The impact data gaps have on parameter estimation

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Why care?

The data stream

data = Signal + Noise

Signal → Deterministic

Noise → Probabilistic

Probabilistic quantity $\Longleftrightarrow$ Probabilistic Models

The Whittle likelihood

- Stationary, Gaussian noise: $\implies \Sigma_n(f, f') = \frac{1}{2}\delta(f - f')S_n(f')$.
- $\log \mathscr{L} \propto -(\hat{\boldsymbol{d}} - \hat{\boldsymbol{h}})^{\dagger}\Sigma_n^{-1}(\hat{\boldsymbol{d}} - \hat{\boldsymbol{h}})^{\dagger} \approx -2\Delta f \sum_i \frac{|\hat{d}(f_i) - \hat{h}(f_i; \boldsymbol{\theta})|^2}{S_n(f_i)}$
- **Reality strikes**: Data gaps, noise artefacts (glitches), instrumentation malfunctions, **aliens** etc.

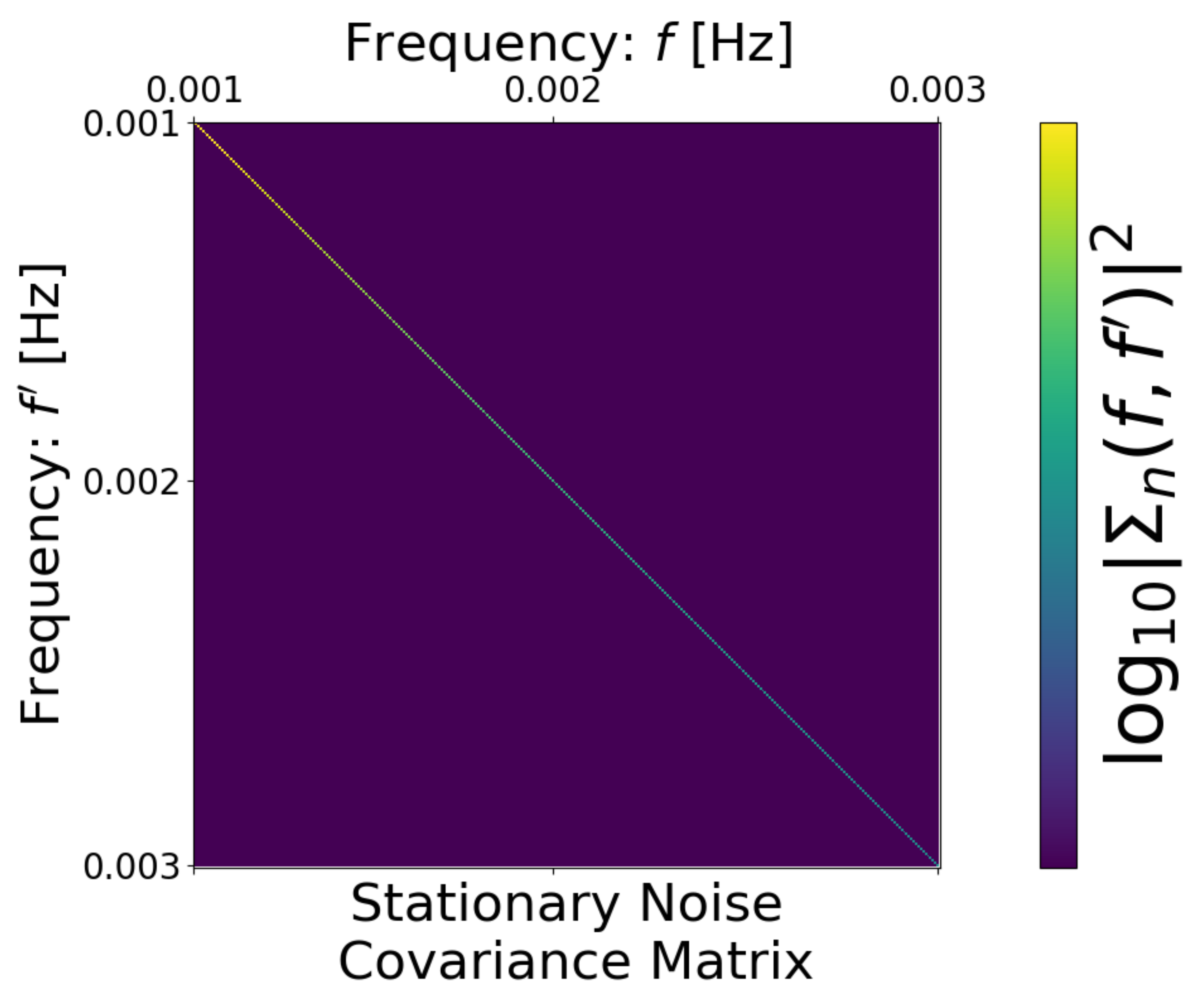
These tamper with $n(t)$ and **must** be accounted for

Modifying the noise

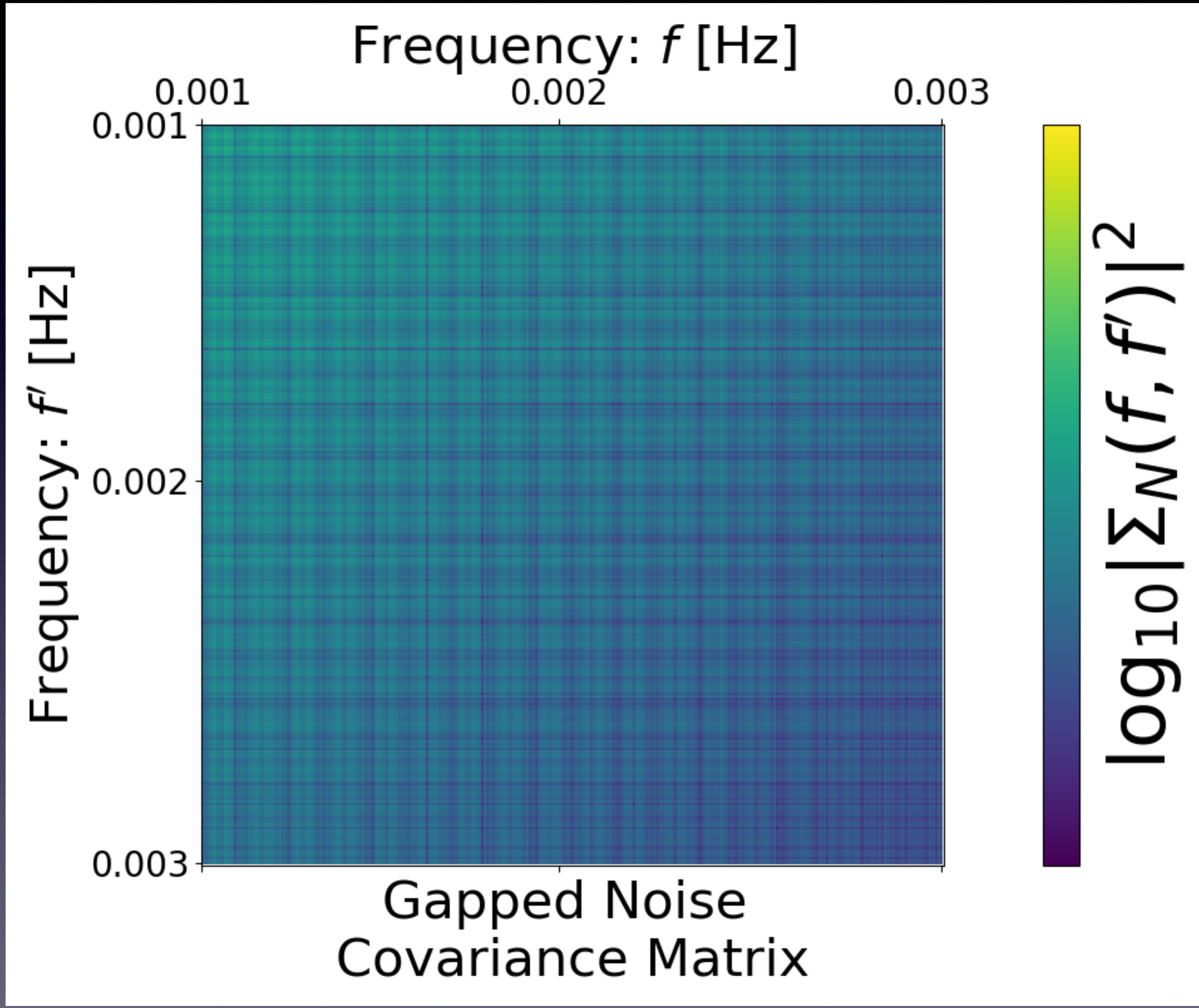
- Stationary case: $\Sigma_n(f,f') = \langle \hat{n}(f)\hat{n}^\star(f')\rangle = \frac{1}{2}\delta(f-f')S_n(f')$.
- Consider: $N(t) = w(t)n(t)$, assume $n(t)$ stationary, gaussian.
- $\Sigma_N(f,f') = \langle \tilde{N}(f)\tilde{N}^\star(f')\rangle = \frac{1}{2}\int_0^\infty \tilde{w}(f-u)\tilde{w}^\star(f'-u)S_n(u)\ du$
- **Non - diagonal** covariance $\Longrightarrow$ correlations between frequencies

Talbot *et al*, arXiv:2106.13785v2

Noise Covariance Matrix



30 hour stationary LISA noise



30 hour stationary LISA noise
With 15 hour gap applied

A simple case study

- Gapped data steam: $D(t) = w(t)h(t;\boldsymbol{\theta}) + w(t)n(t) = H(t;\boldsymbol{\theta}) + N(t)$
- Consider $h(t;a) = a \cdot 10^{-20}\sin(2\pi \cdot f(t) \cdot t), \quad f(t) = 10^{-5}\sqrt{t}$
- Time series 30 hours long with 15 hour gap in the middle.
- Likelihood: $\log \mathscr{L}(D\,|\,a,\Sigma) = -\,(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})^{\dagger}\Sigma^{-1}(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})$

$$\underbrace{\Sigma := \Sigma_N}_{\text{Gaps}} \qquad \underbrace{\Sigma := \Sigma_n}_{\text{No Gaps}}$$

Correct modelling of the noise

Gapped Data stream:

$$D(t) = H(t) + N(t)$$

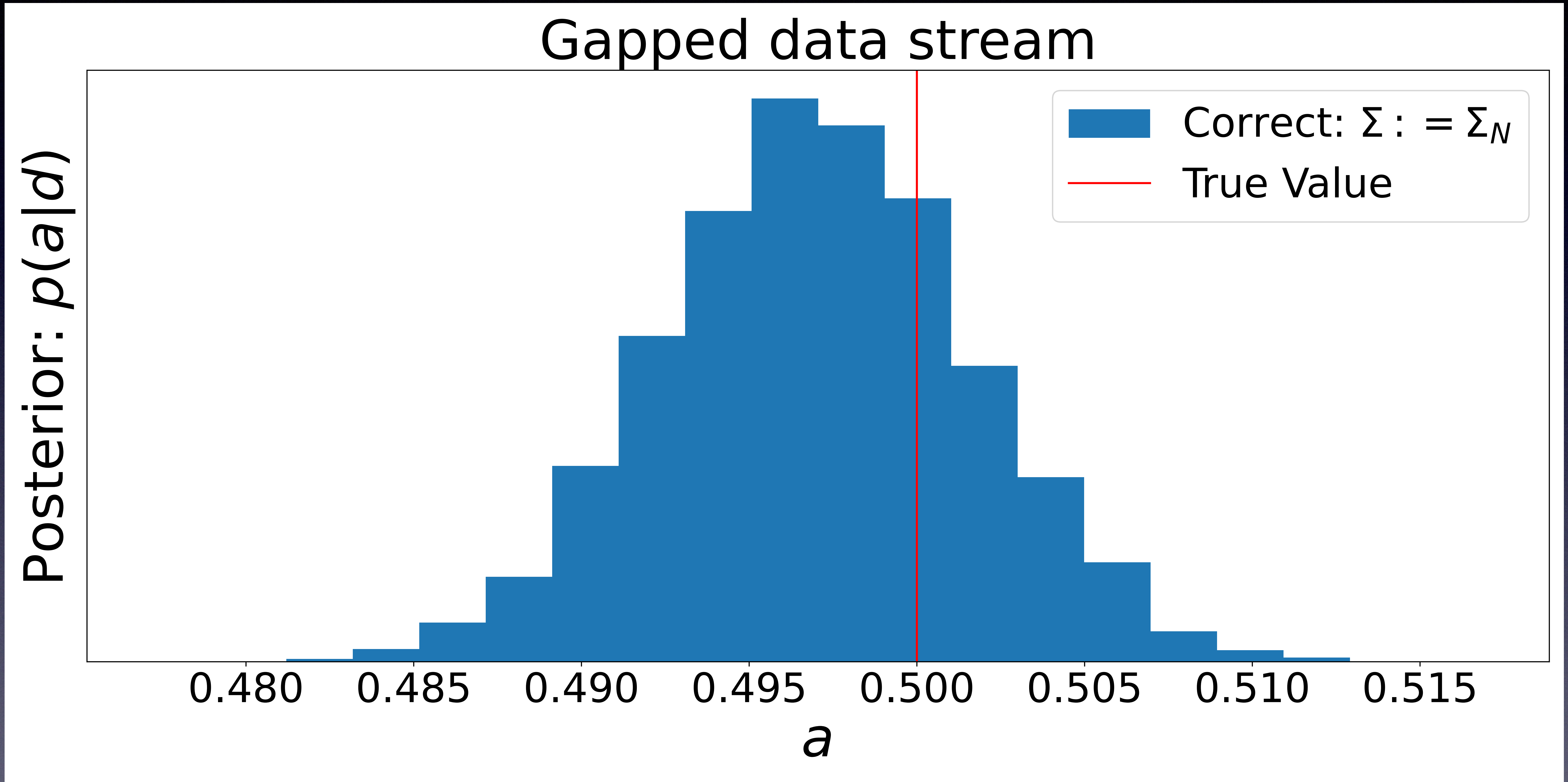
$$\log p(D \mid a, \Sigma) = -(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})^{\dagger} \underbrace{\Sigma_N^{-1}}_{\text{gaps}} (\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})$$

Correct modelling of the noise

Gapped Data stream:

$$D(t) = H(t) + N(t)$$

Blue Posterior : Correct model covariance



$$\log p(D\,|\,a,\Sigma) = -(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})^{\dagger} \underbrace{\Sigma_N^{-1}}_{\text{gaps}} (\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})$$

Incorrect modelling of the noise

Gapped Data stream:

$$D(t) = H(t) + N(t)$$

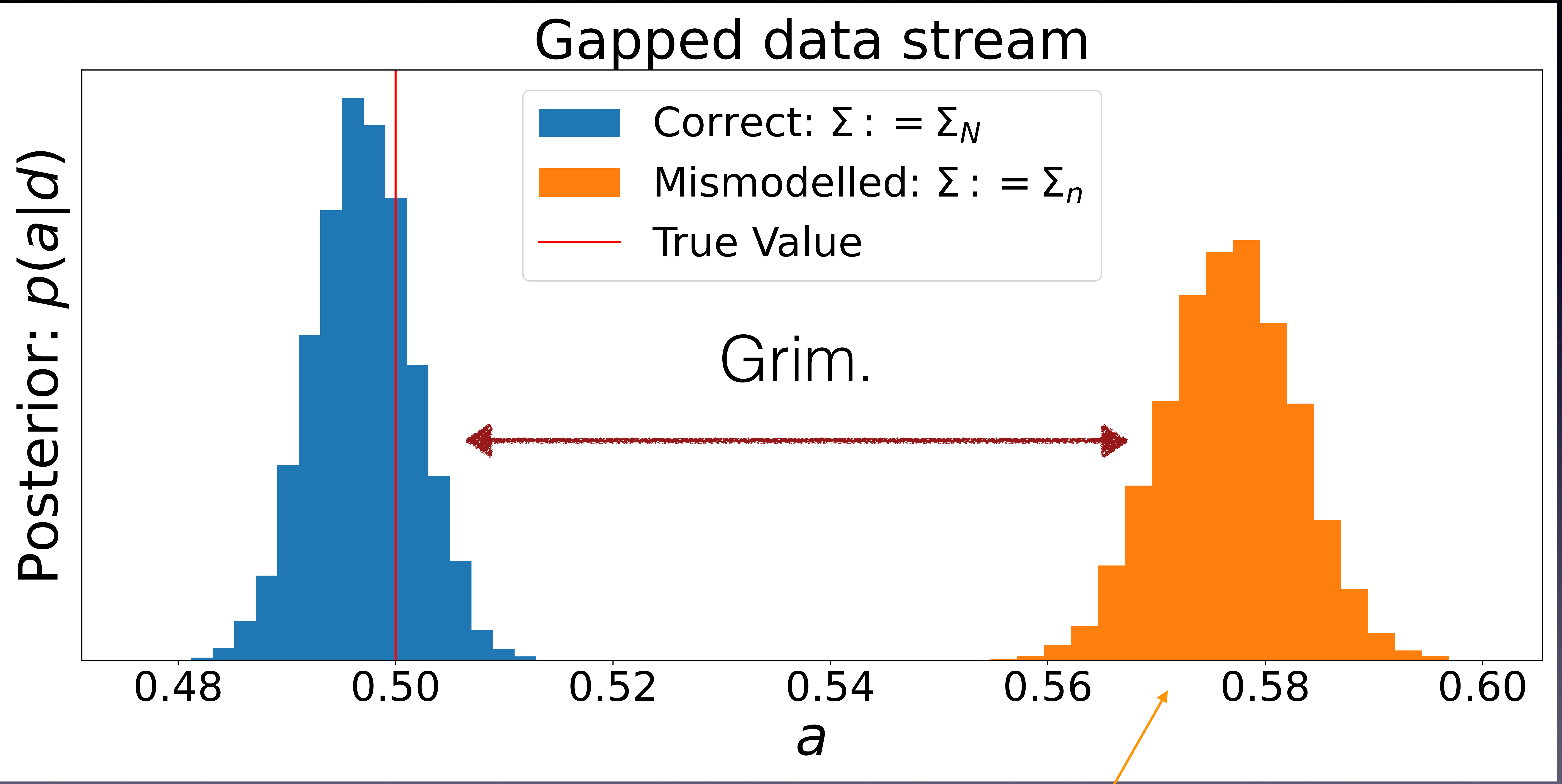
$$\log p(D \,|\, a, \Sigma) = -\,(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})^{\dagger} \underbrace{\Sigma_n^{-1}}_{\text{whittle}} (\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})$$

Incorrect modelling of the noise

Gapped Data stream:

$$D(t) = H(t) + N(t)$$

Orange Posterior : Incorrect model covariance



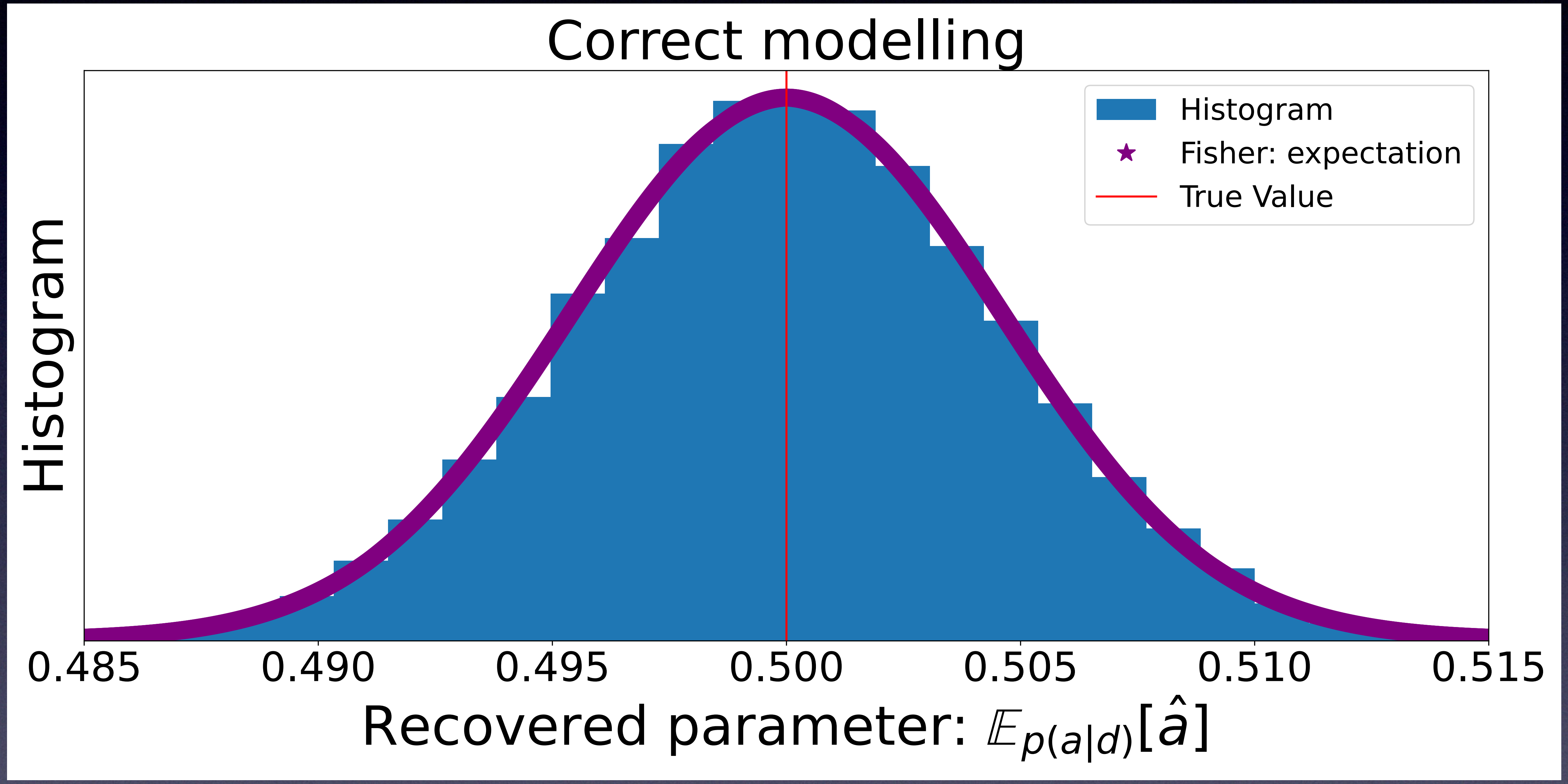
$$\log p(D\,|\,a, \Sigma) = -\,(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})^{\dagger}\, \underbrace{\Sigma_n^{-1}}_{\text{whittle}}\, (\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})$$

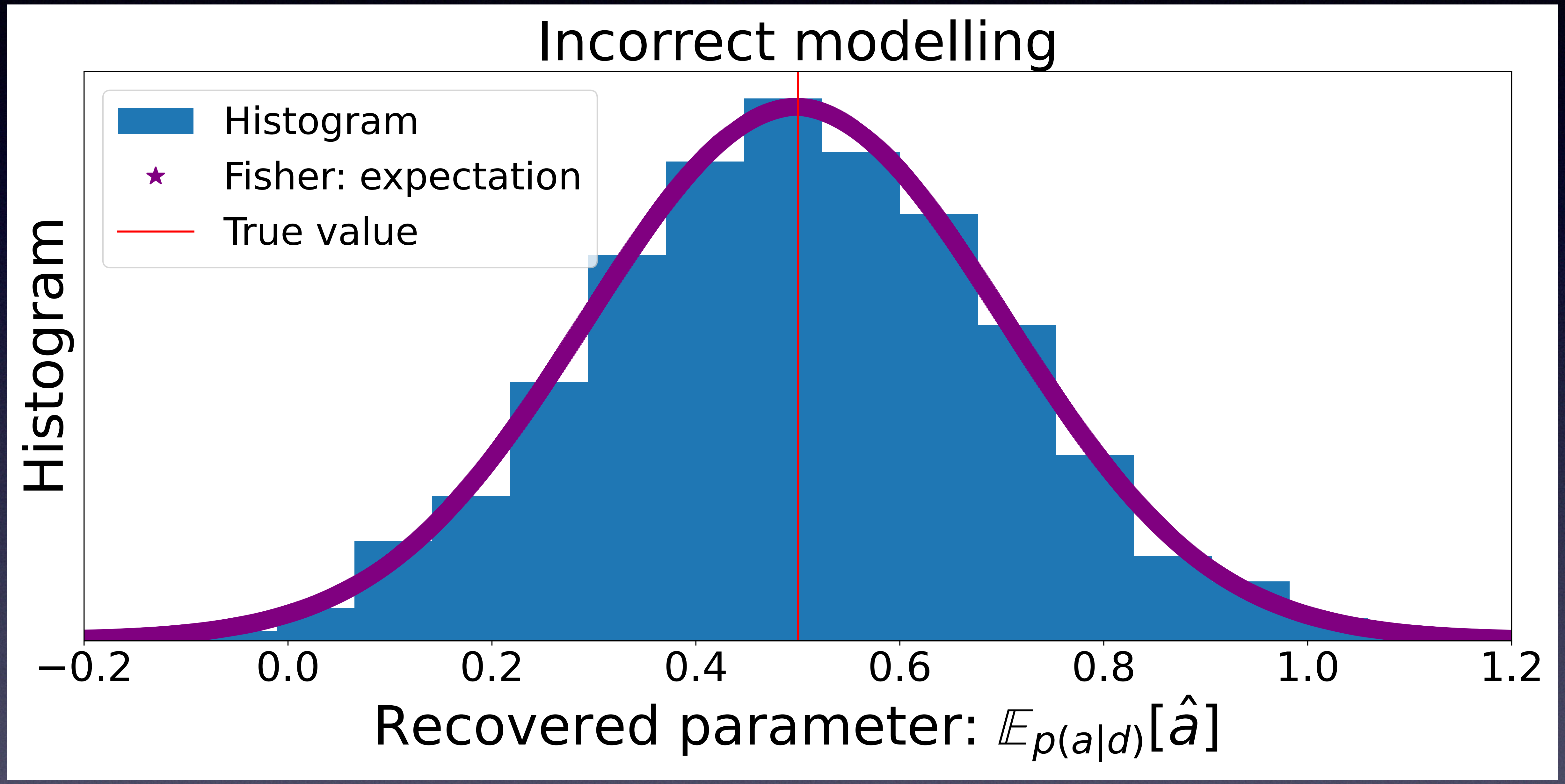
Expectation: mis-modelling

- Easily shown: $\mathbb{E}[\Delta\theta^i_{\text{bf}}] = 0 \iff$ unbiased.
- What about covariance $\mathbb{E}[\Delta\boldsymbol{\theta}^i\Delta\boldsymbol{\theta}^j]$?
- $\mathbb{E}[\Delta\boldsymbol{\theta}^i\Delta\boldsymbol{\theta}^j] = 2(\Gamma^{-1})^{ik}\text{Re}\left[\partial_k\hat{H}\Sigma_N^{-1}\Sigma\Sigma_N^{-1}\partial_p\hat{H}\right](\Gamma^{-1})^{pj}$
- Σ := Model and Σ_N := consistent with noise process
- We can **predict** the $\sim 1\sigma$ statistical uncertainty when subject to noise mis-modelling errors.

Edy et al, arXiv: 2101.07743v2

Repeated inference





$$\log p(D\,|\,a, \Sigma) = -\,(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})^{\dagger} \underbrace{\Sigma_N^{-1}}_{\text{gaps}} (\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})$$

$$\log p(D\,|\,a, \Sigma) = -\,(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})^{\dagger} \underbrace{\Sigma_n^{-1}}_{\text{whittle}} (\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})$$

Take home message

Accounting for gaps is crucial for LISA science

Where next?

- Massive black holes — nearly finished!
- Extreme mass-ratio inspirals
- Have a bash at the Spritz data set

Cheers

Extra stuff

General Likelihood

$$\log \mathscr{L}(\hat{N}) \propto -\frac{1}{2}\begin{pmatrix}\hat{\boldsymbol{N}}^{\star} & \boldsymbol{N}\end{pmatrix}\begin{pmatrix}\Sigma_N & R_N \\ R_N & \Sigma_N\end{pmatrix}^{-1}\begin{pmatrix}\hat{\boldsymbol{N}} \\ \boldsymbol{N}^{\star}\end{pmatrix}$$

$\Sigma_N =$ Covariance Matrix $= \mathbb{E}[\boldsymbol{N}\boldsymbol{N}^{\dagger}] = \langle \hat{N}(f)\hat{N}^{\star}(f') \rangle$

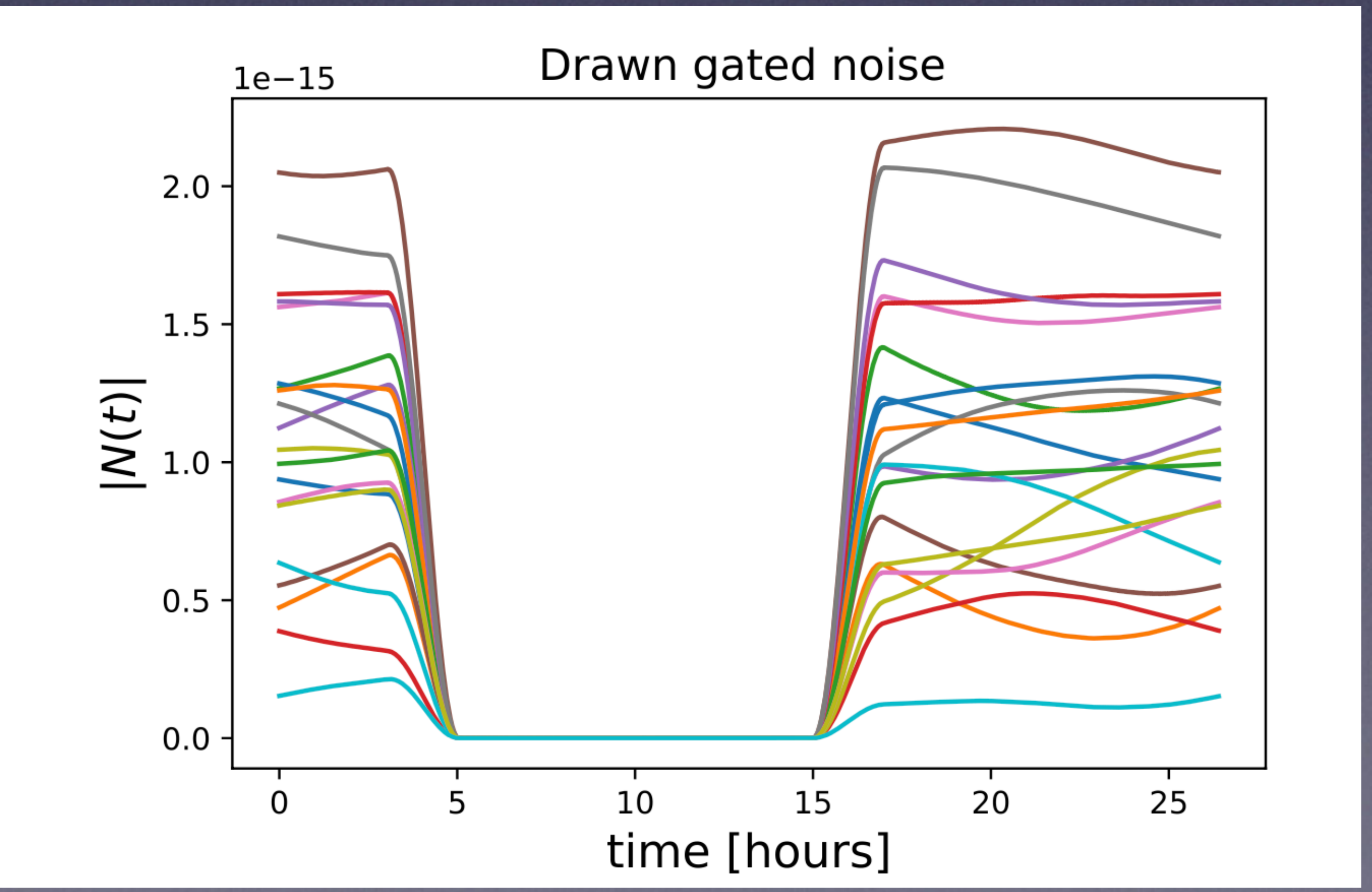
$R_N =$ Relation Matrix $= \mathbb{E}[\boldsymbol{N}\boldsymbol{N}^{T}] = \langle \hat{N}(f)\hat{N}(f') \rangle$

General analogues

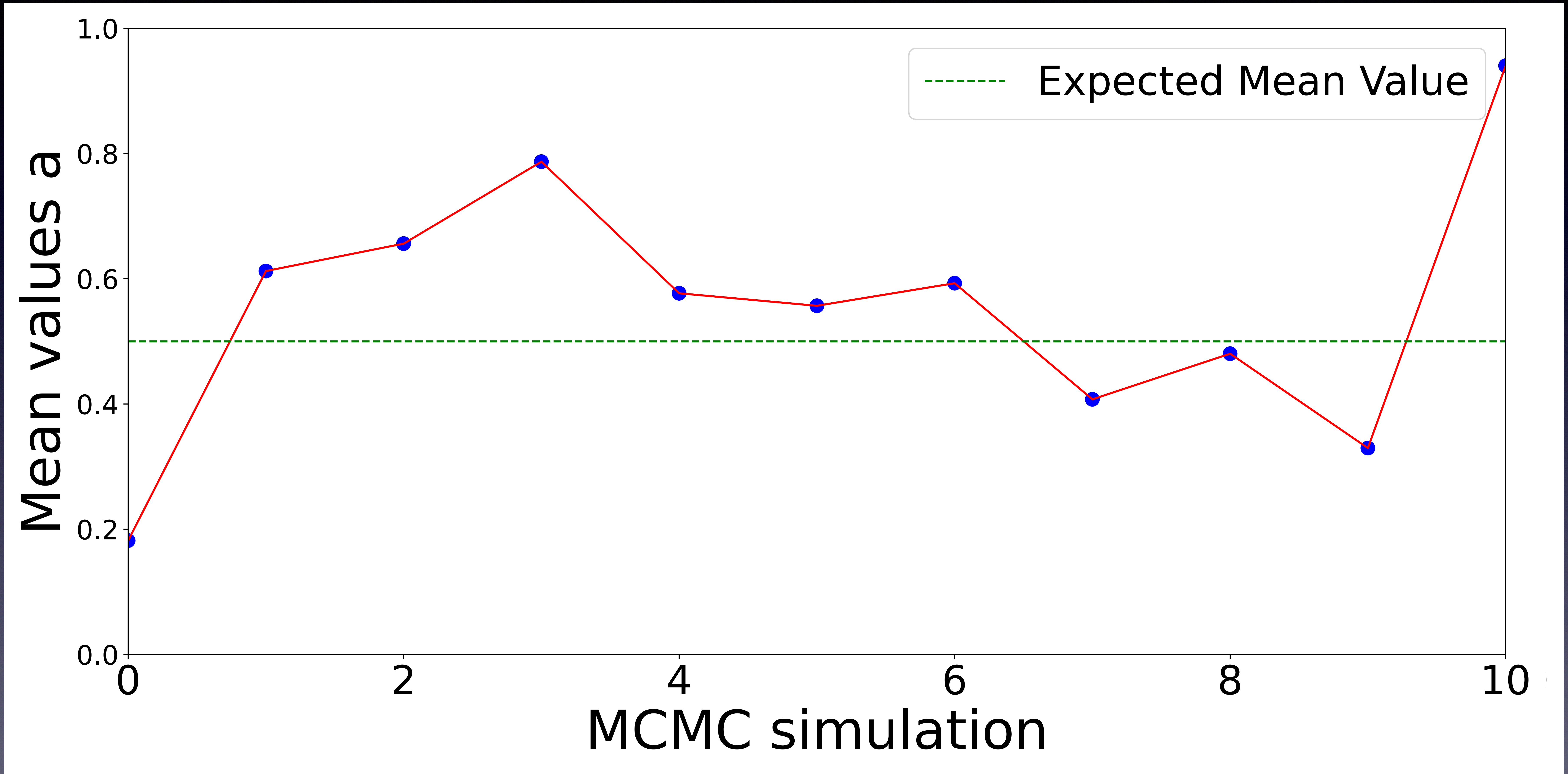
- Likelihood: $p(D\,|\,\boldsymbol{\theta}, \Sigma) = -\dfrac{1}{2}(D - H\,|\,D - H)_\Sigma$
- With (general) inner product $(a\,|\,b)_\Sigma = \boldsymbol{a}^\dagger \Sigma^{-1} \boldsymbol{b} + \boldsymbol{b} \Sigma^{-1} \boldsymbol{a}^\dagger = 2\mathrm{Re}(\boldsymbol{a}^\dagger \Sigma^{-1} \boldsymbol{b})$
- SNR : $\rho^2 = (H\,|\,H)_\Sigma$
- Fisher Matrix: $\Gamma_{ij} = (\partial_i H\,|\,\partial_j H)_\Sigma$
- Statistical fluctuation: $\Delta\theta^i_{\mathrm{bf}} = (\Gamma^{-1})^{ij}(\partial_j H\,|\,N)_\Sigma$

Simulating non-stationary realisations

- Matrix $\Sigma_N(f, f')$ has **complex** off-diagonal elements.
- Consider $\hat{N}(f) = \hat{N}_1(f) + i\hat{N}_2(f)$
- Draw $\begin{pmatrix} \hat{N}_1 \\ \hat{N}_2 \end{pmatrix} = N\left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \frac{1}{2}\begin{pmatrix} \mathrm{Re}(\Sigma_N) & \mathrm{Im}(\Sigma_N) \\ \mathrm{Im}(\Sigma_N) & \mathrm{Re}(\Sigma_N) \end{pmatrix}\right]$

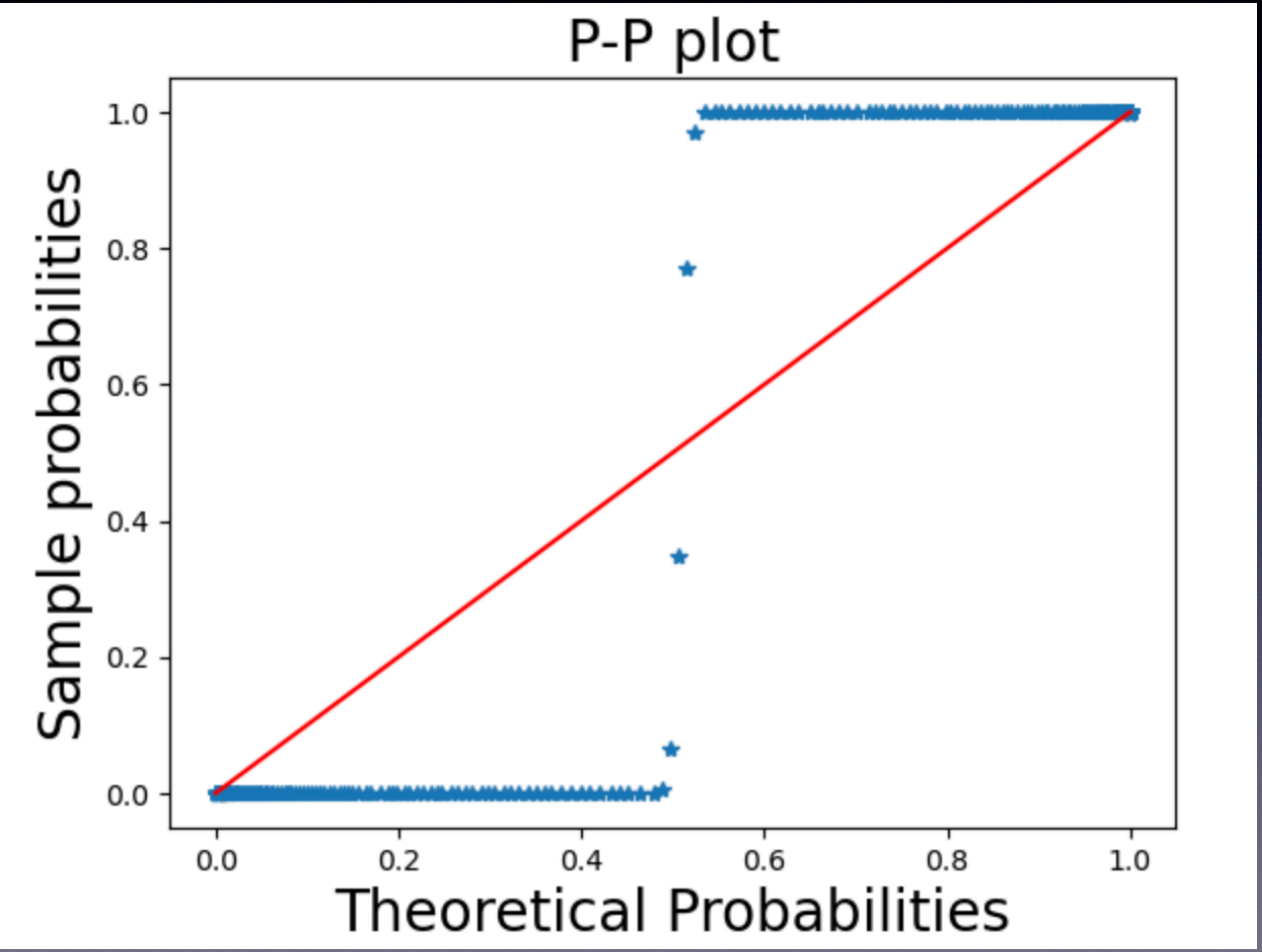


Repeated inference



$$\log p(D\,|\,a,\Sigma) = -\,(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})^{\dagger}\Sigma_n^{-1}(\hat{\boldsymbol{D}} - \hat{\boldsymbol{H}})$$

P-P Plot



P-P plot
Sample probabilities
Theoretical Probabilities
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0