Love and EMRIs in SPA*

Gabriel Andres Piovano, Andrea Maselli and Paolo Pani

Sapienza, University of Rome & INFN Rome & University College Dublin

LIDA: from classical methods to machine learning, Toulouse
21-25 November 2022

*Based on arXiv:2207.07452

Intro

- EMRIs: capture of stellar-mass objects by a supermassive BH
- Primary targets of future space-borne observatory (LISA)
- Binary properties can be measured with great accuracy
- Response to tidal field reveals structure of a body

Tidal deformability is zero for a BH...

... while it is tiny but non-zero for an exotic object

All tidal effects can be encoded in the Tidal Love Numbers k

<u>Key question:</u>

can we measure k of a supermassive exotic object in EMRIs?

We performed a parameter estimation with Fisher matrix approach using the Stationary Phase Approximation (SPA) plus a few tricks

Outline of the presentation

1. Modeling the inspiral
2. Parameter estimation with the SPA
3. Conclusions and future perspective

Modeling the inspiral

Tidal response in the extreme-mass-ratio limits

For $q = \mu/M \ll 1$, the (dimensionless) tidal deformabilities are

$$\lambda_1 = \frac{2}{3} k_1 \quad \text{primary} \qquad \lambda_2 = \frac{2}{3} q^5 k_2 \quad \text{secondary}$$

k_1 enters at adiabatic order in the gravitational wave phase[1]

TLN binding energy at 6PN

$$E_{TLN} = \frac{1}{2}\left(\frac{6}{r^6} + \frac{88}{3}\frac{1}{r^7}\right) k_1$$

TLN flux at 6PN

$$\mathcal{F}_{TLN}(r) = \frac{128}{5}\left(\frac{1}{r^{10}} - \frac{22}{21}\frac{1}{r^{11}}\right) k_1$$

[1] Pani and Maselli, 2019

Radiation reaction equations

We consider circular, equatorial orbits with spins aligned.

Small corrections due secondary spin χ and TLN k_1

$$E = E^0 + \sigma E^1 + E_{TLN} \qquad \Omega(r) = \Omega^0(r) + \sigma\Omega^1(r)$$

with $\sigma := S/(\mu M) = \chi q$.

Gravitational wave fluxes

$$\mathcal{F}(r,\Omega) = \mathcal{F}^0(r,\Omega^0) + \mathcal{F}_{TLN}(r) + \sigma\mathcal{F}^1(r,\Omega^0,\Omega^1)$$

$\mathcal{F}^0$ and $\mathcal{F}^1$ computed with Teukolsky formalism[2]

Evolution equations (expanded in q)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -q\mathcal{F}(r,\Omega)\left(\frac{\mathrm{d}E}{\mathrm{d}r}\right)^{-1} \qquad \frac{\mathrm{d}\Phi_{\mathrm{GW}}}{\mathrm{d}t} = 2\Omega(r(t))$$

[2]Piovano+,2021

Parameter estimation with the SPA

Error estimate with Fisher Information matrices

In the high SNR limit

$$\Sigma_{ij} = (\Gamma^{-1})_{ij} \qquad \text{covariance matrix}$$

$$\Gamma_{ij} = \sum_{\alpha=I,II} \left(\frac{d\tilde{h}_\alpha}{dx^i} \middle| \frac{d\tilde{h}_\alpha}{dx^j} \right)_{\vec{x}=\vec{x}_0} \qquad \text{Fisher matrix}$$

$$(p_\alpha|q_\alpha) = 4\mathrm{Re}\int_{f_{\min}}^{f_{\max}} \frac{df}{S_n(f)} \tilde{p}^*_\alpha(f)\tilde{q}_\alpha(f) \qquad \text{scalar product}$$

12 parameters for circular equatorial orbits with spins (anti)aligned

- intrinsic: masses $(\ln\mu, \ln M)$, spins (a, χ), t_0, k_1
- extrinsic: ϕ_0, angles $(\vartheta_S, \varphi_S, \vartheta_K, \varphi_K)$, distance $\ln D$

Γ_{ij} is ill-conditioned: small error in $\Gamma_{ij} \implies$ large error in Σ_{ij}

Problem exacerbated by computing $d\tilde{h}_\alpha/dx^i$ with finite difference methods

Frequency domain waveform with the SPA

Applicable for strictly monotonic frequency $\Omega(t)$

$$\tilde{h}_\alpha(f) = \frac{\mu}{D}(\pi f M)^{2/3}\mathcal{A}_\alpha(\tilde{t}(f))\sqrt{\frac{\pi}{|\dot{\Omega}(\tilde{t}(f))|}}e^{-i(\tilde{\Phi}_\alpha(\tilde{t}(f))\pm\pi/4)}$$

$$\tilde{\Phi}_\alpha(\tilde{t}(f)) = 2\pi f(\tilde{t}(f) - t_0) - 2(\phi(\tilde{t}(f)) - \phi_0) - \phi^{\mathrm{Dop}}(\tilde{t}(f)) - \phi^{\mathrm{sh}}_\alpha(\tilde{t}(f))$$

$\alpha = I, II$ for the two LISA channels (in *long-wavelength approx*)

$$\phi^{\mathrm{sh}}_\alpha(t) = \arctan\left(-\frac{A^\times_\alpha(t)}{A^+_\alpha(t)}\right) \qquad \mathcal{A}_\alpha(t) = \sqrt{(A^+_\alpha(t))^2 + (A^\times_\alpha(t))^2}$$

$$A^+_\alpha(t) = (1+\cos^2\vartheta)F^+_\alpha(t) \qquad A^\times_\alpha(t) = -2\cos\vartheta F^\times_\alpha(t)$$

$\tilde{t}(f)$ is solution of

$$\Omega(t) = \pi f$$

Semi-analytic derivatives of waveforms with the SPA

Functions $\tilde{t}(f;\vec{x})$, $\phi(\tilde{t}(f;\vec{x});\vec{x})$ and $r(\tilde{t};\vec{x})$ depends implicitly on parameters $\vec{x} = (\ln M, \ln \mu, a, \chi, k_1)$ for SPA waveforms

By the theorem of the implicit functions

$$\frac{\partial \tilde{t}(f;\vec{x})}{\partial x^i} = -\frac{1}{\dot{\Omega}(t;\vec{x})} \frac{\partial \Omega(t;\vec{x})}{\partial x^i}\bigg|_{t=\tilde{t}(f;\vec{x})}$$

$\phi(\tilde{t}(f;\vec{x});\vec{x})$ and $r(\tilde{t};\vec{x})$ are solutions of ODEs like

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathscr{G}(t, y; \alpha) \qquad y(0) = y_0$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial y(t;\alpha)}{\partial \alpha}\right) = \frac{\partial \mathscr{G}(t,y,\alpha)}{\partial y}\frac{\partial y(t;\alpha)}{\partial \alpha} + \frac{\partial \mathscr{G}(t,y,\alpha)}{\partial \alpha} \qquad \frac{\partial y(0;\alpha)}{\partial \alpha} = 0$$

Testing the SPA waveforms

$$\mathscr{F}(h_\alpha^{\rm SPA}, h_\alpha^{\rm FFT}) = \max_{t_0,\phi_0} \frac{(h_\alpha^{\rm SPA}|h_\alpha^{\rm FFT})}{\sqrt{(h_\alpha^{\rm SPA}|h_\alpha^{\rm SPA})(h^{\rm FFT}|h^{\rm FFT})}}$$

$\hat{a}$	channel	$\mathscr{F}$
0.9	I	0.9931
	II	0.9970
0.99	I	0.9942
	II	0.9971

Table: Faithfulness $\mathscr{F}$ between FFT $h_\alpha^{\rm FFT}$ and SPA $h_\alpha^{\rm SPA}$ waveforms

$\mathscr{F}$ better than 0.993 for 1 year observation down to the Kerr ISCO[3]

$(1 - \sigma_{\rm SPA}/\sigma_{\rm FFT})$ of FFT and SPA standard deviations in the worst cases:

- no prior for χ: 15%
- prior for χ: 2%

[3] In agreement with Hughes+,2021

Fisher-matrix errors for the primary TLN

$M = 10^6 M_\odot, \mu = 10 M_\odot, \chi = 0, k_1 = 0$. Sky location fixed.

a/M	prior	$\ln M$	$\ln \mu$	a/M	χ	t_0/M	k_1	ϕ_0
0.9	no	-4.9	-4.1	-3.8	1.6	0.48	-1.5	0.74
	yes	-5.7	-4.2	-4.1	0.57	0.48	-1.6	0.74
0.99	no	-5.2	-4.6	-4.4	1.2	0.21	-2.7	0.74
	yes	-5.7	-4.8	-4.9	0.61	0.21	-3.1	0.74

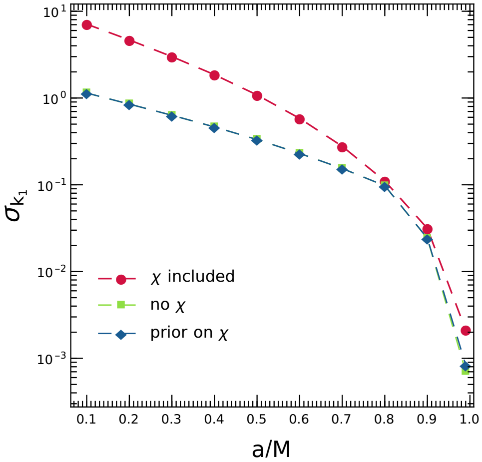
Table: $\log_{10}$ of the errors with and without imposing a prior on χ.

a/M	prior	$\ln D$	$\Delta\Omega_S$	$\Delta\Omega_K$
0.9	no	-0.069	6.2×10^{-4}	7.5
	yes	-0.069	5.9×10^{-4}	7.5
0.99	no	-0.071	2.7×10^{-4}	6.7
	yes	-0.071	2.7×10^{-4}	6.7

Table: errors on $\ln D$ and sky location of source ($\Delta\Omega_S$) and primary spin ($\Delta\Omega_K$).

TLN of the primary is measurable

Probing corrections at the horizon scale



Error on TLN k_1 for primary spins $a = 0.1, 0.2 \ldots 0.9, 0.99$

distance δ of exotic compact object surface from its Schwarzschild radius [1]

$$\delta \sim 2Me^{-1/k_1}$$

Valid for e.g. a wormhole or a gravastar

For $M \sim 10^6 M_\odot$ and $k \sim 5 \times 10^{-3}$ $\implies \delta \sim \ell_{\text{Planck}}$

Potentially achievable with $a = 0.99$!

[1] Maselli+,2018 and 2019

Conclusions and future perspective

Conclusions

We estimated the error on the TLN k_1 of a supermassive object in EMRIs with a Fisher matrix approach and kludge waveforms

Summary:

- k_1 can be measured with great accuracy
- Potential to distinguish supermassive BH from a class of ECOs
- Probe Planck correction at the horizon for fast spinning primaries?

But there are some caveats:

- Focused on circular, equatorial orbits with spins aligned
- Neglected important self-force effects
- Assumed the PN terms of the TLN are still valid in strong regime
- How general is relation between the TLN and δ?[4]

[4] Datta, 2022

Future work

- Consider general orbits (eccentric and inclined)
- Include post-adiabatic corrections
- Test quadrupole deviations together with TLN[5]
- Use our approach for Fisher matrices to do parameter estimation with second order SF waveform[6]

Most importantly

- Learn how to use FEW[7] and play with it
- Speed-up everything in my life with GPUs and ML!

[5] Barack and Cutler,2006
[6] Wardell+,2021
[7] Katz+,2021

Final notes and acknowledgments

- code and data will be soon available at the GitHub repository https://web.uniroma1.it/gmunu/resources
- this work makes use of the BHPToolkit https://bhptoolkit.org/
- Feel free to contact me at gabriel.piovano@ucd.ie

Thank you for you attention!

Backup slides

Gaussian prior

In the high SNR limit, the posterior distribution is approximated by a Gaussian distribution

$$\log p(\vec{\theta}|s) \propto \log p_0(\vec{\theta}) - \frac{1}{2}(\vec{\theta} - \vec{\theta}_{\rm true})^t \Gamma (\vec{\theta} - \vec{\theta}_{\rm true}) \equiv f(\vec{\theta})$$

Gaussian prior on χ, with $\sigma_{0,\chi} = 1$ and centered in $\chi = 0$

$$\log p_0(\vec{\theta}) = -\frac{1}{2}\chi^2 \delta_{\chi i} \quad \text{with } \delta_{\chi i} \text{ delta Kronecker}$$

We can then rewrite $f(\vec{\theta})$ as single quadratic form plus a constant term

$$f(\vec{\theta}) = -\frac{1}{2}\,(\vec{\theta} - \Gamma_{\rm pos}^{-1}\vec{b})^t \Gamma_{\rm pos} (\vec{\theta} - \Gamma_{\rm pos}^{-1}\vec{b}) + R$$

where

$$\Gamma_{\rm pos} = \Gamma + \delta_{\chi i} \qquad \vec{b} = \vec{\theta}_{\rm true}\Gamma \qquad R = \frac{1}{2}\left(\vec{b}^t \Gamma_{\rm pos}^{-1}\vec{b} - \vec{\theta}_{\rm true}\Gamma\vec{\theta}_{\rm true}\right)$$

Radiation reaction effects and balance laws

GW fluxes and waveforms computed with the Teukolsky formalism

$$\sigma := \frac{S}{\mu M} = \frac{S}{\mu^2} q = \chi q \qquad \text{with } q \ll \sigma \ll 1 \text{ in EMRIs}$$

At first order in the secondary spin[8]

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{GW}} = -\frac{\mathrm{d}E}{\mathrm{d}t} \qquad \left(\frac{\mathrm{d}J_z}{\mathrm{d}t}\right)_{\mathrm{GW}} = -\frac{\mathrm{d}J_z}{\mathrm{d}t}$$

Technical assumptions:

- no radiation reaction on $S^{\mu\nu}$ and assumed $S = cost$
- orbits remain circular and equatorial

We employed the numerical routines of the BHPToolkit [9]

[8] Akcay+, 2019

[9] Black Hole Perturbation Toolkit https://bhptoolkit.org