cb}|_{\text{incl}} \times 10^3 = 41.69 \pm 0.63$$

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EXCLUSIVE   *INCLUSIVE*

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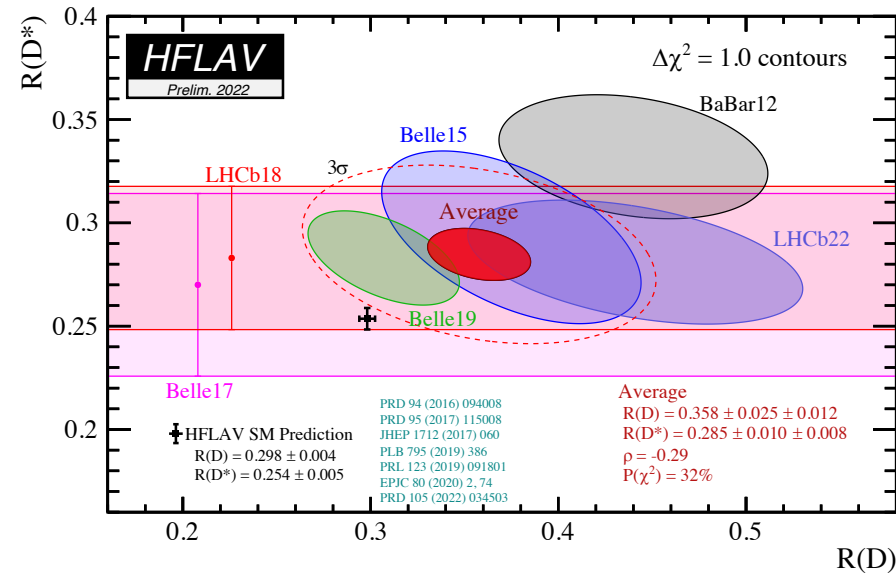$$|V_{cb}| \times 10^3 = 39.36(68) \quad \textbf{VS} \quad |V_{cb}| \times 10^3 = 42.00(65)$$

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- $R_{D(*)}$ anomalies:

$$\mathcal{R}(D) = \frac{\mathcal{B}(B \to D\tau\nu_\tau)}{\mathcal{B}(B \to D\ell\nu_\ell)},$$

$$\mathcal{R}(D^*) = \frac{\mathcal{B}(B \to D^*\tau\nu_\tau)}{\mathcal{B}(B \to D^*\ell\nu_\ell)}$$

$$\boxed{\begin{array}{c} 3.2\sigma \\ \text{discrepancy!} \end{array}}$$

HFLAV Collaboration, arXiv:2206.07501 [hep-ex] (updated plot)

The central role of the Form Factors (FFs) in excl. semil. *B* decays

- Production of a pseudoscalar meson (*i.e. D*) in case of massless lepton:

$$\frac{d\Gamma}{dq^2} \simeq \frac{G_F^2 |V_{cb}|^2 \eta_{EW}^2}{24\pi^3} |\vec{p}_D|^3 \boxed{|f^+(q^2)|^2}$$

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- Production of a vector meson (*i.e. D**) in case of massless leptons:

$$\frac{d\Gamma(B \to D^*(\to D\pi)\ell\nu)}{dw\, d\cos\theta_\ell\, d\cos\theta_v\, d\chi} = \frac{G_F^2 |V_{cb}|^2 \eta_{EW}^2}{4(4\pi)^4} 3 m_B m_{D^*}^2 \sqrt{w^2 - 1}$$

$$\times B(D^* \to D\pi)\{(1 - \cos\theta_\ell)^2 \sin^2\theta_v |H_+|^2$$

$$+ (1 + \cos\theta_\ell)^2 \sin^2\theta_v |H_-|^2 + 4\sin^2\theta_\ell \cos^2\theta_v |H_0|^2$$

$$H_\pm(w) = \boxed{f(w)} \mp m_B m_{D^*} \sqrt{w^2 - 1} \boxed{g(w)}$$

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Relation between the momentum transfer and the recoil:

$$\boxed{q^2 = m_B^2 + m_P^2 - 2 m_B m_P\, w}$$

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Two FFs coupled to the lepton mass: $\boxed{f_0(w)}$ (pseudoscalar), $\boxed{P_1(w)}$ (vector)

The Dispersive Matrix (DM) method

Our goal is to describe the FFs using a novel, non-perturbative and model independent approach: starting from the available LQCD computations of the FFs in the high-q^2 (or low-w) regime, we extract the FFs behaviour in the low-q^2 (or high-w) region!

- Pioneering works from S. Okubo [PRD, 3 (1971); PRD, 4 (1971)], C.'Bourrely et al [NPB, 189 (1981)] and L. Lellouch [NPB, 479 (1996)]
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- **can be applied to theoretical calculations of the FFs**, but also **to experimental data**

- keep **theoretical calculations and experimental data separated**

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How does it work?

The DM method

Let us focus on a generic FF f: we can define

$$\mathbf{M} = \begin{pmatrix}
\chi & \phi f & \phi_1 f_1 & \phi_2 f_2 & \cdots & \phi_N f_N \\
\phi f & \frac{1}{1-z^2} & \frac{1}{1-zz_1} & \frac{1}{1-zz_2} & \cdots & \frac{1}{1-zz_N} \\
\phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & \cdots & \frac{1}{1-z_1 z_N} \\
\phi_2 f_2 & \frac{1}{1-z_2 z} & \frac{1}{1-z_2 z_1} & \frac{1}{1-z_2^2} & \cdots & \frac{1}{1-z_2 z_N} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \frac{1}{1-z_N z_2} & \cdots & \frac{1}{1-z_N^2}
\end{pmatrix}$$

$$\phi_i f_i \equiv \phi(z_i) f(z_i) \ (\text{with } i = 1, 2, ...N)$$

$$\begin{pmatrix} z(t) = \dfrac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1} \\ t_\pm \equiv (m_B \pm m_D)^2 \\ \textit{t: momentum transfer} \end{pmatrix}$$

The DM method

L. Vittorio (LAPTh & CNRS, Annecy)

Non-perturbative values of the susceptibilities from the dispersion relations (see PRD '21 (2105.07851))

Estimates of the FFs, computed on the lattice

$$\mathbf{M} = \begin{pmatrix} \chi & \phi f & \phi_1 f_1 & \phi_2 f_2 & ... & \phi_N f_N \\ \phi f & \frac{1}{1-z^2} & \frac{1}{1-zz_1} & \frac{1}{1-zz_2} & ... & \frac{1}{1-zz_N} \\ \phi_1 f_1 & \frac{1}{1-z_1 z} & \frac{1}{1-z_1^2} & \frac{1}{1-z_1 z_2} & ... & \frac{1}{1-z_1 z_N} \\ \phi_2 f_2 & \frac{1}{1-z_2 z} & \frac{1}{1-z_2 z_1} & \frac{1}{1-z_2^2} & ... & \frac{1}{1-z_2 z_N} \\ ... & ... & ... & ... & ... & ... \\ \phi_N f_N & \frac{1}{1-z_N z} & \frac{1}{1-z_N z_1} & \frac{1}{1-z_N z_2} & ... & \frac{1}{1-z_N^2} \end{pmatrix}$$

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One can show that

$$\det \mathbf{M} \geq 0$$

$$\downarrow$$

$$f_{\text{lo}}(z) \leq f(z) \leq f_{\text{up}}(z)$$

Values of the momentum transfer @ which FFs are computed on the lattice

The simplest example: semileptonic B → D decays

In **PRD '21 (arXiv:2105.08674)**, our DM method has been applied to B → D decays:

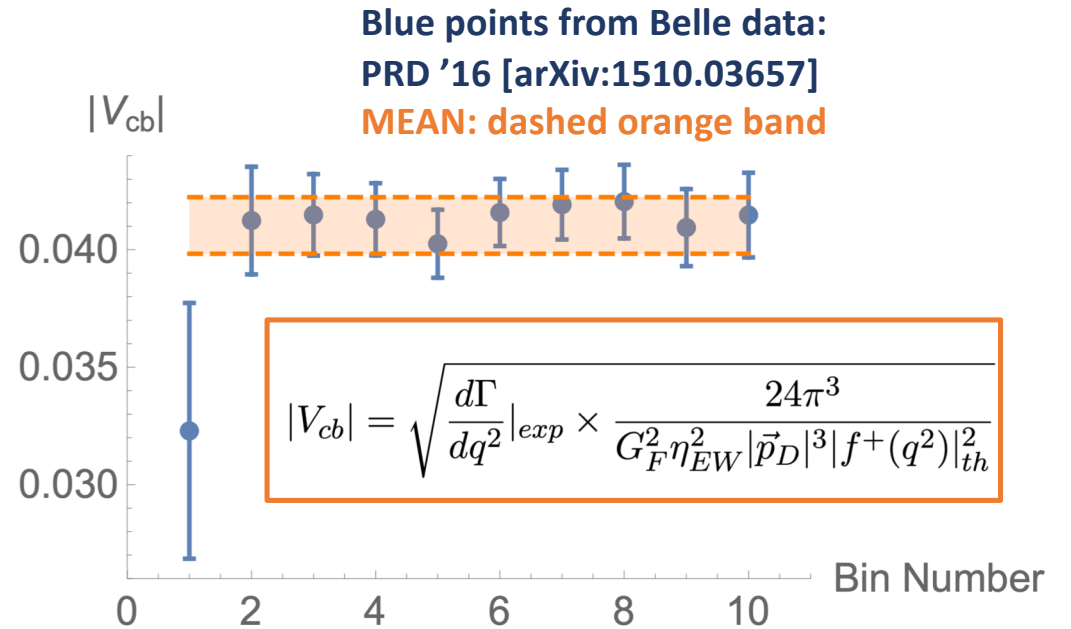- **3 FNAL/MILC data for each FF**: final results contained in **PRD '15 (arXiv:1503.07237)**

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$f_+(z)$

$f_0(z_{max}) = f_+(z_{max})$

$f_0(z)$

KC at maximum recoil
for each bootstrap
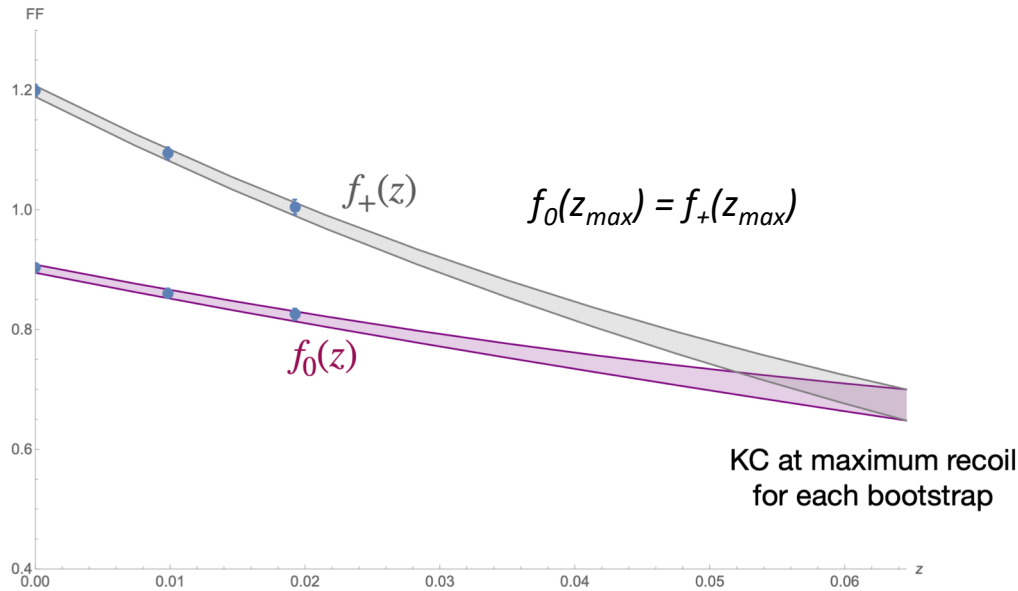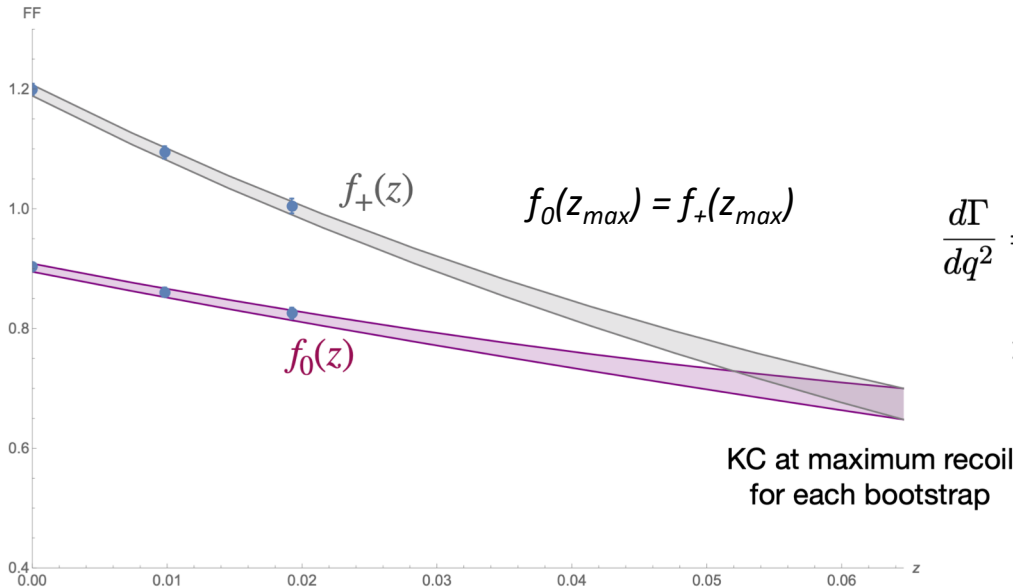
Recalling that for production of a pseudoscalar meson
*(i.e. D) in case of **massive** lepton:*

$$\frac{d\Gamma}{dq^2} = \frac{G_F^2 |V_{cb}|^2 \eta_{EW}^2}{24\pi^3} \left(1 - \frac{m_\ell^2}{q^2}\right)^2$$

$$\times \left[|\vec{p}_D|^3 \left(1 + \frac{m_\ell^2}{2q^2}\right) |f^+(q^2)|^2 + m_B^2 |\vec{p}_D| \left(1 - \frac{m_D^2}{m_B^2}\right)^2 \frac{3m_\ell^2}{8q^2} |f^0(q^2)|^2 \right]$$

FULLY-THEORETICAL
ESTIMATE!

$$R(D) = 0.296 \pm 0.008$$

L. Vittorio (LAPTh & CNRS, Annecy)   5

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Blue points from Belle data:
PRD '16 [arXiv:1510.03657]
MEAN: dashed orange band

$f_0(z_{max}) = f_+(z_{max})$

$f_+(z)$

$f_0(z)$

KC at maximum recoil for each bootstrap

$$|V_{cb}| = \sqrt{\frac{d\Gamma}{dq^2}\Big|_{exp} \times \frac{24\pi^3}{G_F^2 \eta_{EW}^2 |\vec{p}_D|^3 |f^+(q^2)|_{th}^2}}$$
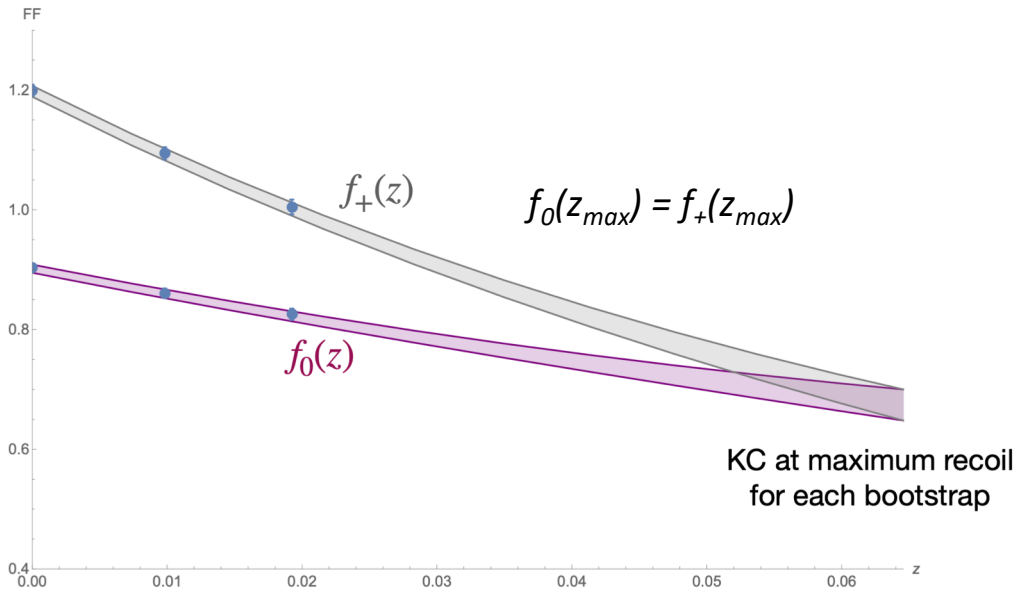
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$$|V_{cb}| \times 10^3 = 41.0 \pm 1.2$$

The simplest example: semileptonic B → D decays

Important recent novelty: new data on semileptonic B → D decays by **Belle II Collaboration** (see arXiv:2210.13143)!

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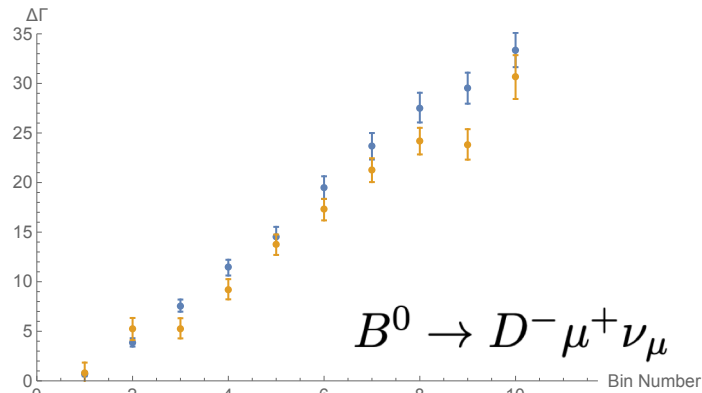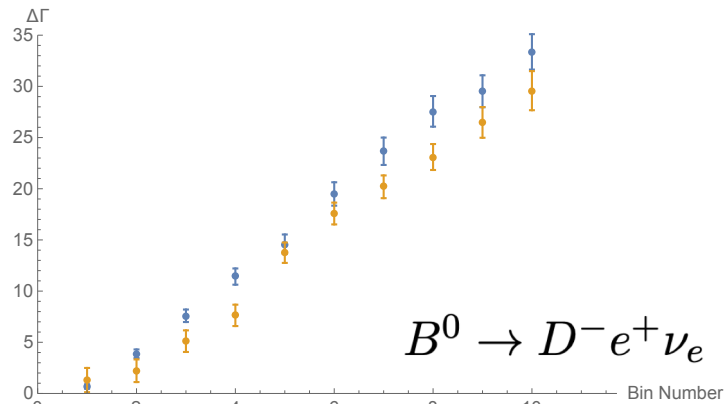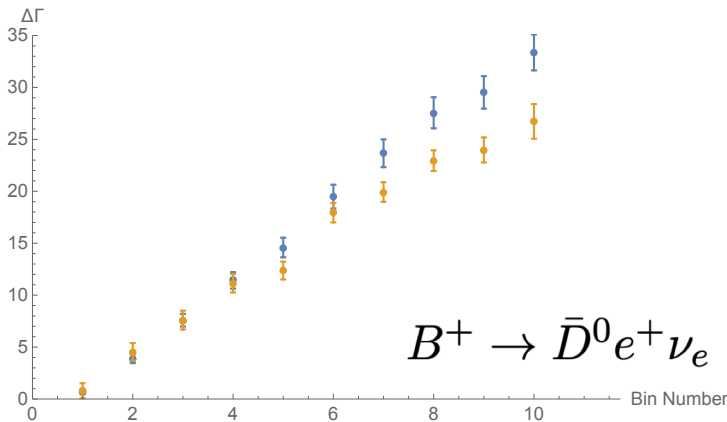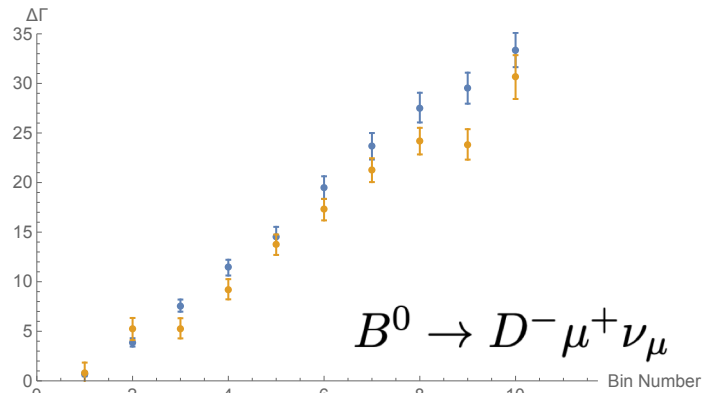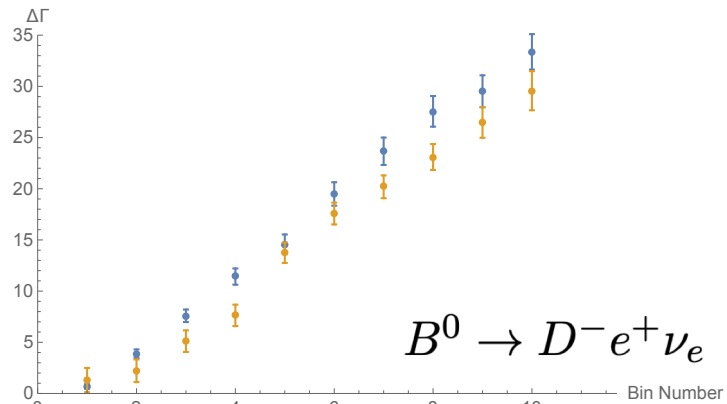

- **Blue points from
PRD '16 [arXiv:1510.03657]
(average case)**
- **Orange points from
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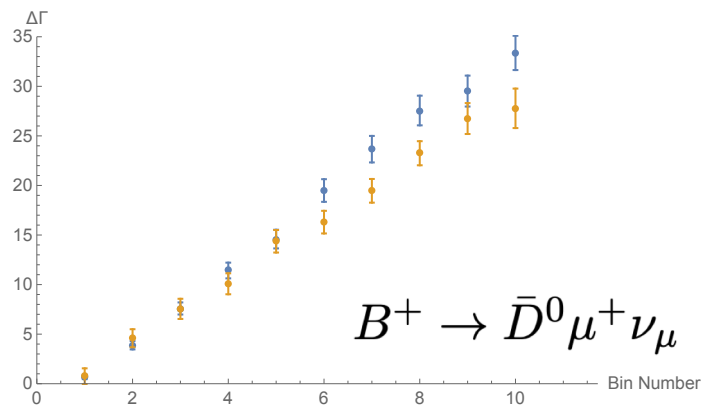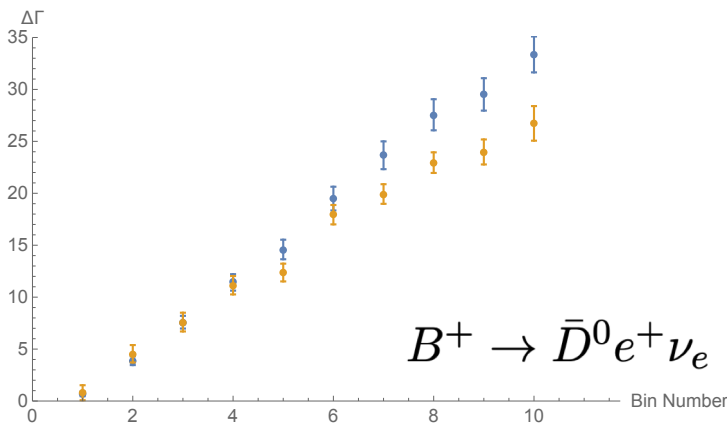
The simplest example: semileptonic B → D decays

Important recent novelty: new data on semileptonic B → D decays by **Belle II Collaboration** (see arXiv:2210.13143)!

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Two fundamental issues:
1. **(More or less) slightly different shapes of the data implies different bin-per-bin values of Vcb**
2. **Why the covariance matrix is not available?**

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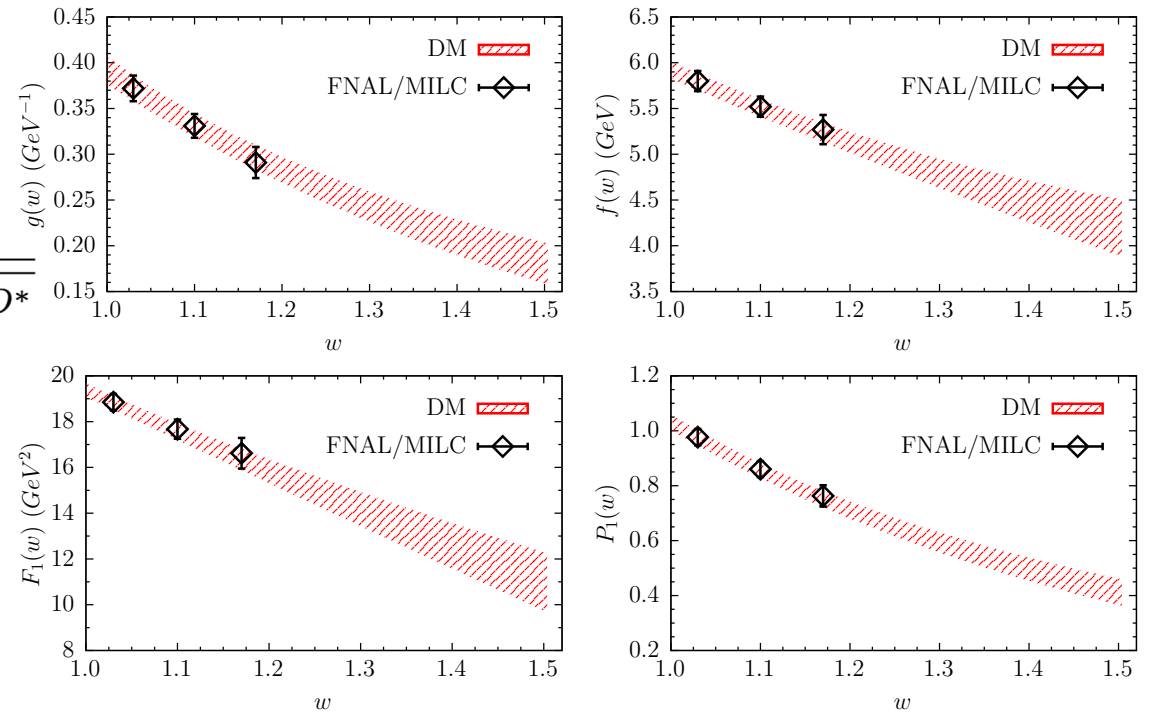
The "problematic" semileptonic B → D* channel

In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

- **3 FNAL/MILC data (diamonds) for each FF**: final results contained in **arXiv:2105.14019 [hep-lat]**

Two kinematical constraints (KCs):

$$\mathcal{F}_1(1) = (m_B - m_{D^*})f(1)$$

$$P_1(w_{max}) = \frac{\mathcal{F}_1(w_{max})}{(1 + w_{max})(m_B - m_{D^*})\sqrt{m_B m_{D^*}}}$$

The "problematic" semileptonic B → D* channel

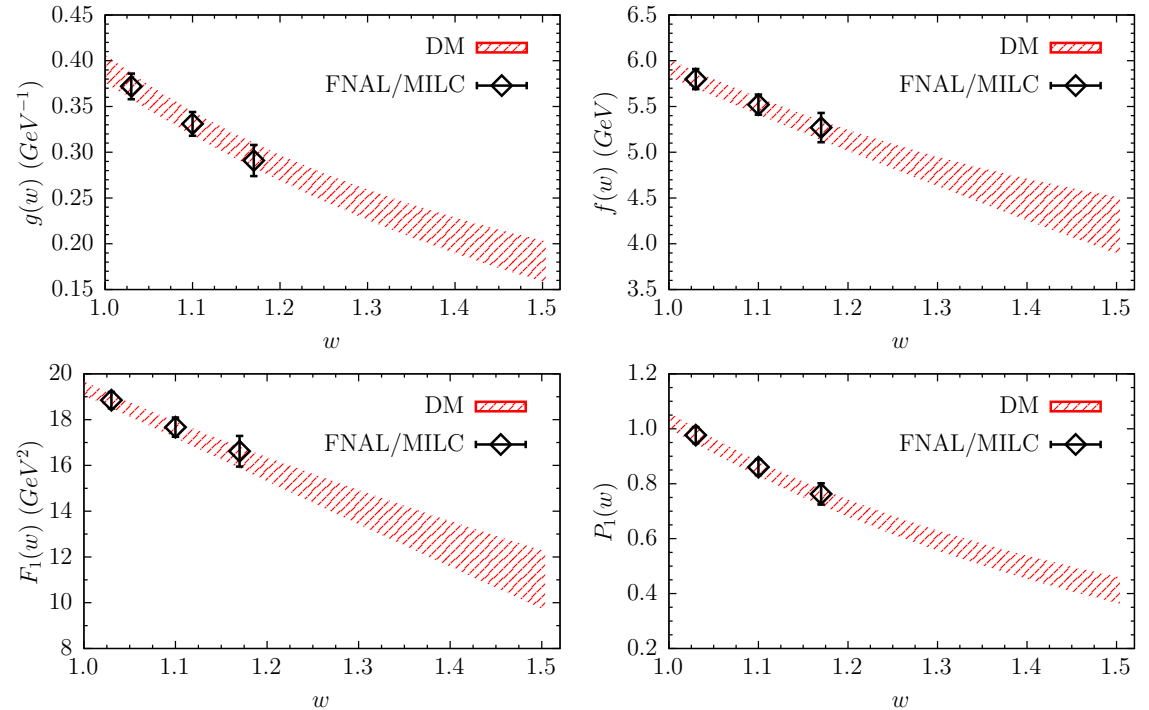In **arXiv:2109.15248**, we have studied the final results of the FNAL/MILC computations of the FFs

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Using the unitary bands of the FFs, we can compute new fully-theoretical expectation values of the anomaly R(D):*

DM prediction

$$R(D^*) = 0.275 \pm 0.008$$

$$R(D^*)|_{\exp} = 0.285 \pm 0.010 \pm 0.008$$

HFLAV Coll. (https://hflav-eos.web.cern.ch/hflav-eos/semi/fall22/html/RDsDsstar/RDRDs.html)

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Less than 1σ compatibility!

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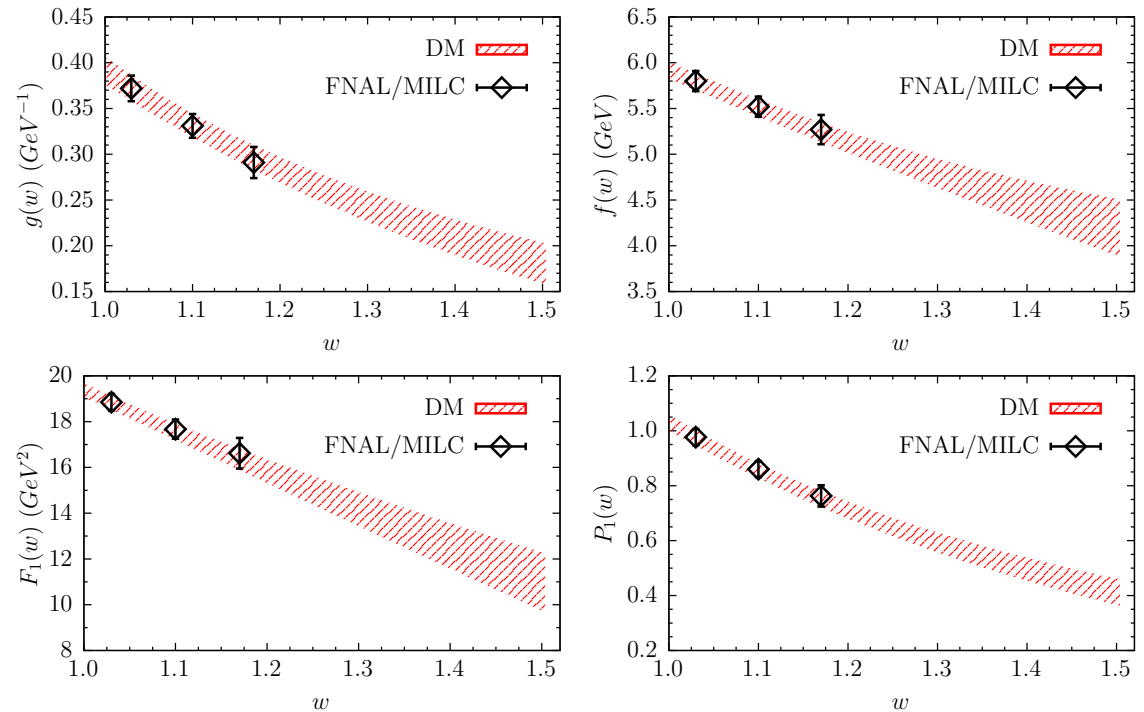
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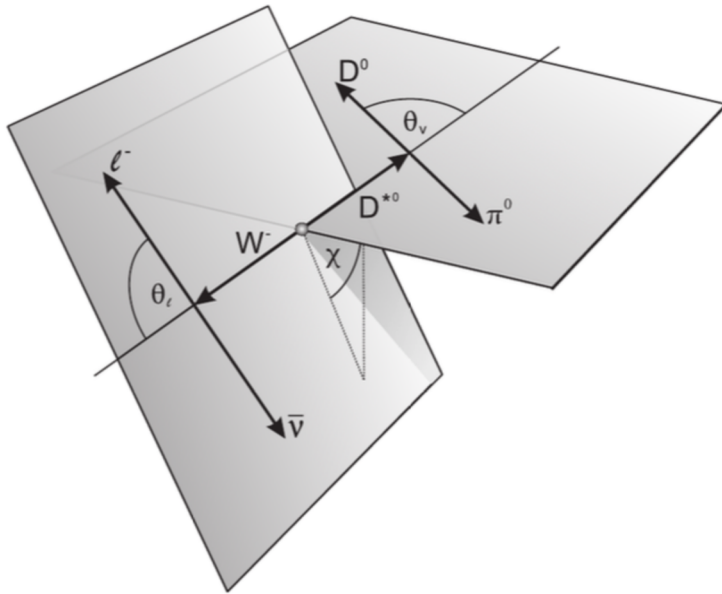
How to deal with experimental data?

Starting from the FFs bands, we propose to use the experimental data **only** to compute bin-per-bin estimates of Vcb.

NB: the experimental data do NOT enter in the determination of the bands of the FFs

To do it, it is sufficient to compare the two sets of measurements of the differential decay widths

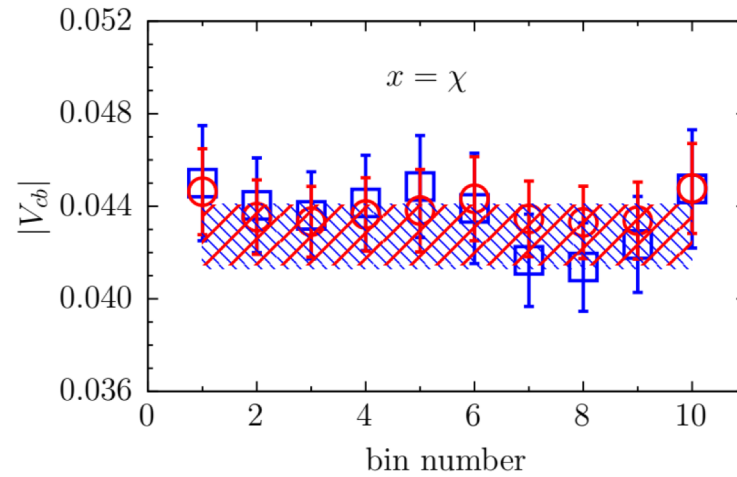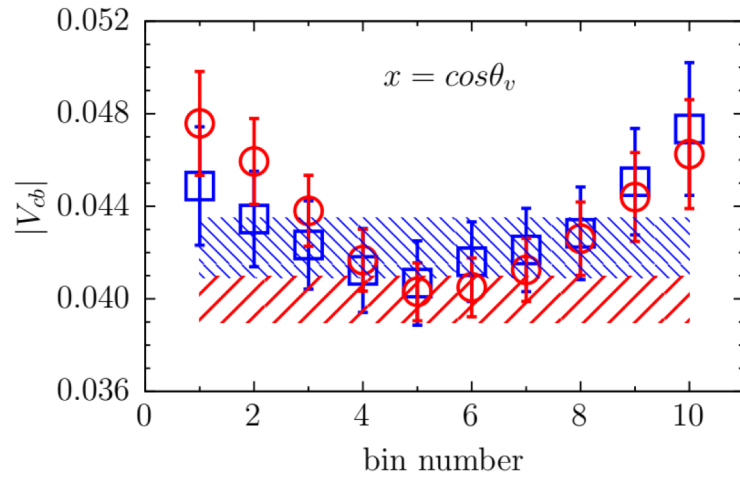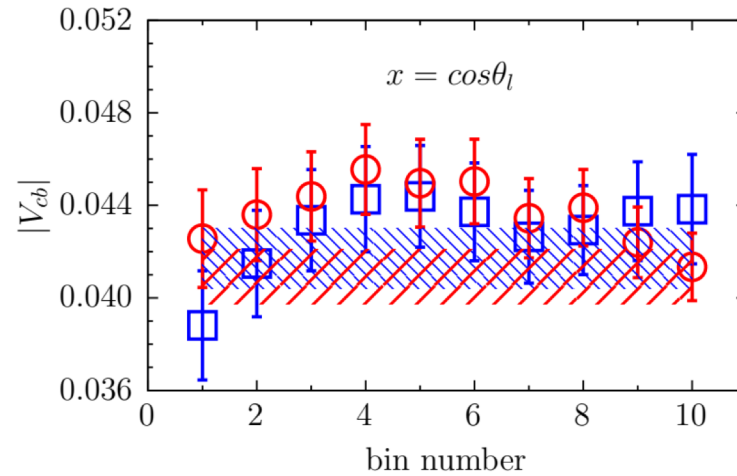$$d\Gamma/dx, \quad x = w, \cos\theta_l, \cos\theta_v, \chi$$
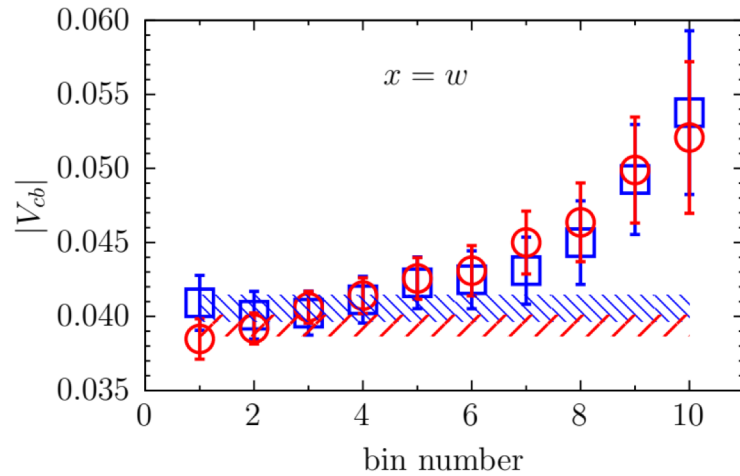
by the Belle Collaboration **(arXiv:1702.01521, arXiv:1809.03290)** with their *theoretical estimate*, computed through the unitarity bands shown before.

$$|V_{cb}|_i \equiv \sqrt{\frac{(d\Gamma/dx)_i^{exp.}}{(d\Gamma/dx)_i^{th.}}}$$

$$\frac{d\Gamma(B \to D^*(\to D\pi)\ell\nu)}{dw \, d\cos\theta_\ell \, d\cos\theta_v \, d\chi} = \frac{G_F^2 |V_{cb}|^2 \eta_{EW}^2}{4(4\pi)^4} 3m_B m_{D^*}^2 \sqrt{w^2 - 1}$$

$$\times B(D^* \to D\pi)\{(1 - \cos\theta_\ell)^2 \sin^2\theta_v |H_+|^2$$

$$+ (1 + \cos\theta_\ell)^2 \sin^2\theta_v |H_-|^2 + 4\sin^2\theta_\ell \cos^2\theta_v |H_0|^2$$

$$- 2\sin^2\theta_\ell \sin^2\theta_v \cos 2\chi H_+ H_-$$

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$$+ 4\sin\theta_\ell(1 + \cos\theta_\ell)\sin\theta_v \cos\theta_v \cos\chi H_- H_0\},$$

L. Vittorio (LAPTh & CNRS, Annecy)

8

Exclusive Vcb determination through unitarity

Blue squares: arXiv:1702.01521

Red points: arXiv:1809.03290

Exclusive Vcb determination through unitarity

experiment	$|V_{cb}|(x=w)$	$|V_{cb}|(x=\cos\theta_l)$	$|V_{cb}|(x=\cos\theta_v)$	$|V_{cb}|(x=\chi)$
Ref. [14]	0.0405 (9)	0.0417 (13)	0.0422 (13)	0.0427 (14)
$\chi^2/$(d.o.f.)	1.01	0.89	0.66	0.72
Ref. [15]	0.0394 (7)	0.0409 (12)	0.0400 (10)	0.0427 (13)
$\chi^2/$(d.o.f.)	1.21	1.36	1.99	0.38

Exclusive Vcb determination through unitarity

To compute the final average of these Vcb estimates:

$$\mu_x = \frac{1}{N} \sum_{k=1}^{N} x_k,$$

$$\sigma_x^2 = \frac{1}{N} \sum_{k=1}^{N} \sigma_k^2 + \frac{1}{N} \sum_{k=1}^{N} (x_k - \mu_x)^2$$

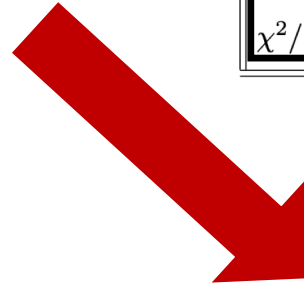| experiment	$|V_{cb}|(x=w)$	$|V_{cb}|(x=\cos\theta_l)$	$|V_{cb}|(x=\cos\theta_v)$	$|V_{cb}|(x=\chi)$
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Final DM estimate

$$\boxed{|V_{cb}| \times 10^3 = 41.3 \pm 1.7}$$

Compatible with the (most recent) **inclusive** values!

$$|V_{cb}|_{\rm incl} \times 10^3 = 42.16 \pm 0.50$$

Bordone et al., Phys.Lett.B [2107.00604]

$$|V_{cb}|_{\rm incl} \times 10^3 = 41.69 \pm 0.63$$

Bernlochner et al., arXiv:2205.10274

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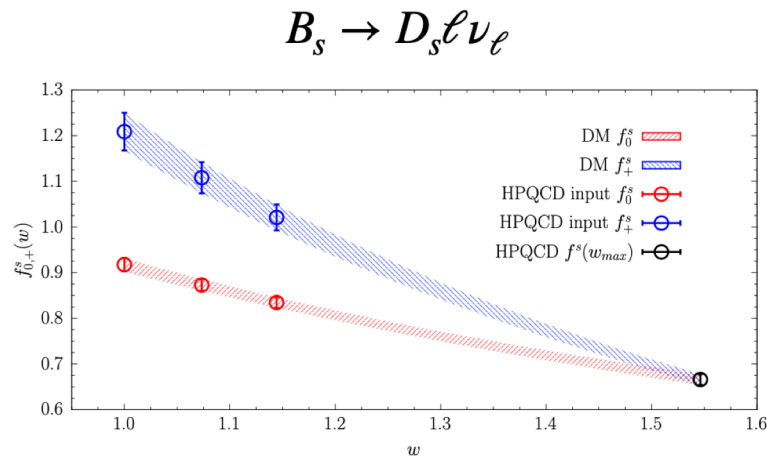| experiment	$|V_{cb}|(x=w)$	$|V_{cb}|(x=\cos\theta_l)$	$|V_{cb}|(x=\cos\theta_v)$	$|V_{cb}|(x=\chi)$
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MOST IMPORTANT MESSAGE OF THE TALK: we want to avoid any bias by treating experimental and LQCD data differently in the DM approach to the hadronic FFs

Final DM estimate

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Quick parenthesis: semileptonic $B_s \to D_s^{(*)}$ decays

In **PRD '22 [arXiv:2204.05925],** our DM method has been applied to **semileptonic $B_s \to D_s^{(*)}$ decays**. LQCD form factors taken from the results of the fits preformed by the HPQCD Collaboration in **PRD '20 [arXiv:1906.00701]** ($B_s \to D_s$) and **PRD '22 [arXiv:2105.11433]** ($B_s \to D_s^*$): we extract 3 data points for the FFs at small values of the recoil and apply the DM approach.

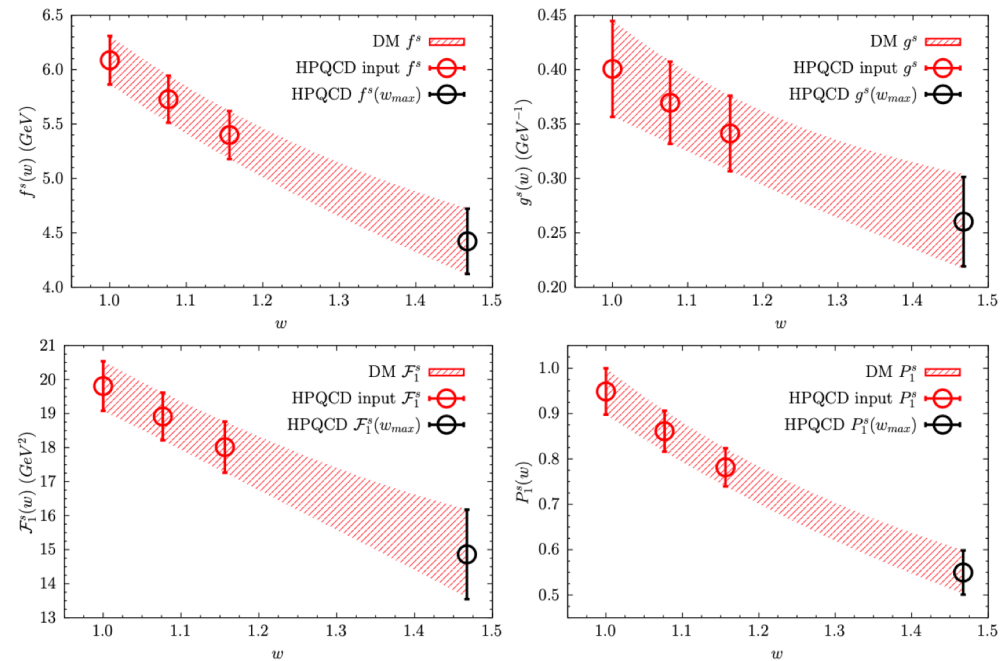# Quick parenthesis: semileptonic $B_s \to D_s^{(*)}$ decays

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Quick parenthesis: semileptonic $B_s \to D_s^{(*)}$ decays

Without entering in the details of this analysis, phenomenological applications give the results

$$|V_{cb}|^{\text{DM}} \cdot 10^3 = 41.7 \pm 1.9 \qquad \text{from } B_s \to D_s \ell \nu_\ell \text{ decays}$$

$$= 40.7 \pm 2.4 \qquad \text{from } B_s \to D_s^* \ell \nu_\ell \text{ decays}$$
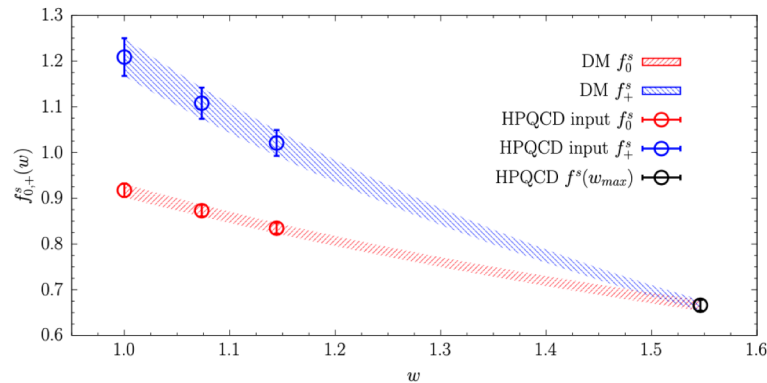
$$R(D_s) = 0.298 \ (5)$$
$$R(D_s^*) = 0.250 \ (6)$$

fully-theoretical

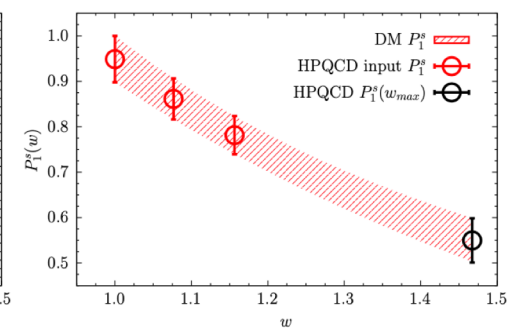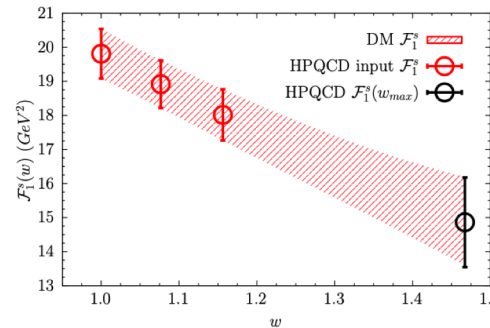$$B_s \to D_s^* \ell \nu_\ell$$

through available exps. data by LHCb Collaboration
(PRD '20 [2001.03225], JHEP '20 [2003.08453])

$$B_s \to D_s \ell \nu_\ell$$

* nice agreement in the whole kinematical range
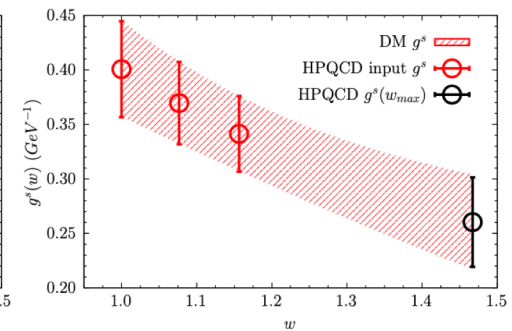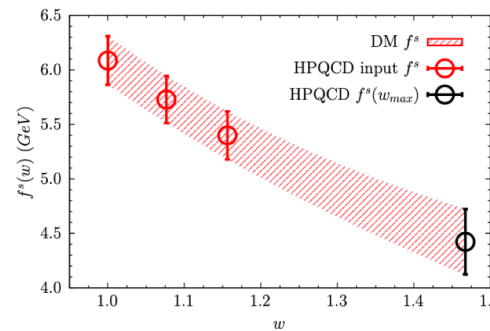
L. Vittorio (LAPTh & CNRS, Annecy)   11

Summary of all the DM results
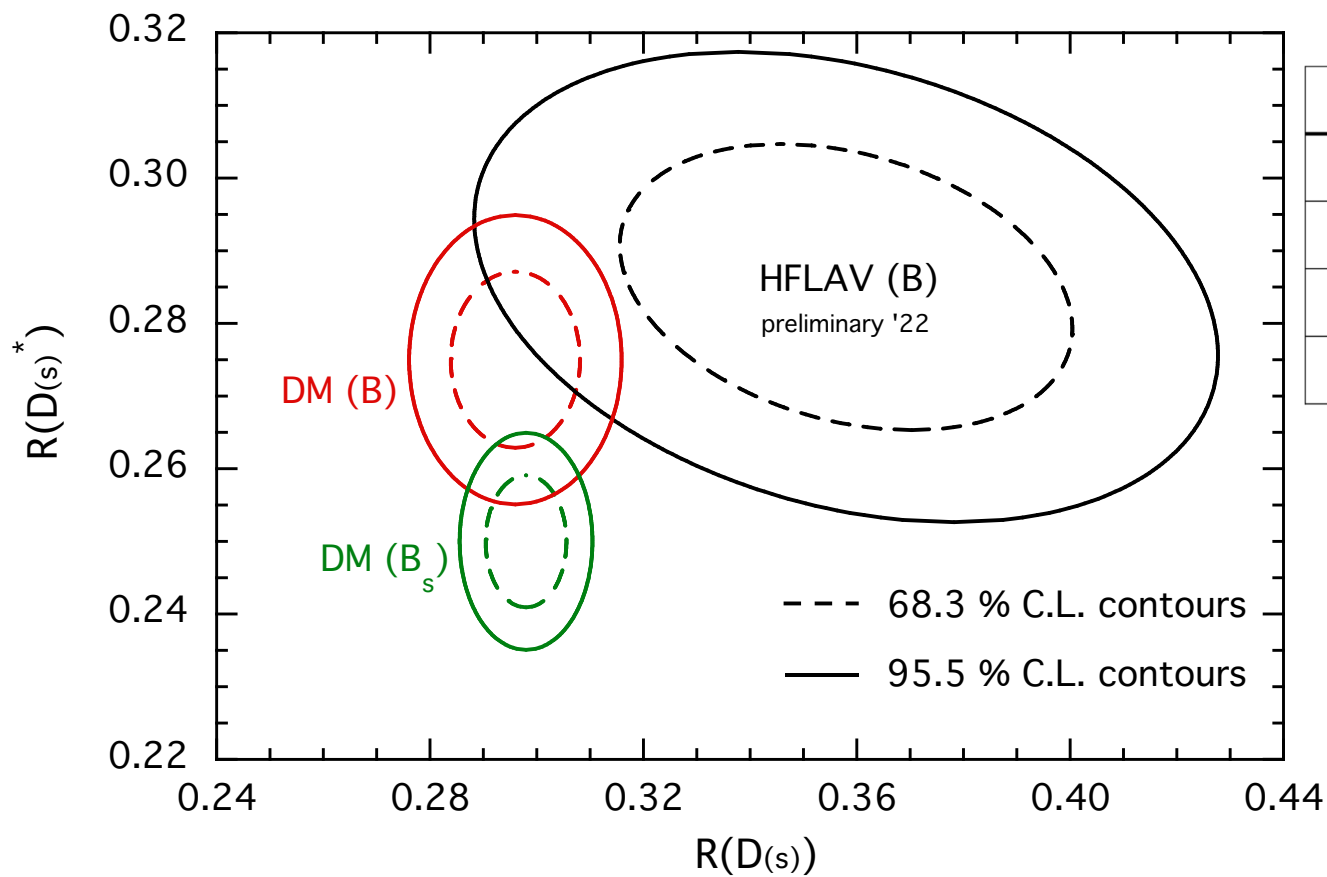
LFU observables

	DM	HFLAV '22
R(D)	0.296 (8)	0.358 (25) (12)
R(D*)	0.275 (8)	0.285 (10) (8)
R(D$_s$)	0.298 (5)	
R(D$_s$*)	0.250 (6)	

Summary of all the DM results

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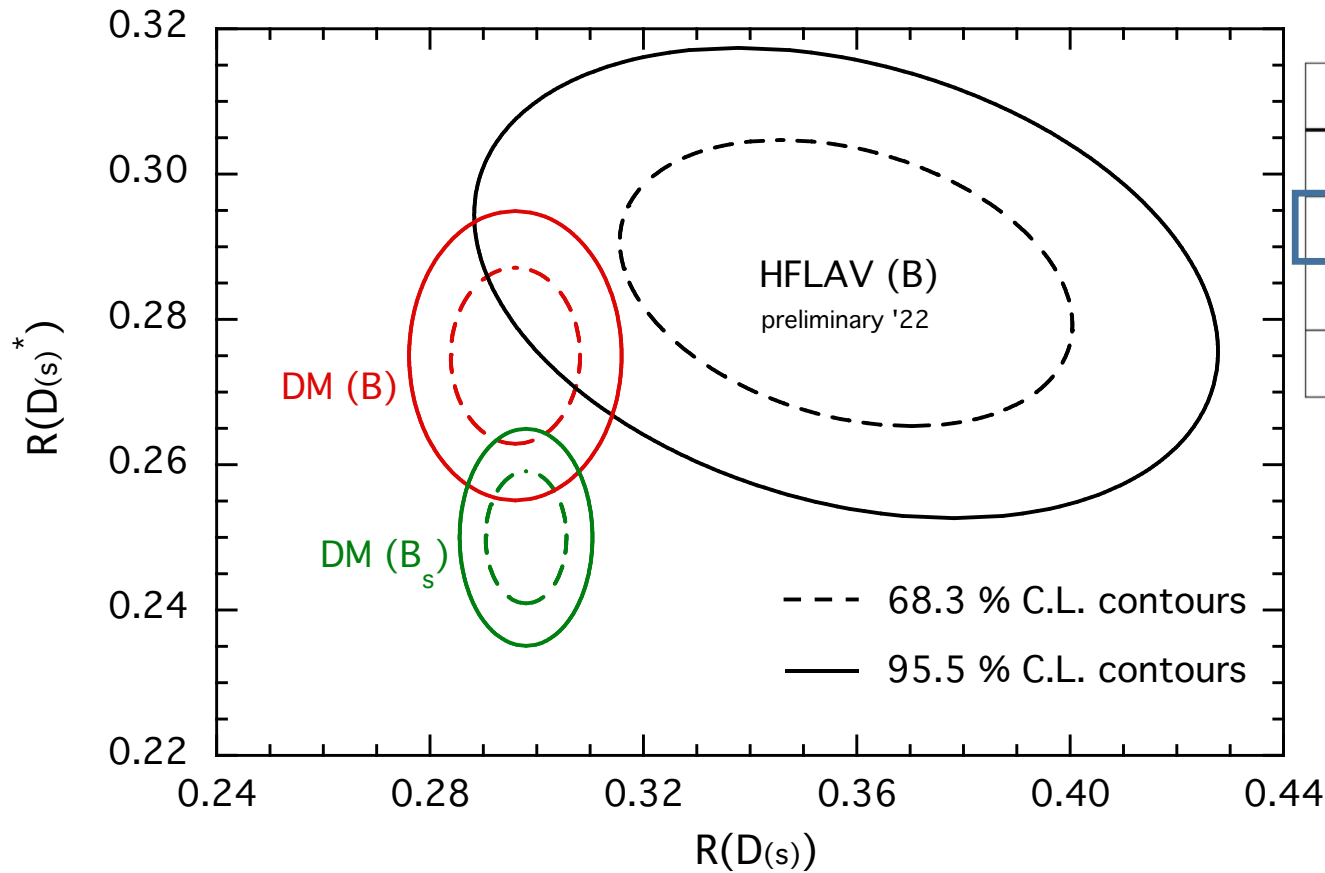
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**By using (and trusting) FNAL/MILC lattice data, this discrepancy is reduced at less than 1σ level**

Summary of all the DM results
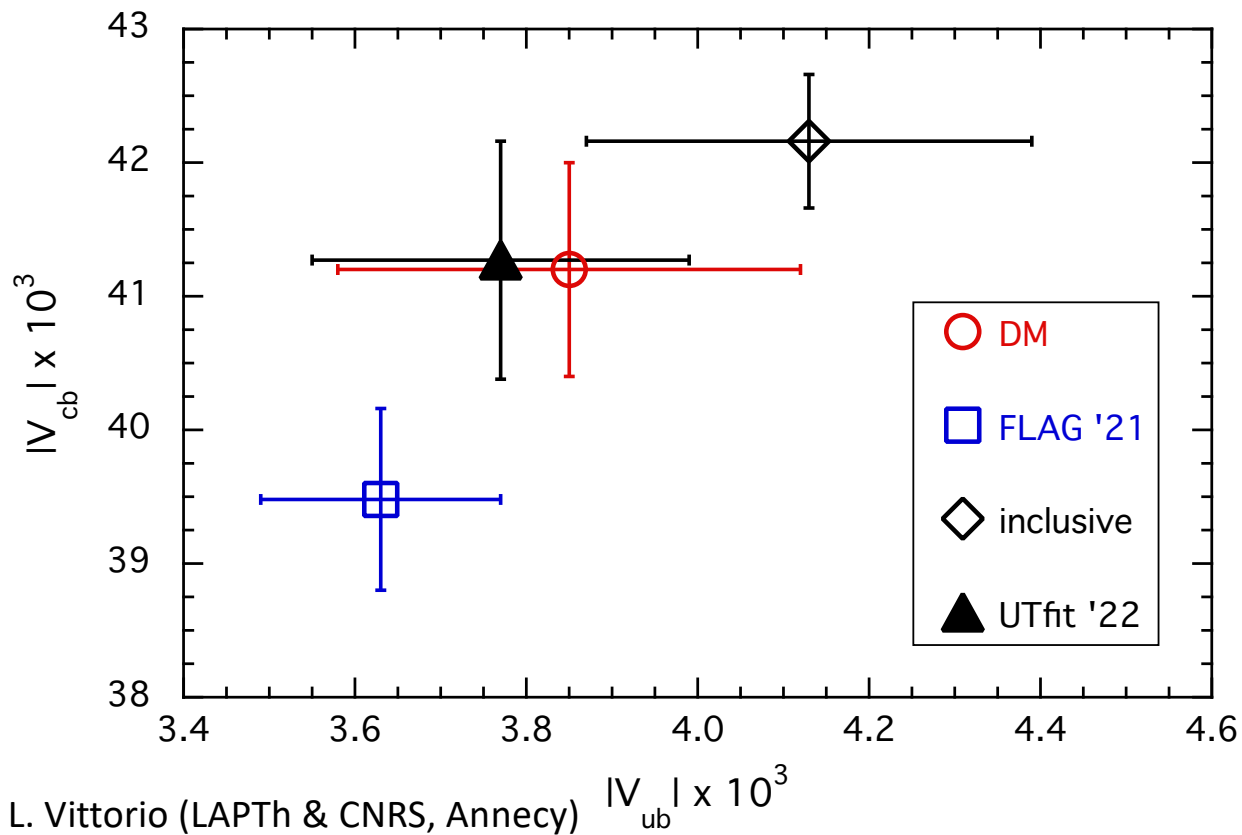
CKM parameters Vub and Vcb

Summary of all the DM results

CKM parameters Vub and Vcb

	decays	DM	FLAG '21	inclusive
$|V_{cb}| \cdot 10^3$	$B_{(s)} \to D_{(s)}^{(*)}$	41.2 (8)	39.48 (68)	42.16 (50)
$|V_{ub}| \cdot 10^3$	$B_{(s)} \to \pi, K$	3.85 (27)	3.63 (14)	4.13 (26)
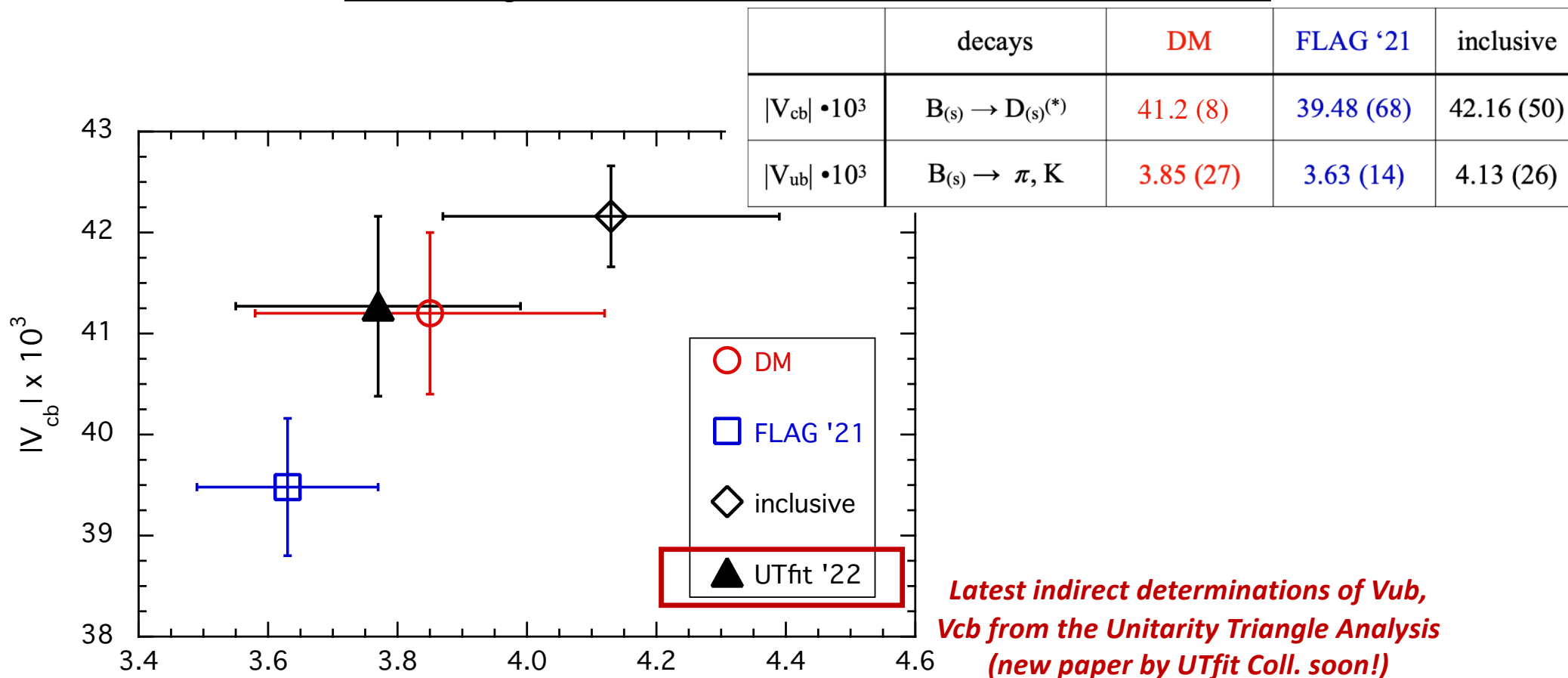
Latest indirect determinations of Vub, Vcb from the Unitarity Triangle Analysis (new paper by UTfit Coll. soon!)

Summary of all the DM results

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For the first time, a very precise exclusive determination of $|V_{cb}|$ is compatible with the inclusive determination (average of 4 decay channels!)

Latest indirect determinations of Vub, Vcb from the Unitarity Triangle Analysis (new paper by UTfit Coll. soon!)
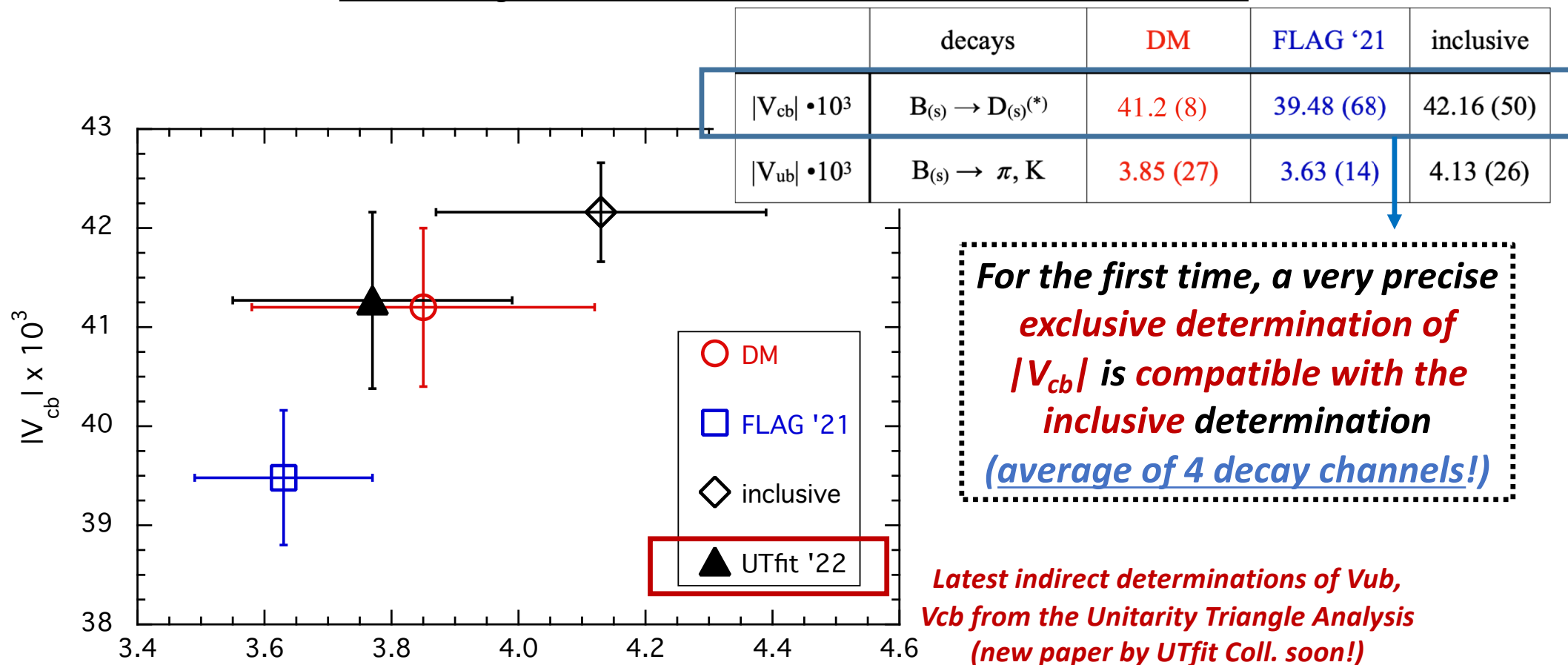
# THANKS FOR YOUR ATTENTION!

BACK-UP SLIDES

How to deal with experimental data? 2nd possibility

In principle, one can use **also** experimental data (in addition to the LQCD ones) to constrain the shape of the FFs…

VERY DANGEROUS: in this case, you are trying to fix both the normalization (*i.e.* |Vcb|) and the slopes of the FFs at the same time…

Let us see this in detail: *let us consider the BGL fits performed by FNAL/MILC Collaborations in arXiv:2105.14019 [hep-lat]*
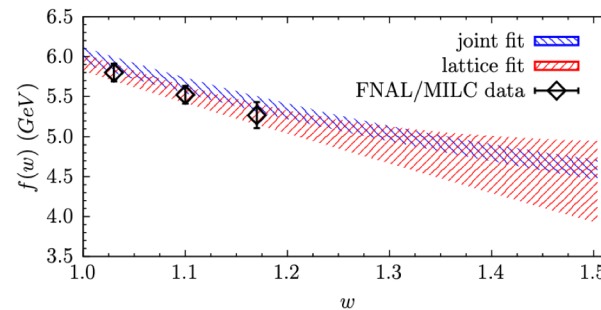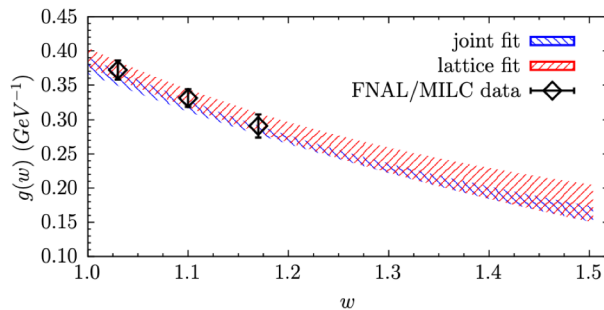
joint fit:
BGL fit of LQCD points +
Belle + BaBar exp. data

$|V_{cb}| \cdot 10^3 = 38.40 \pm 0.74$

$R(D^*) = 0.2483 \pm 0.0013$
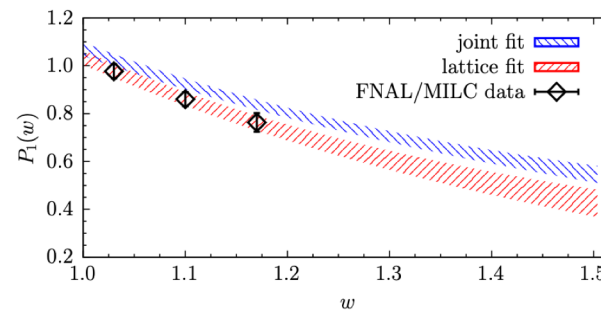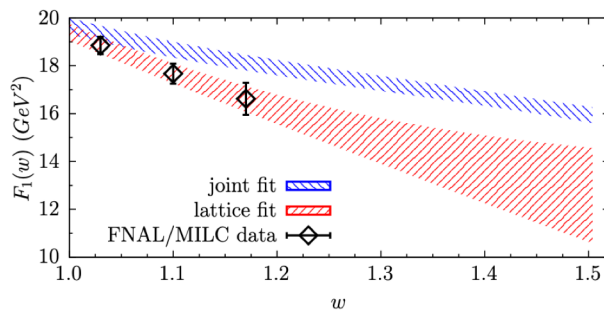
lattice fit:
quadratic BGL fit of LQCD
points only

$|V_{cb}| > |V_{cb}|^{\text{joint fit}}$?

$R(D^*) = 0.265 \pm 0.013$

simultaneous fit of the lattice points and experimental data to determine the shape of the FFs and to extract $|V_{cb}|$

*** slope differences between exp's and theory $\rightarrow$ bias on $|V_{cb}|^{\text{joint fit}}$? ***

Comparison between the two different analysis strategies

$$\frac{d\Gamma}{dw} \propto |V_{cb}|^2 \sqrt{w^2 - 1}\frac{q^2}{M_B^4}\left[H_0^2(w) + H_-^2(w) + H_+^2(w)\right] = |V_{cb}|^2 \sqrt{w^2 - 1}\left\{\left(\frac{\mathcal{F}_1(w)}{M_B^2}\right)^2 + 2\frac{q^2}{M_B^2}\left[\left(\frac{f(w)}{M_B}\right)^2 + r^2(w^2 - 1)\,m_B^2\,g^2(w)\right]\right\} \qquad m_\ell = 0$$

$|V_{cb}|^{latticefit} = 40.5 \cdot 10^{-3}$

$|V_{cb}|^{jointfit} = 38.4 \cdot 10^{-3}$

lattice fit

joint fit
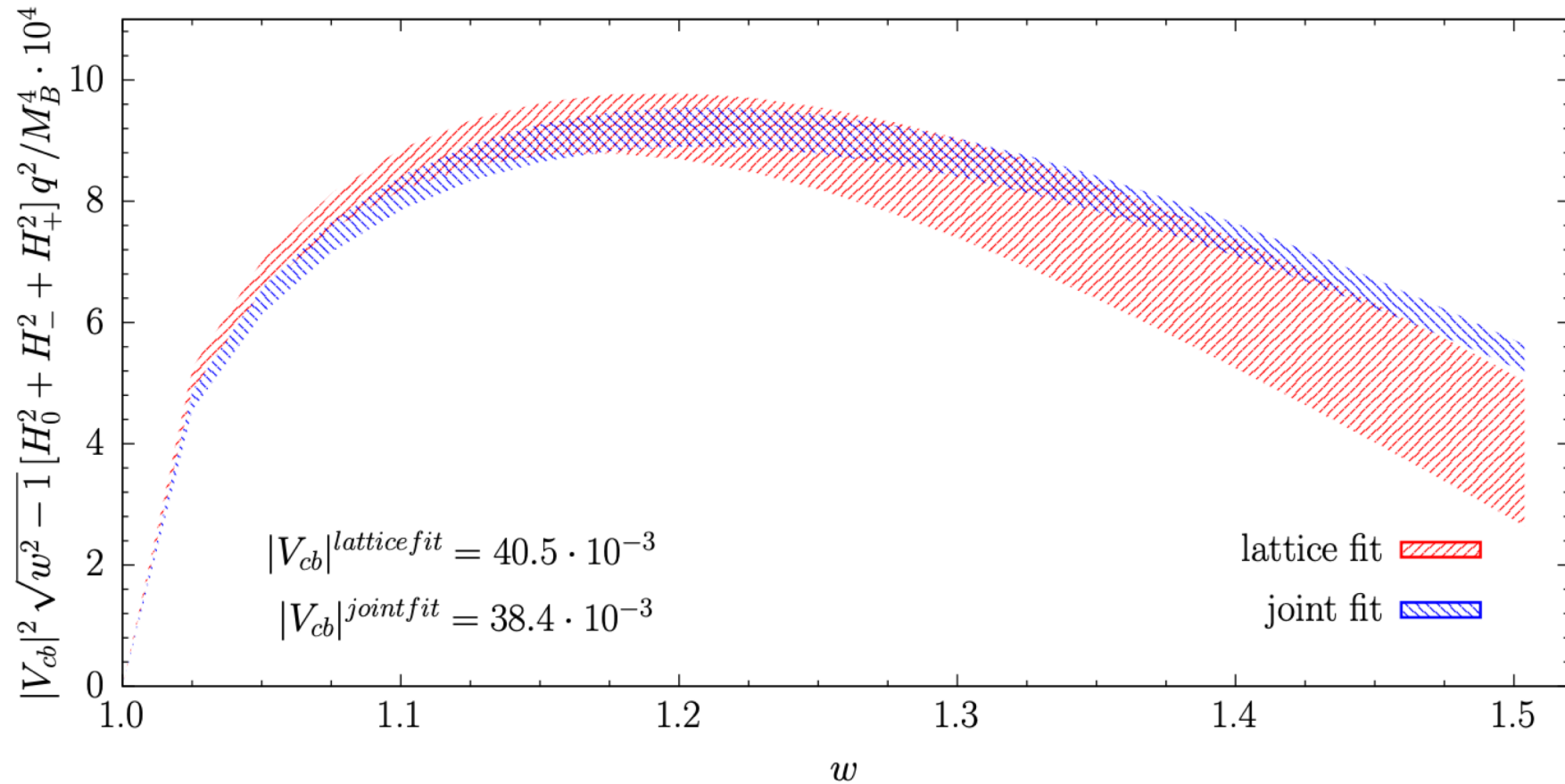
Comparison between the two different analysis strategies

> **Both the curves reproduce the data!**
> But they induce *different values of the (|Vcb|,R(D*)) couple*...

$|V_{cb}|^{lattice fit} = 40.5 \cdot 10^{-3}$

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joint fit
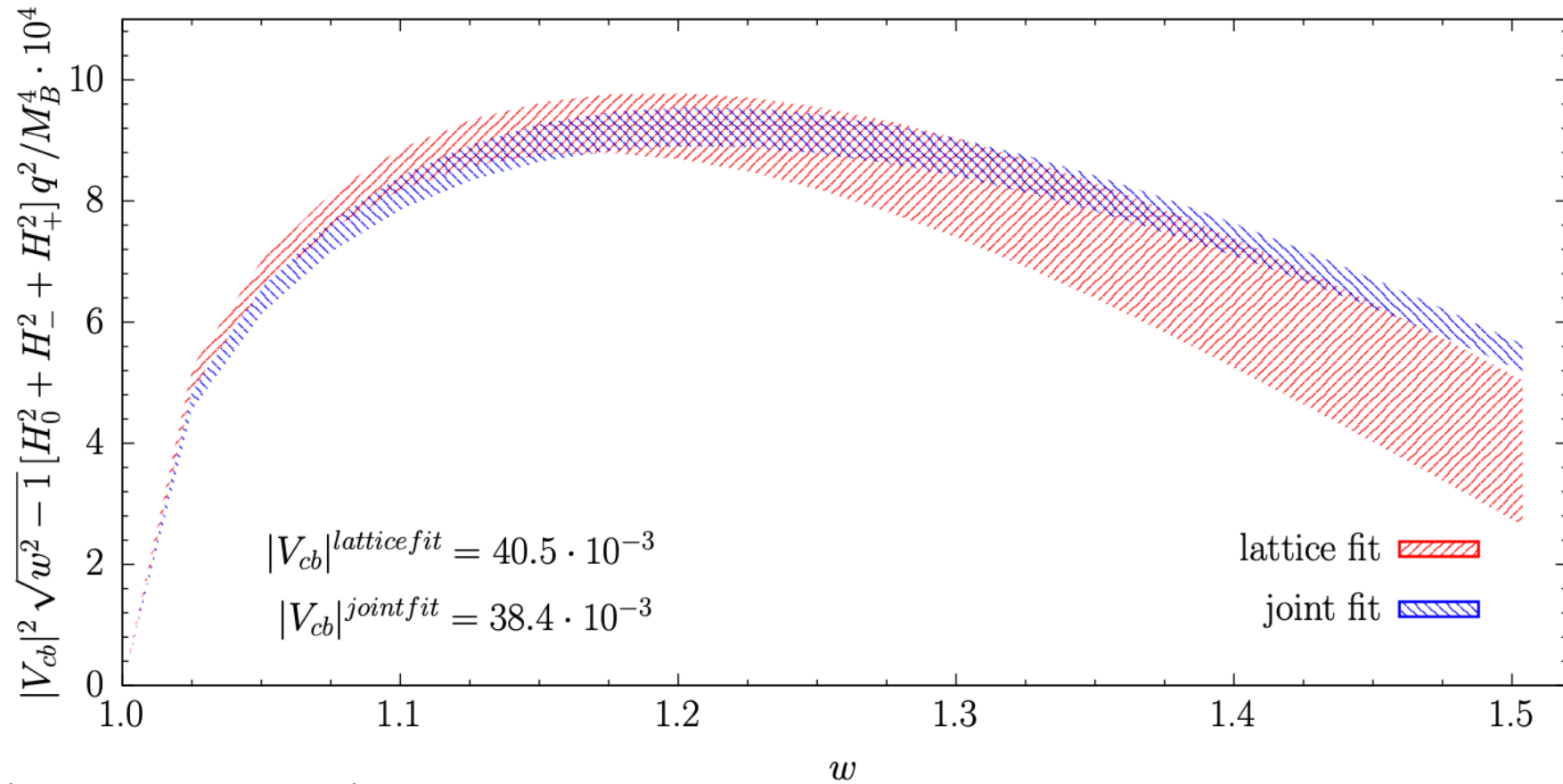
Future perspectives for LQCD data

JLQCD (prelim.)
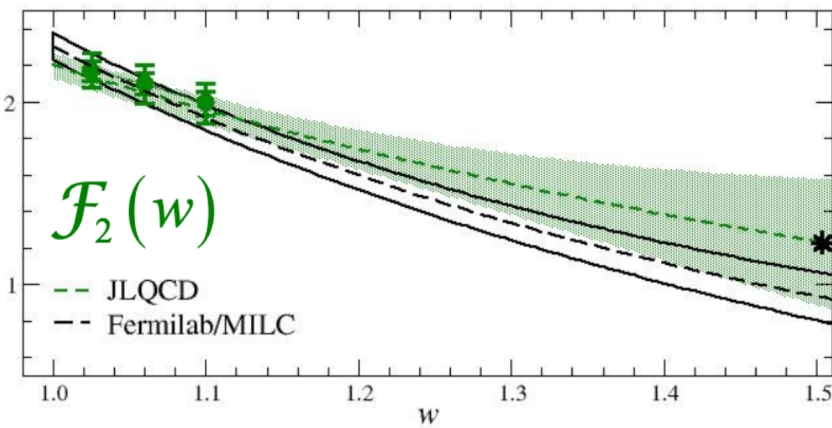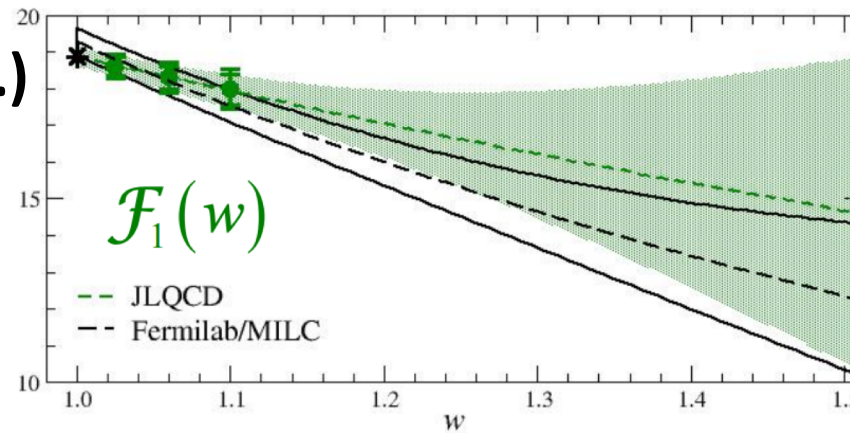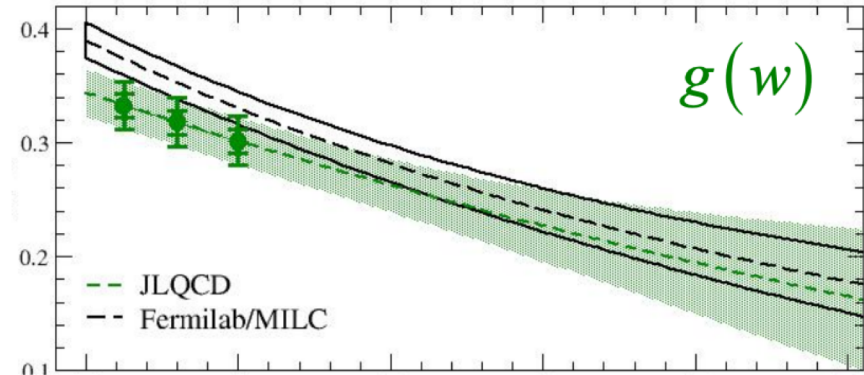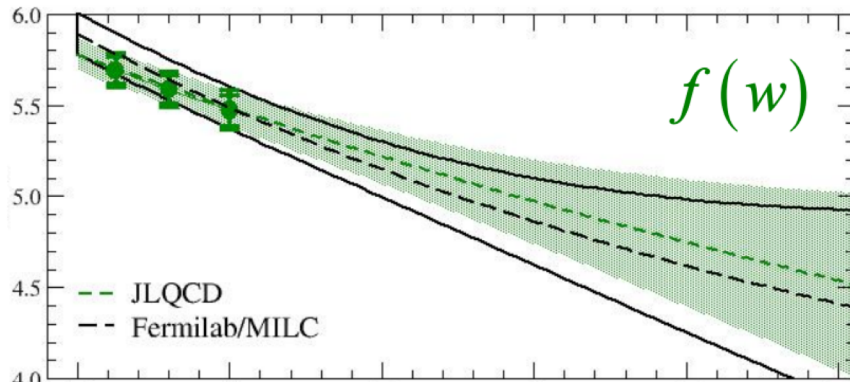
Kaneko's talk @ "Challenges in Semileptonic B decays 2022" Workshop
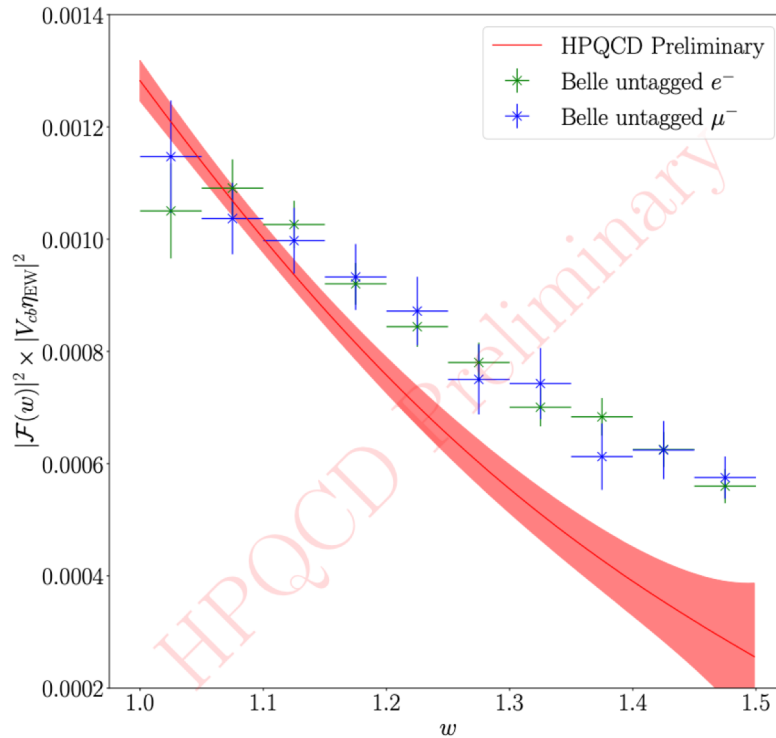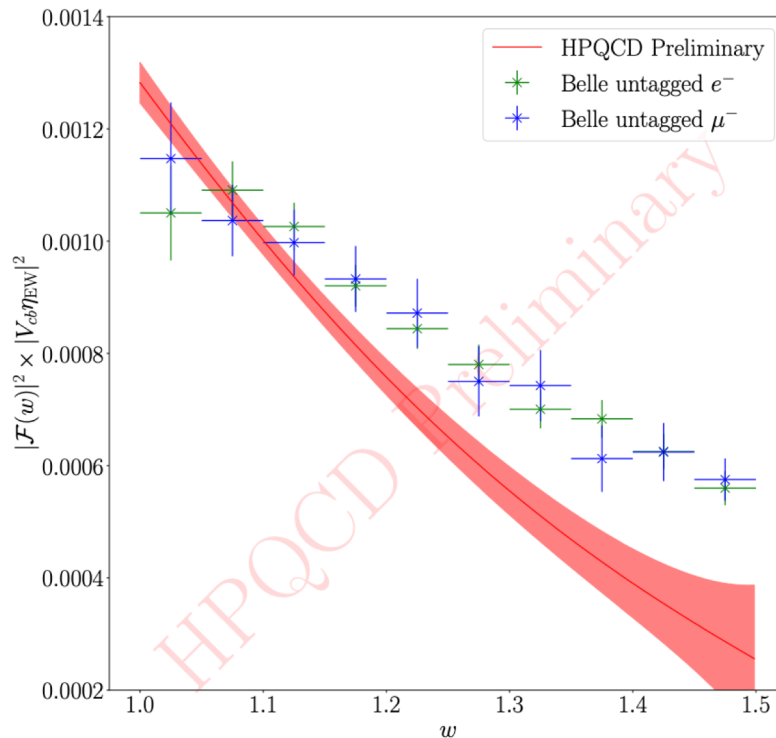
Future perspectives for LQCD data

HPQCD (prelim.)

FNAL/MILC

$$\frac{\mathrm{d}\Gamma(\bar{B} \to D^* \ell \, \bar{\nu})}{\mathrm{d}w} = \frac{G_F^2 |V_{cb}|^2}{48\pi^3} (m_B - m_{D^*})^2 \, m_{D^*}^3 \sqrt{w^2 - 1} \, (w+1)^2$$

$$\times \left[1 + \frac{4w}{w+1} \frac{m_B^2 - 2w \, m_B m_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2} \right] |\mathcal{F}(w)|^2$$

Future perspectives for LQCD data

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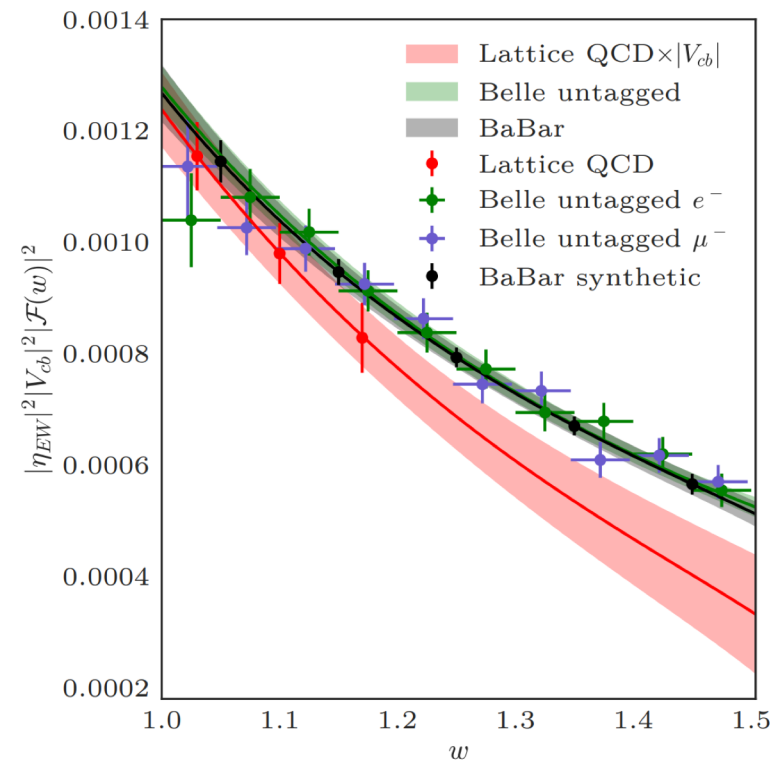
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CONCLUSION: FNAL/MILC and HPQCD have similar shape, which is different from Belle & different from JLQCD (which is affected by higher uncertainties...)

Vaquero's talk @ First LatticeNET Workshop (Benasque 2022)

L. Vittorio (LAPTh & CNRS, Annecy)

Summary of the DM study of LQCD and exps. data

According to our prescription, **the shape of the FFs have to be constrained by using _only_ the results of the LQCD computations** on the lattice. In this way:

- the estimate of R(D*) is fully-theoretical
- |Vcb| can be exctracted by a direct comparison with the experimental data, which do not introduce any bias

Remark 1

The value of $|V_{cb}|$ exhibits some dependence on the specific w-bin. The value obtained adopting a constant fit is dominated by the bins at small values of the recoil w, where direct lattice data are available and the lenght of the momentum extrapolation is limited.

Remark 2

The value of $|V_{cb}|$ deviates from a constant fit for $x = \cos(\theta_v)$. If we try a quadratic fit of the form

$$|V_{cb}| \left[1 + \delta B \cos^2(\theta_v) \right]$$

we get $\delta B \neq 0$ (2-3σ level) and $|V_{cb}|$ more consistent between the two sets of Belle data, but still in agreement with the value of $|V_{cb}|$ obtained with a constant fit

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OTHER DATA ARE FUNDAMENTAL!

Super-quick parenthesis: semileptonic $B_s \rightarrow D_s^{(*)}$ decays

$B_{(s)} \rightarrow D_{(s)} \ell \nu_\ell$

$B_{(s)} \rightarrow D^*_{(s)} \ell \nu_\ell$

red: u/d spectator quark

blue: strange spectator quark

Super-quick parenthesis: semileptonic $B_s \to D_s^{(*)}$ decays

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red: u/d spectator quark

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SU(3)$_F$ symmetry breaking effects?
Interesting question for the future!

The DM method: explicit expressions

To be more specific, one can demonstrate that:

LOWER bound $\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1 - z_j^2}{z - z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1 - z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1 - z_i^2)(1 - z_j^2)}{1 - z_i z_j} \right]$$

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The functions d_i are simply kinematical terms

The DM method: explicit expressions

To be more specific, one can demonstrate that:

PRD '21 (2105.02497)

LOWER bound $\qquad \boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}} \qquad$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1-z_j^2}{z-z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1-z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1-z_i^2)(1-z_j^2)}{1-z_i z_j} \right]$$

UNITARITY FILTER: **unitarity is satisfied if γ is semipositive definite, namely if**

$$\chi \geq \sum_{i,j=1}^{N} N f_i f_j \phi_i \phi_j d_i d_j \frac{(1-z_i^2)(1-z_j^2)}{1-z_i z_j}$$

The DM method: explicit expressions

To be more specific, one can demonstrate that:

LOWER bound $\boxed{\beta - \sqrt{\gamma}} \leq f(z) \leq \boxed{\beta + \sqrt{\gamma}}$ *UPPER bound*

$$\beta = \frac{1}{d(z)\phi(z)} \sum_{j=1}^{N} f_j \phi_j d_j \frac{1-z_j^2}{z-z_f} \qquad \gamma = \frac{1}{d^2(z)\phi^2(z)} \frac{1}{1-z^2} \left[\chi - \sum_{i,j=1}^{N} f_i f_j \phi_i \phi_j d_i d_j \frac{(1-z_i^2)(1-z_j^2)}{1-z_i z_j} \right]$$

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This is a **parametrization-independent unitarity test** of the LQCD input data

An instructive example: semileptonic D → K decays

The semileptonic D → K decays have a great advantage. A computation on the lattice of the FFs in the whole kinematical range has been performed by V. Lubicz et al in **PRD '17 [arXiv:1706.03017 [hep-lat]].**

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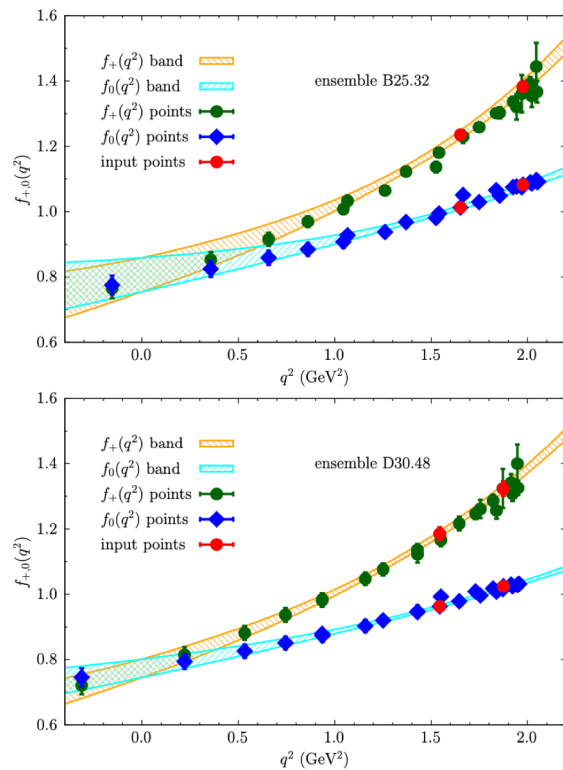
The figures show the bands obtained by using as inputs only the red points and the rest of the lattice points that are not used as input in our analysis in the case of the ETMC ensembles B25.32 and D30.48.

The red points are the only data used as input for the DM method!!

The agreement is excellent!

These results suggest that it will be possible to obtain quite precise determinations of the form factors for B decays by combining form factors at large q^2 with the non perturbative calculation of the susceptibilities.

L. Vittorio (LAPTh & CNRS, Annecy)

The Dispersive Matrix (DM) method

Let us examine the case of the production of a pseudoscalar meson (as for the $B \to D$ case). Supposing to have *n* LQCD data for the FFs at the quadratic momenta $\{t_1, \cdots, t_n\}$ (hereafter $t \equiv q^2$), we define

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

$$\langle h_1 | h_2 \rangle = \int_{|z|=1} \frac{dz}{2\pi i z} \bar{h}_1(z) h_2(z)$$

$$g_t(z) \equiv \frac{1}{1 - \bar{z}(t) z}$$

The conformal variable z is related to the momentum transfer as:

$$z(t) = \frac{\sqrt{\frac{t_+ - t}{t_+ - t_-}} - 1}{\sqrt{\frac{t_+ - t}{t_+ - t_-}} + 1}$$

$$t_\pm \equiv (m_B \pm m_D)^2$$

CENTRAL REQUIREMENT:

$$\det \mathbf{M} \geq 0$$

Two advantages:
1. *z is real*
2. *1-to-1 correspondence:*

$$[0, t_{max} = t_-] \Rightarrow [z_{max}, 0]$$

A lot of work in the past:
L. Lellouch, NPB, 479 (1996), p. 353-391
C. Bourrely, B. Machet, and E. de Rafael, NPB, 189 (1981), pp. 157 – 181
E. de Rafael and J. Taron, PRD, 50 (1994), p. 373-380

L. Vittorio (LAPTh & CNRS, Annecy)

The DM method

$$\mathbf{M} = \begin{pmatrix} \langle\phi f|\phi f\rangle & \langle\phi f|g_t\rangle & \langle\phi f|g_{t_1}\rangle & \cdots & \langle\phi f|g_{t_n}\rangle \\ \langle g_t|\phi f\rangle & \langle g_t|g_t\rangle & \langle g_t|g_{t_1}\rangle & \cdots & \langle g_t|g_{t_n}\rangle \\ \langle g_{t_1}|\phi f\rangle & \langle g_{t_1}|g_t\rangle & \langle g_{t_1}|g_{t_1}\rangle & \cdots & \langle g_{t_1}|g_{t_n}\rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n}|\phi f\rangle & \langle g_{t_n}|g_t\rangle & \langle g_{t_n}|g_{t_1}\rangle & \cdots & \langle g_{t_n}|g_{t_n}\rangle \end{pmatrix}$$

We also have to define the kinematical functions

$$\phi_0(z, Q^2) = \sqrt{\frac{2n_I}{3}}\sqrt{\frac{3t_+t_-}{4\pi}}\frac{1}{t_+ - t_-}\frac{1+z}{(1-z)^{5/2}}\left(\beta(0) + \frac{1+z}{1-z}\right)^{-2}\left(\beta(-Q^2) + \frac{1+z}{1-z}\right)^{-2},$$

$$\phi_+(z, Q^2) = \sqrt{\frac{2n_I}{3}}\sqrt{\frac{1}{\pi(t_+ - t_-)}}\frac{(1+z)^2}{(1-z)^{9/2}}\left(\beta(0) + \frac{1+z}{1-z}\right)^{-2}\left(\beta(-Q^2) + \frac{1+z}{1-z}\right)^{-3}, \quad \beta(t) \equiv \sqrt{\frac{t_+ - t}{t_+ - t_-}}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, @ *{t₁,...,tₙ}*: from Cauchy's theorem (for generic *m*)

$$\langle g_{t_m}|\phi f\rangle = \phi(t_m, Q^2)f(t_m)$$
LQCD data!

$$\langle g_{t_m}|g_{t_l}\rangle = \frac{1}{1 - \bar{z}(t_l)z(t_m)}$$

- non-perturbative values of the susceptibilities, since from the dispersion relations (calling Q^2 the Euclidean quadratic momentum)

$$\chi(Q^2) \geq \boxed{\langle\phi f|\phi f\rangle}$$

Since the susceptibilities are computed on the lattice, we can in principle use whatever value of Q^2 !

L. Vittorio (LAPTh & CNRS, Annecy)

The DM method

$$\mathbf{M} = \begin{pmatrix} \langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle \\ \langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle \\ \langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle \end{pmatrix}$$

In the presence of **poles @** $t_{P1}, t_{P2}, \cdots ..., t_{PN}$:

$$\phi(z, q^2) \to \phi_P(z, q^2) \equiv \phi(z, q^2) \times \frac{z - z(t_{P1})}{1 - \bar{z}(t_{P1})z} \times \cdots \times \frac{z - z(t_{PN})}{1 - \bar{z}(t_{PN})z}$$

Thus, we need **these external inputs** to implement our method:

- estimates of the FFs, computed on the lattice, @ *{t₁,...,tₙ}*: from Cauchy's theorem (for generic *m*)

$$\boxed{\langle g_{t_m} | \phi f \rangle} = \phi(t_m, Q^2) f(t_m)$$
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$$\chi(Q^2) \geq \boxed{\langle \phi f | \phi f \rangle}$$

L. Vittorio (LAPTh & CNRS, Annecy)

Statistical and systematic uncertainties

How can we finally combine all the N_U lower and upper bounds of both the FFs??

One bootstrap event case:

after a single extraction, we have one value of the lower bound f_L and one value of the upper one f_U for each FF. Assuming that the true value of each FF can be **everywhere inside the range $(f_U - f_L)$ with equal probability**, we associate to the FFs a *flat* distribution

$$P(f_{0(+)}) = \frac{1}{f_{U,0(+)} - f_{L,0(+)}} \Theta(f_{0(+)} - f_{L,0(+)}) \Theta(f_{U,0(+)} - f_{0(+)})$$

Many bootstrap events case:

how to mediate over the whole set of bootstrap events? Since the lower and the upper bounds of a generic FF are deeply correlated, we will assume a multivariate Gaussian distribution:

$$P(f_L, f_U) = \frac{\sqrt{\det \rho}}{2\pi} \exp\left[-\frac{\rho_{up,up}(f_U - \langle f_U \rangle)^2 + \rho_{lo,lo}(f_L - \langle f_L \rangle)^2 + 2\rho_{lo,up}(f_U - \langle f_U \rangle)(f_L - \langle f_L \rangle)}{2} \right]$$

In conclusion, we can combine the bounds of each FF in a final mean value and a final standard deviation, defined as

$$\langle f \rangle = \frac{\langle f_L \rangle + \langle f_U \rangle}{2},$$

NO PARAMETRIZATION ADOPTED!!!

$$\sigma_f = \frac{1}{12}(\langle f_U \rangle - \langle f_L \rangle)^2 + \frac{1}{3}(\sigma_{f_{lo}}^2 + \sigma_{f_{up}}^2 + \rho_{lo,up}\sigma_{f_{lo}}\sigma_{f_{up}})$$

L. Vittorio (LAPTh & CNRS, Annecy)

Kinematical Constraints (KCs)

REMINDER: after the unitarity filter we were left with *N_U < N survived events!!!*

Let us focus on the pseudoscalar case. Since by construction the following *kinematical constraint* holds

$$f_0(0) = f_+(0)$$

we will filter only the *N_{KC} < N_U* events for which the two bands of the FFs intersect each other @ *t = 0*.
Namely, for each of these events we also define

$$\phi_{lo} = \max[F_{+,lo}(t=0), F_{0,lo}(t=0)]$$

$$\phi_{up} = \min[F_{+,up}(t=0), F_{0,up}(t=0)]$$

From WE theorem

$$\langle D(p_D)| V^\mu |B(p_B)\rangle = f_+(p_B + p_D)^\mu + f_-(p_B - p_D)^\mu$$

One then defines

$$f_0(q^2) = f_+(q^2) + \frac{q^2}{m_B^2 - m_D^2} f_-(q^2)$$

$$\langle D(p_D)|V^\mu|B(p_B)\rangle = f^+(q^2)\left(p_B^\mu + p_D^\mu - \frac{m_B^2 - m_D^2}{q^2}q^\mu\right) + f^0(q^2)\frac{m_B^2 - m_D^2}{q^2}q^\mu$$

Kinematical Constraints (KCs)

We then consider a **modified matrix**

$$
\mathbf{M_C} =
\begin{pmatrix}
\langle \phi f | \phi f \rangle & \langle \phi f | g_t \rangle & \langle \phi f | g_{t_1} \rangle & \cdots & \langle \phi f | g_{t_n} \rangle & \langle \phi f | g_{t_{n+1}} \rangle \\
\langle g_t | \phi f \rangle & \langle g_t | g_t \rangle & \langle g_t | g_{t_1} \rangle & \cdots & \langle g_t | g_{t_n} \rangle & \langle g_t | g_{t_{n+1}} \rangle \\
\langle g_{t_1} | \phi f \rangle & \langle g_{t_1} | g_t \rangle & \langle g_{t_1} | g_{t_1} \rangle & \cdots & \langle g_{t_1} | g_{t_n} \rangle & \langle g_{t_1} | g_{t_{n+1}} \rangle \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\langle g_{t_n} | \phi f \rangle & \langle g_{t_n} | g_t \rangle & \langle g_{t_n} | g_{t_1} \rangle & \cdots & \langle g_{t_n} | g_{t_n} \rangle & \langle g_{t_n} | g_{t_{n+1}} \rangle \\
\langle g_{t_{n+1}} | \phi f \rangle & \langle g_{t_{n+1}} | g_t \rangle & \langle g_{t_{n+1}} | g_{t_1} \rangle & \cdots & \langle g_{t_{n+1}} | g_{t_n} \rangle & \langle g_{t_{n+1}} | g_{t_{n+1}} \rangle
\end{pmatrix}
$$

with $t_{n+1} = 0$. Hence, we compute the new lower and upper bounds of the FFs in this way. For each of the N_{KC} events, we extract $N_{KC,2}$ values of $f_0(0) = f_+(0) \equiv f(0)$ with uniform distribution defined in the range $[\phi_{lo}, \phi_{up}]$. Thus, for both the FFs and for each of the N_{KC} events we define

$$
\begin{aligned}
F_{lo}(t) &= \min[F_{lo}^1(t), F_{lo}^2(t), \cdots, F_{lo}^{N_{KC,2}}(t)], \\
F_{up}(t) &= \max[F_{up}^1(t), F_{up}^2(t), \cdots, F_{up}^{N_{KC,2}}(t)]
\end{aligned}
$$

L. Vittorio (LAPTh & CNRS, Annecy)

Non-perturbative computation of the susceptibilities

In **arXiv:2105.07851**, we have presented the results of the first computation on the lattice of the susceptibilities for the $b \to c$ quark transition, using the N_f=2+1+1 gauge ensembles generated by ETM Collaboration.

How are they defined? The starting point is the HVP tensor:

$$\Pi_{\mu\nu}^{V}(Q) = \int d^4x \; e^{-iQ\cdot x} \langle 0|T\left[\bar{b}(x)\gamma_{\mu}^{E}c(x)\; \bar{c}(0)\gamma_{\nu}^{E}b(0)\right]|0\rangle$$

$$= -Q_{\mu}Q_{\nu}\Pi_{0+}(Q^2) + (\delta_{\mu\nu}Q^2 - Q_{\mu}Q_{\nu})\Pi_{1-}(Q^2)$$

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:

$$\chi_{0+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0+}(Q^2)\right] = \int_0^{\infty} dt \; t^2 j_0(Qt) \; C_{0+}(t) \,,$$

$$\chi_{1-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1-}(Q^2)\right] = \frac{1}{4}\int_0^{\infty} dt \; t^4 \frac{j_1(Qt)}{Qt} \; C_{1-}(t)$$

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$$C_{0+}(t) = \int d^3x \langle 0|T\left[\bar{b}(x)\gamma_0 c(x)\; \bar{c}(0)\gamma_0 b(0)\right]|0\rangle \,,$$

$$C_{1-}(t) = \frac{1}{3}\sum_{j=1}^{3}\int d^3x \langle 0|T\left[\bar{b}(x)\gamma_j c(x)\; \bar{c}(0)\gamma_j b(0)\right]|0\rangle \,,$$

$$C_{0-}(t) = \int d^3x \langle 0|T\left[\bar{b}(x)\gamma_0\gamma_5 c(x)\; \bar{c}(0)\gamma_0\gamma_5 b(0)\right]|0\rangle \,,$$

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$$\chi_{0+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0+}(Q^2)\right] = \int_0^\infty dt \; t^2 j_0(Qt) \; C_{0+}(t) , \quad \xrightarrow{\text{W. I.}} \quad \frac{1}{4}\int_0^\infty dt' \; t'^4 \; \frac{j_1(Qt')}{Qt'} \left[(m_b - m_c)^2 C_S(t') + Q^2 C_{0+}(t')\right]$$

$$\chi_{1-}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1-}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt \; t^4 \frac{j_1(Qt)}{Qt} \; C_{1-}(t)$$

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$$\chi_{1+}(Q^2) \equiv -\frac{1}{2}\frac{\partial^2}{\partial^2 Q^2}\left[Q^2\Pi_{1+}(Q^2)\right] = \frac{1}{4}\int_0^\infty dt \; t^4 \frac{j_1(Qt)}{Qt} \; C_{1+}(t)$$

L. Vittorio (LAPTh & CNRS, Annecy)

Non-perturbative computation of the susceptibilities

The possibility to compute the χs
on the lattice allows us
to choose *whatever value of Q² !!!!*
(i.e. near the region of production
of the resonances)

$\Longrightarrow$

NOT POSSIBLE IN PERTURBATION THEORY!!!

$$(m_b + m_c)\Lambda_{QCD} << (m_b + m_c)^2 - q^2$$

POSSIBLE IMPROVEMENT IN THE STUDY
OF THE FFs through our method!

Work in progress...

To compute the susceptibilities on the lattice, we start from the Euclidean correlators:
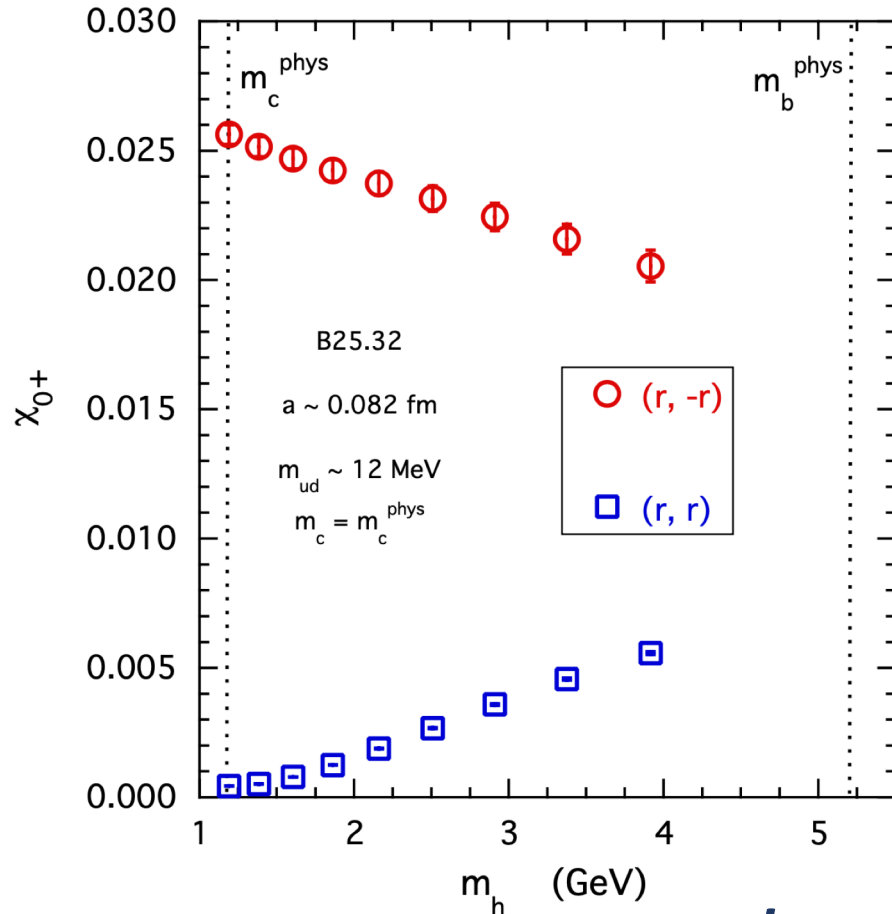
$$\chi_{0+}(Q^2) \equiv \frac{\partial}{\partial Q^2}\left[Q^2\Pi_{0+}(Q^2)\right] = \int_0^\infty dt\, t^2 j_0(Qt)\, C_{0+}(t) , \quad \xrightarrow{W.\,I.} \quad \frac{1}{4}\int_0^\infty dt'\, t'^4\, \frac{j_1(Qt')}{Qt'}\left[(m_b - m_c)^2 C_S(t') + Q^2 C_{0+}(t')\right]$$

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L. Vittorio (LAPTh & CNRS, Annecy)

Non-perturbative computation of the susceptibilities

Following set of masses:

$$m_h(n) = \lambda^{n-1}\, m_c^{phys} \qquad \text{for } n = 1, 2, \ldots$$

$$m_h = a\mu_h/(Z_P a)$$

$$\lambda \equiv [m_b^{phys}/m_c^{phys}]^{1/10} = [5.198/1.176]^{1/10} \simeq 1.1602.$$

Nine masses values!

$$m_h(1) = m_c^{phys}$$

$$m_h(9) \simeq 3.9\ \text{GeV} \simeq 0.75\, m_b^{phys}$$

r: **Wilson parameter**

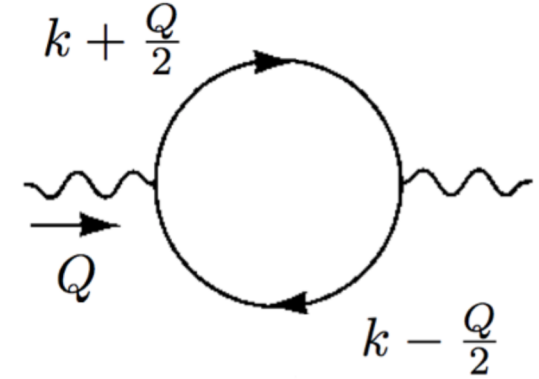***Large discretisation effects and contact terms***

L. Vittorio (LAPTh & CNRS, Annecy)

Contact terms & perturbative subtraction

In **twisted mass LQCD:**

$$\Pi_V^{\alpha\beta} = \int_{-\pi/a}^{+\pi/a} \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[\gamma^\alpha G_1(k + \frac{Q}{2})\gamma^\beta G_2(k - \frac{Q}{2})\right],$$

$$G_i(p) = \frac{-i\gamma_\mu \mathring{p}_\mu + \mathcal{M}_i(p) - ir_i\mu_{q,i}\gamma_5}{\mathring{p}_\mu^2 + \mathcal{M}_i^2(p) + \mu_{q,i}^2}$$

$$\mathring{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu), \qquad \mathcal{M}_i(p) \equiv m_i + \frac{r_i}{2}a\hat{p}_\mu^2, \qquad \hat{p} \equiv \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right).$$

$k + \frac{Q}{2}$

Q

$k - \frac{Q}{2}$

$$\Pi_V^{\alpha\beta} = a^{-2}(Z_1^I + (r_1^2 - r_2^2)Z_2^I + (r_1^2 - r_2^2)(r_1^2 + r_2^2)Z_3^I)g^{\alpha\beta}$$

$$+ (\mu_1^2 Z^{\mu_1^2} + \mu_2^2 Z^{\mu_2^2} + \mu_1\mu_2 Z^{\mu_1\mu_2})g^{\alpha\beta} + (Z_1^{Q^2} + (r_1^2 - r_2^2)Z_2^{Q^2})Q \cdot Q g^{\alpha\beta}$$

$$+ (Z_1^{Q^\alpha Q^\beta} + (r_1^2 - r_2^2)Z_2^{Q^\alpha Q^\beta})Q^\alpha Q^\beta + r_1 r_2 (a^{-2}Z_1^{r_1 r_2}g^{\alpha\beta} + (Z_2^{r_1 r_2} + (r_1^2 + r_2^2)Z_3^{r_1 r_2}$$

$$+ (r_1^4 + r_2^4)Z_4^{r_1 r_2})Q \cdot Q g^{\alpha\beta} + (\mu_1^2 Z_5^{r_1 r_2} + \mu_2^2 Z_6^{r_1 r_2})g^{\alpha\beta}) + O(a^2), \qquad \textit{CONTACT TERMS!!!}$$

F. Burger et al., ETM Coll., JHEP '15 [arXiv:1412.0546]

L. Vittorio (LAPTh & CNRS, Annecy)

Contact terms & perturbative subtraction

In **twisted mass LQCD** (tmLQCD):

$$\Pi_V^{\alpha\beta} = \int_{-\pi/a}^{+\pi/a} \frac{d^4k}{(2\pi)^4} \, \text{Tr}\left[\gamma^\alpha G_1(k + \frac{Q}{2})\gamma^\beta G_2(k - \frac{Q}{2})\right],$$

Thus, by separating the *longitudinal* and the *transverse* contributions, we can compute the susceptibilities for all the spin-parity quantum numbers **in the free theory on the lattice**, *i.e.* at order $\mathcal{O}(\alpha_s^0)$ using twisted-mass fermions!

$k + \frac{Q}{2}$

Q

$k - \frac{Q}{2}$
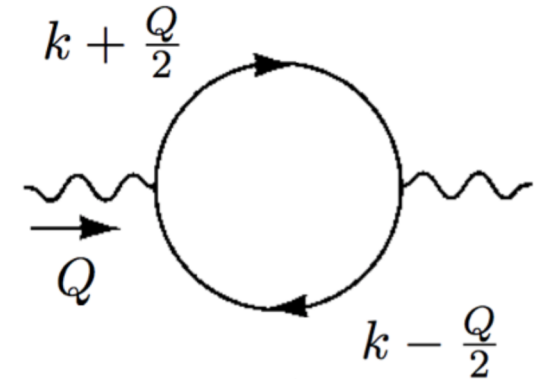
$$\chi_j^{free} = \boxed{\chi_j^{LO}} + \boxed{\chi_j^{discr}}$$
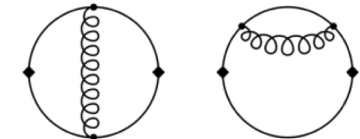
LO term of PT @ $\mathcal{O}(\alpha_s^0)$ contact terms and discretization effects @ $\mathcal{O}(\alpha_s^0 a^m)$ with $m \geq 0$
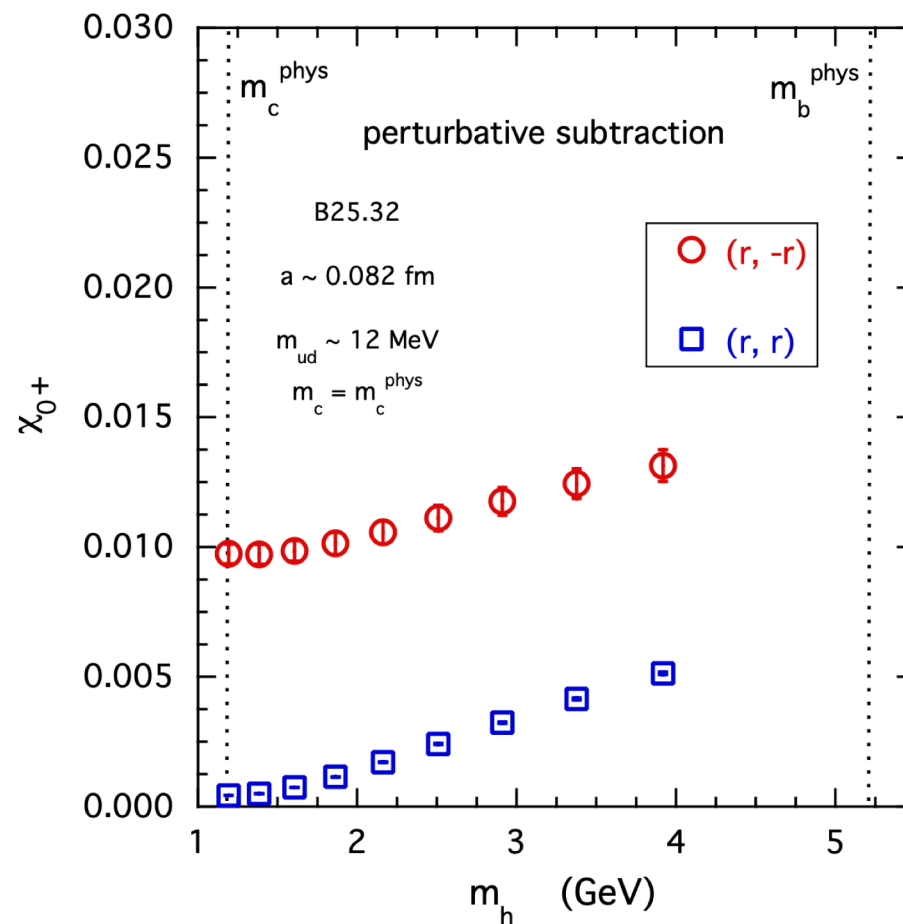
Perturbative subtraction:

Higher order corrections?

$$\chi_j \rightarrow \chi_j - \left[\chi_j^{free} - \chi_j^{LO}\right]$$

Work in progress...

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Contact terms & perturbative subtraction

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Contact terms & perturbative subtraction

ETMC ratio method & final results

For the extrapolation to the physical *b*-quark point we have used the ETMC ratio method:

$$R_j(n; a^2, m_{ud}) \equiv \frac{\chi_j[m_h(n); a^2, m_{ud}]}{\chi_j[m_h(n-1); a^2, m_{ud}]} \boxed{\frac{\rho_j[m_h(n)]}{\rho_j[m_h(n-1)]}} \implies \boxed{\begin{array}{l} \rho_{0+}(m_h) = \rho_{0-}(m_h) = 1 , \\[4pt] \rho_{1-}(m_h) = \rho_{1+}(m_h) = (m_h^{pole})^2 \end{array}}$$

to ensure that $\lim_{n \to \infty} R_j(n) = 1$

All the details are deeply discussed in *arXiv:2105.07851*. In this way, we have obtained **the first lattice QCD determination of susceptibilities of heavy-to-heavy (and heavy-to-light,** *in prep.***) transition current densities:**

<div align="center">

$b \to c$ $b \to u$

</div>

	Perturbative	With subtraction	Non-perturbative	With subtraction	Non-perturbative	With subtraction
$\chi_{V_L} [10^{-3}]$	6.204(81)	—	7.58(59)	—	2.04(20)	—
$\chi_{A_L} [10^{-3}]$	24.1	19.4	25.8(1.7)	21.9(1.9)	2.34(13)	—
$\chi_{V_T} [10^{-4} \text{ GeV}^{-2}]$	6.486(48)	5.131(48)	6.72(41)	5.88(44)	4.88(1.16)	4.45(1.16)
$\chi_{A_T} [10^{-4} \text{ GeV}^{-2}]$	3.894	—	4.69(30)	—	4.65(1.02)	—

Bigi, Gambino PRD '16
Bigi, Gambino, Schacht PLB '17
Bigi, Gambino, Schacht JHEP '17

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$$\boxed{\begin{aligned}\rho_{0^+}(m_h) &= \rho_{0^-}(m_h) = 1, \\ \rho_{1^-}(m_h) &= \rho_{1^+}(m_h) = (m_h^{pole})^2\end{aligned}}$$

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Differences with PT? **~4% for 1⁻, ~7% for 0⁻, ~20 % for 0⁺ and 1⁺**

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