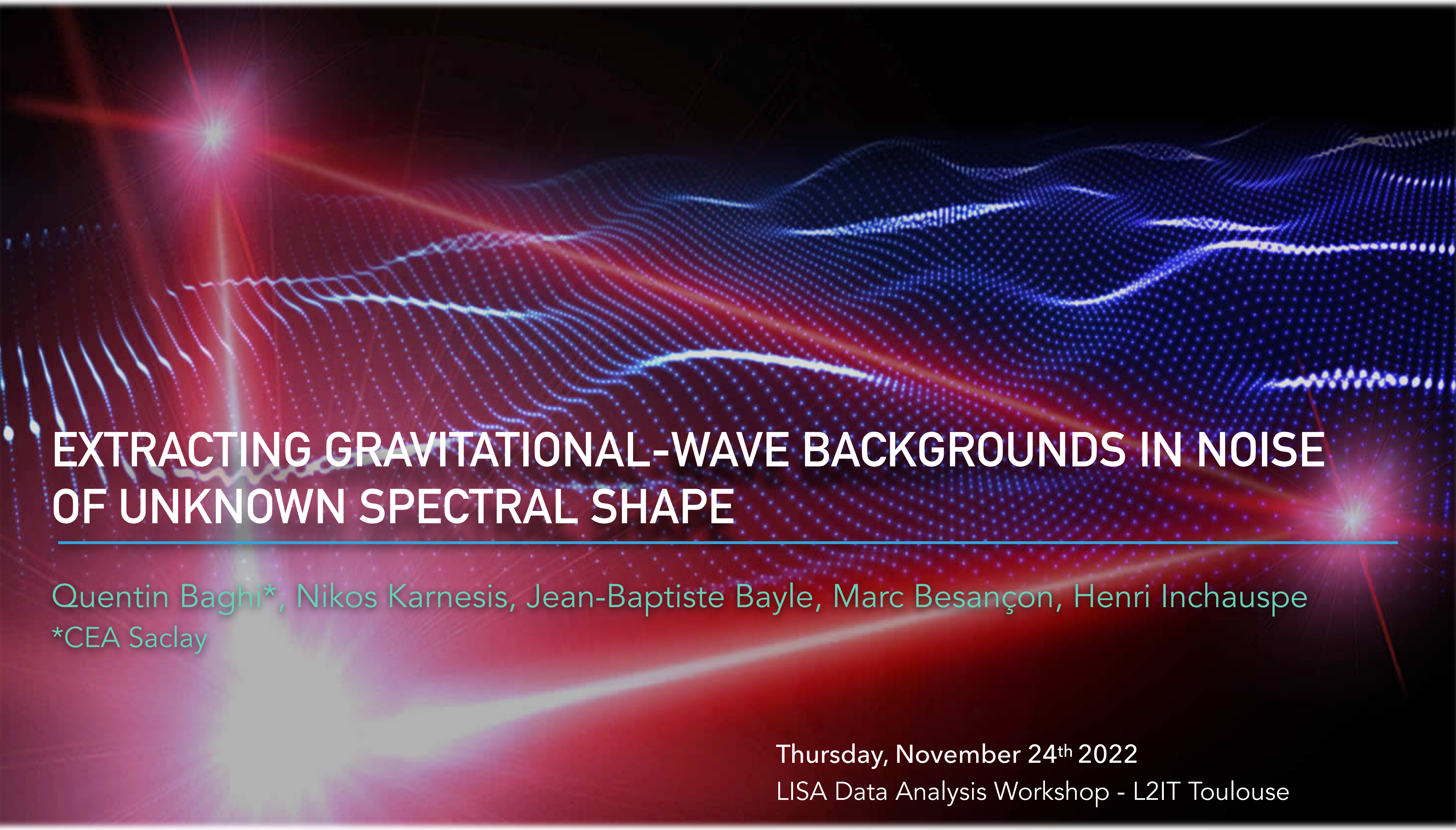


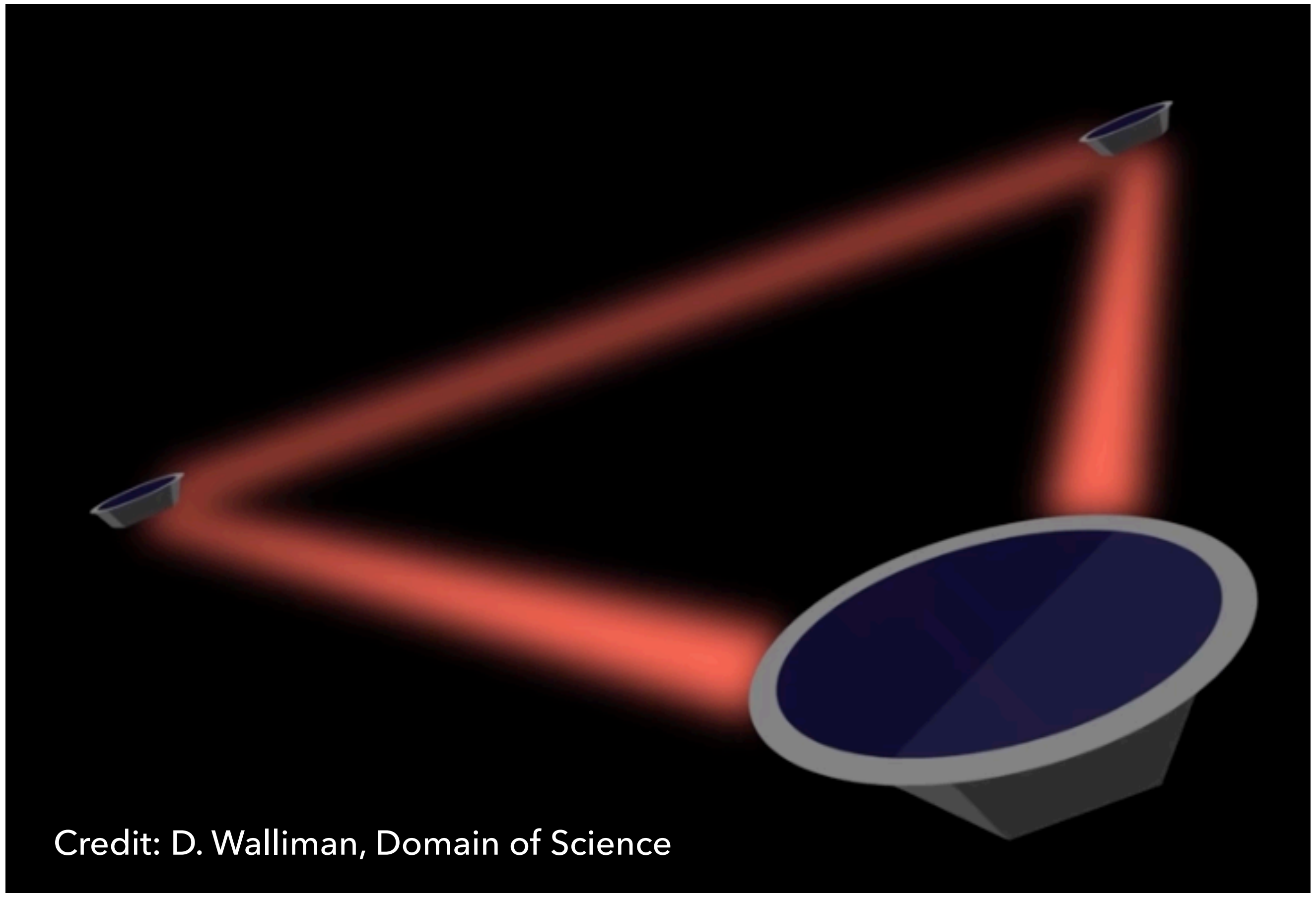
EXTRACTING GRAVITATIONAL-WAVE BACKGROUNDS IN NOISE OF UNKNOWN SPECTRAL SHAPE

Quentin Baghi*, Nikos Karnesis, Jean-Baptiste Bayle, Marc Besançon, Henri Inchauspe
*CEA Saclay

Thursday, November 24th 2022
LISA Data Analysis Workshop - L2IT Toulouse

LAYOUT

1. Context and problem statement
2. A flexible modelling of noise
3. Results of detection tests



Credit: D. Walliman, Domain of Science

CONTEXT AND PROBLEM STATEMENT

- Finding a stochastic gravitational-wave background (SGWB) is one of the main LISA's scientific objectives

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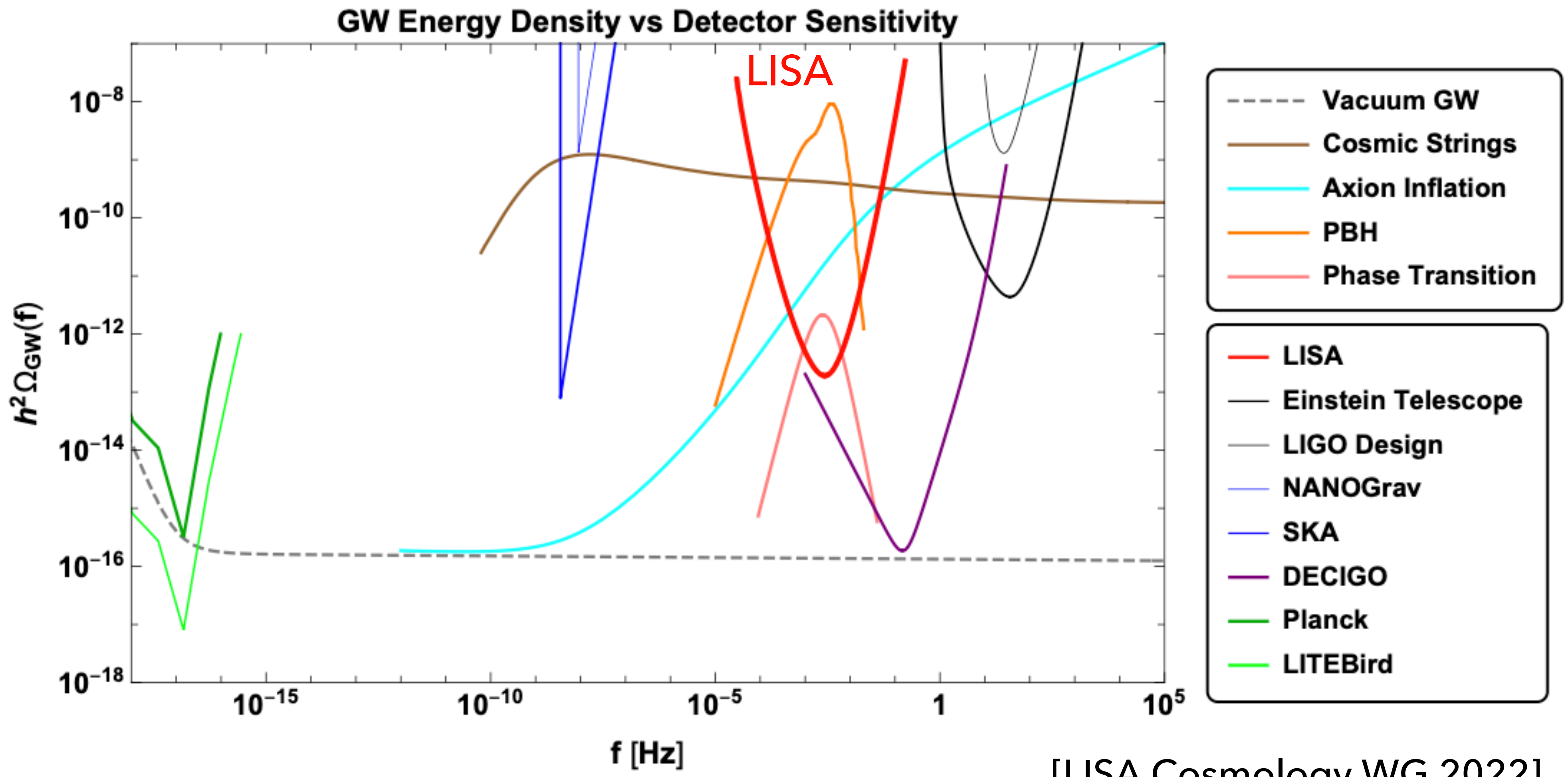

[LISA Cosmology WG 2022]

CONTEXT AND PROBLEM STATEMENT

Reference	Model components	Method	Performance
Cornish and Larson 2001	SGWB + noise	Cross-correlation assuming several detectors, old LISA design	$\Omega_{GW}(f) > 7 \times 10^{-12}$ @3mHz
Adams and Cornish	SGWB + noise	Power law, MCMC	$\Omega_0 > 1.7 \times 10^{-13}$
Caprini et al. 2019	SGWB + noise	Template-free, local fitting of power laws	$\Omega_{2/3} > 5.4 \times 10^{-12}$ with $f_0 = 1$ mHz
Pieroni and Barauss 2020	SGWB + foreground + noise	Template free, principal component analysis	$\Omega_0 > 6 \times 10^{-13}$
Karnesis et al. 2020	SGWB + noise	Excess of power, analytical Bayes factors	$\Omega_{2/3} > 4.2 \times 10^{-12}$ with $f_0 = 1$ mHz
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- Two main limitations:
  - ✦ Assume a fixed noise shape
  - ✦ Use frequency-domain data simulations

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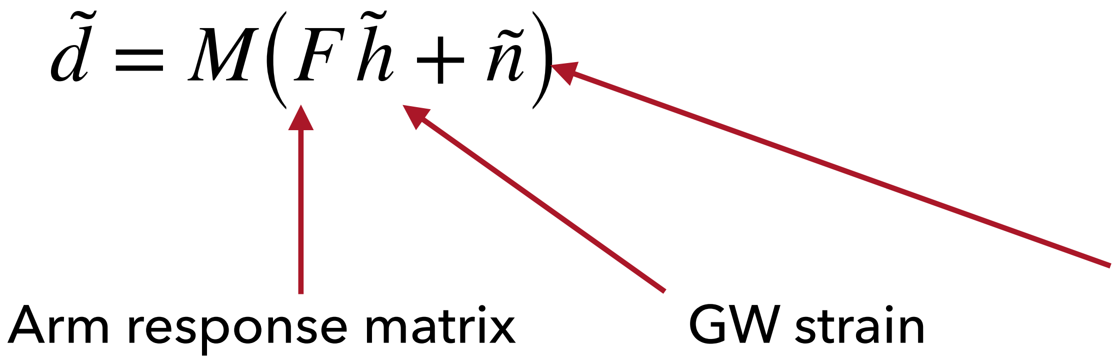

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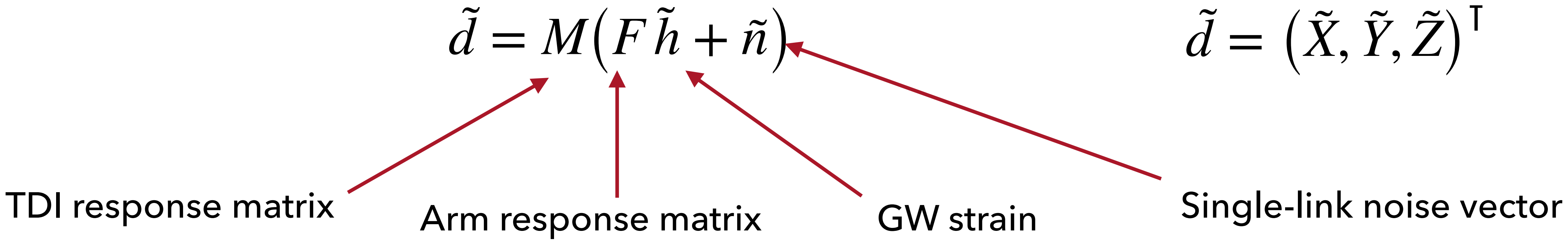

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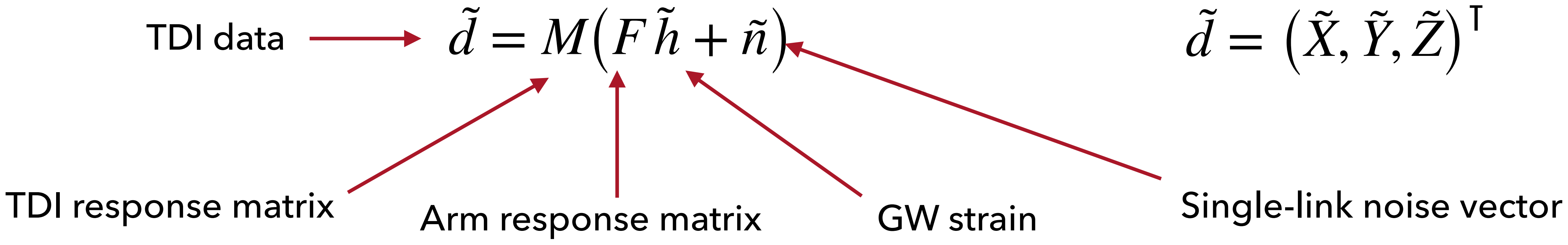

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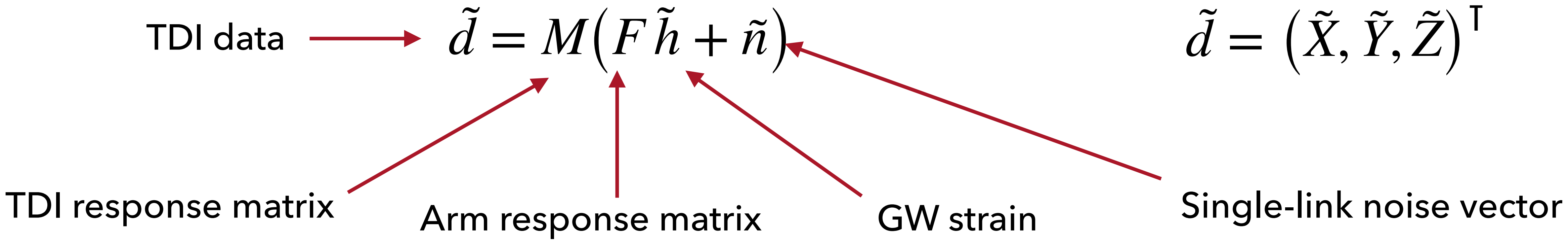

TDI data $\longrightarrow$ $\tilde{d} = M\left(F\tilde{h} + \tilde{n}\right)$

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$$R_n = MM^{\dagger}$$

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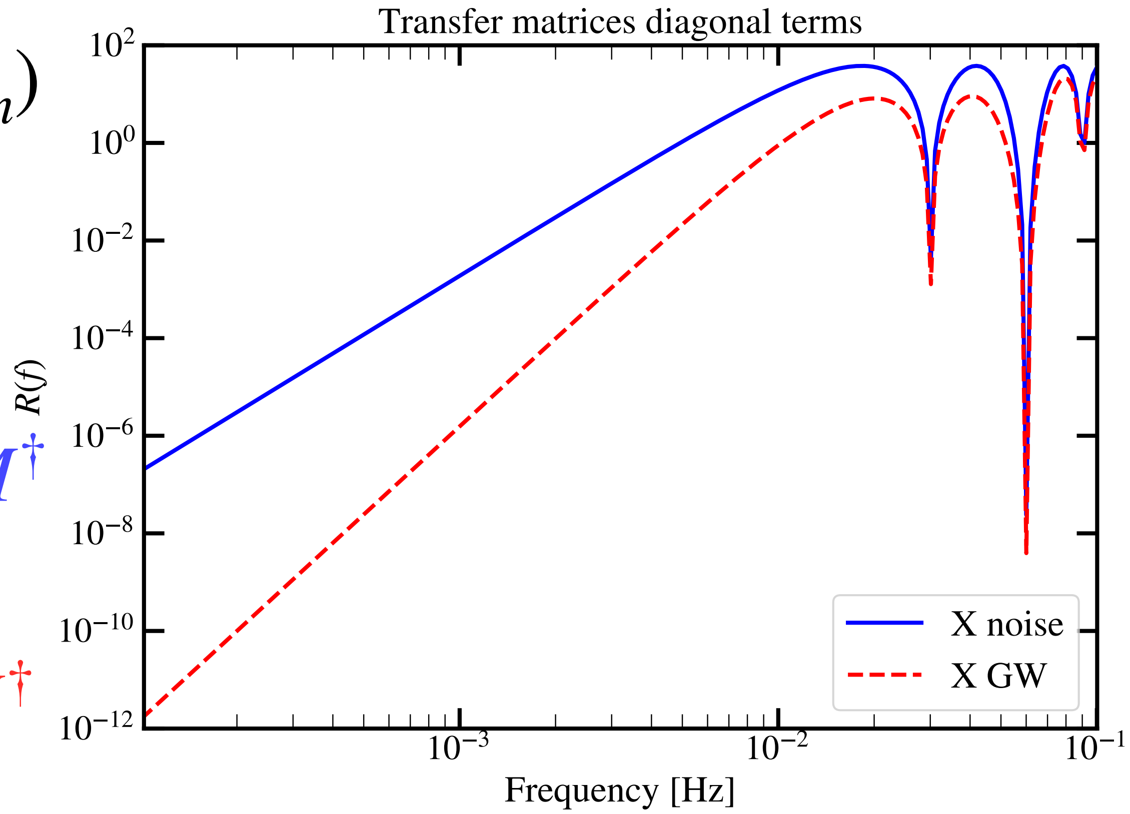

FLEXIBLE MODELLING OF NOISE

- Set of cubic splines to model the single-link noise log-PSD:

$$\log S_n(f) = \sum_{j=0}^{K-1} c_j B_j \left(f, \boldsymbol{\xi}\right)$$

Spline amplitudes

Spline locations

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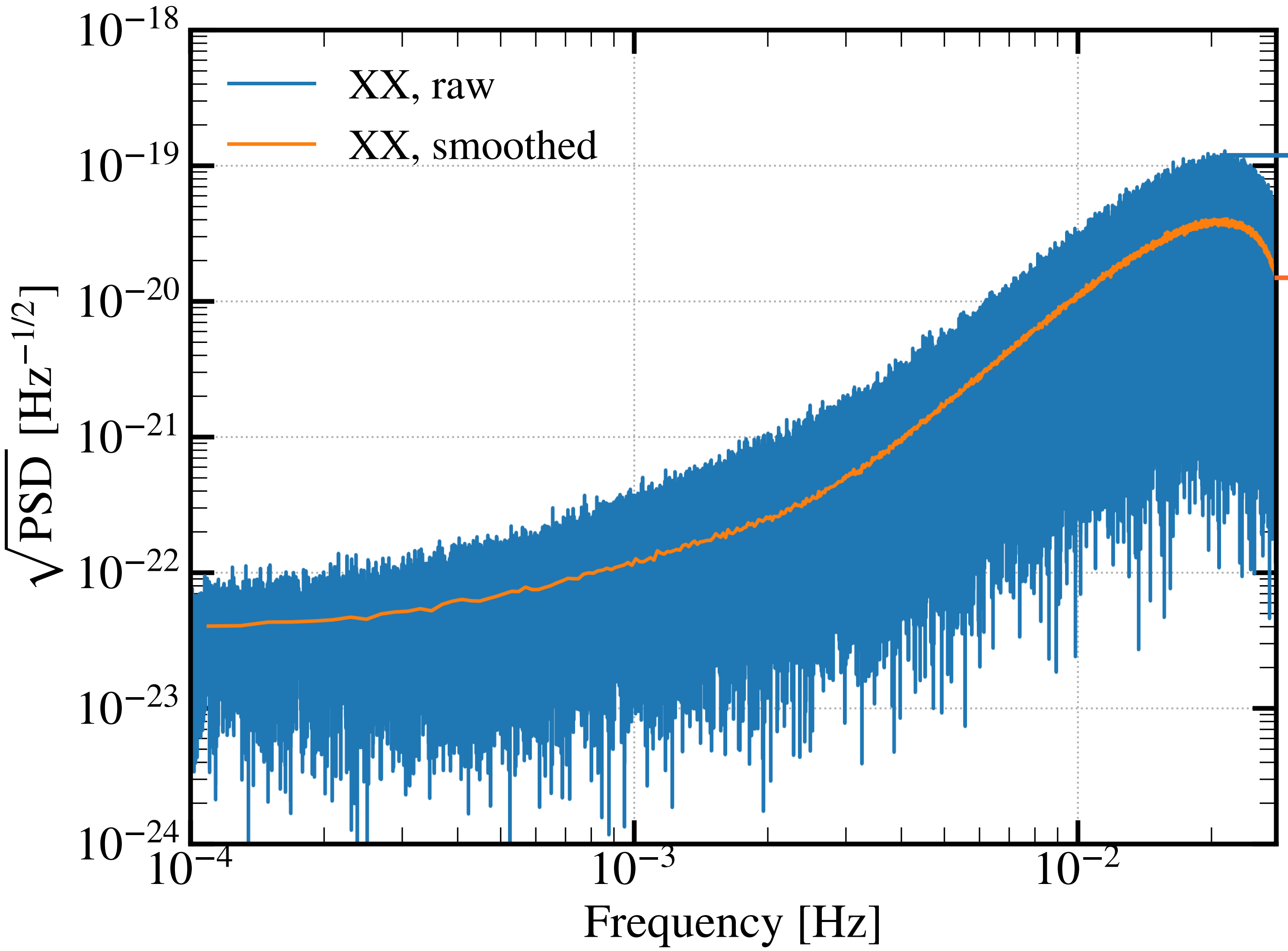
DATA COMPRESSION

- To reduce the data size we average the periodogram (x 100 compression)



$$P(f) = \tilde{d}(f)\tilde{d}(f)^{\dagger}$$

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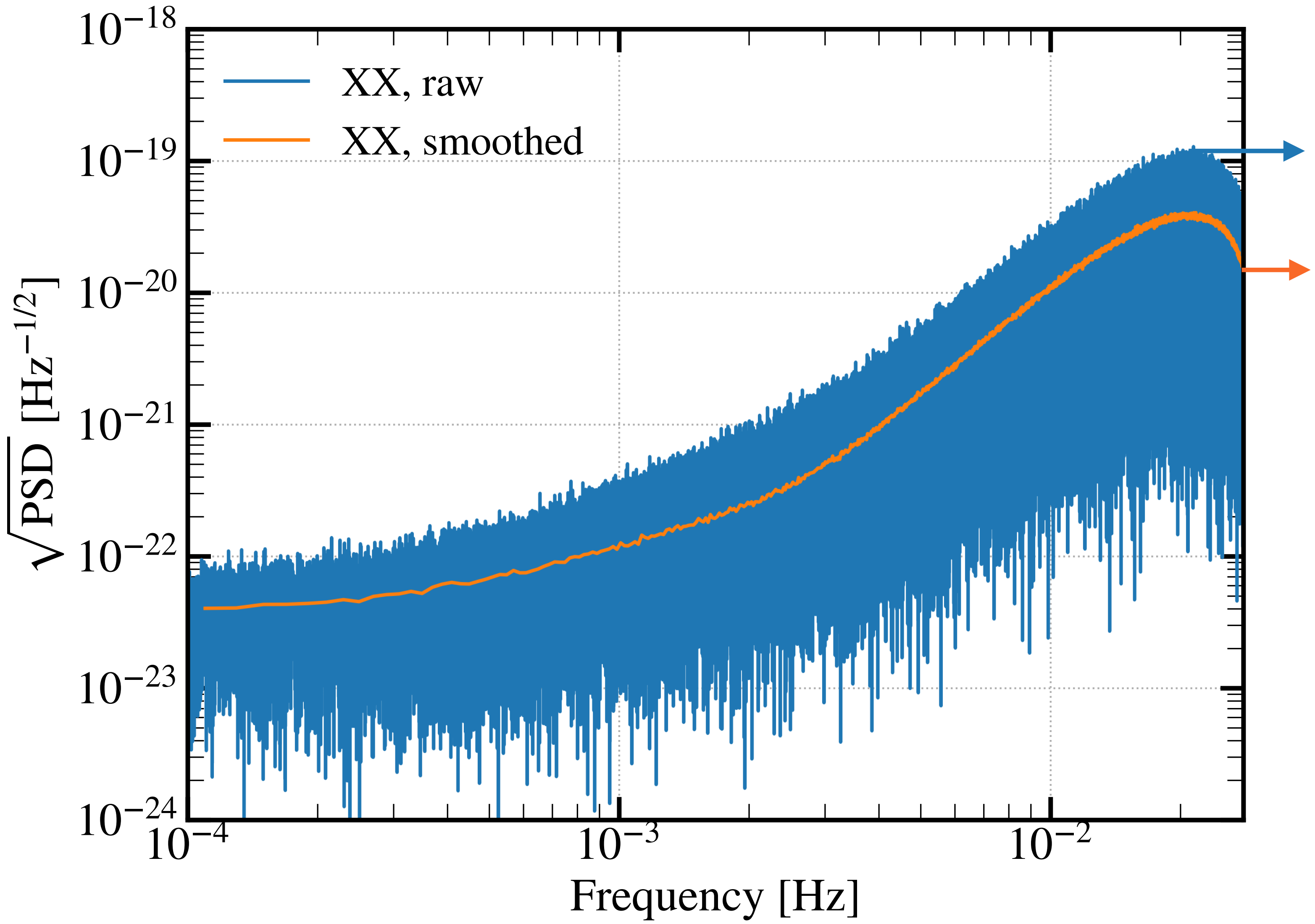

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- The distribution of the averaged periodogram is a complex Wishart distribution

$$\log p(\bar{\mathbf{P}}(f)\,|\,\theta) = -\operatorname{tr}(\nu \mathbf{C}_d^{-1}\bar{\mathbf{P}}(f)) - \nu \log \left|\mathbf{C}_d(f)\right| .$$

DETECTION APPROACH

- Data analysis :
  - ✦ Sampling posterior distributions with parallel tempered MCMC
  - ✦ Uniform priors on GW parameters $\Omega_{m0} \in [10^{-16}, 10^{-14}]$ and $n \in [-5, 7]$
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  - Hypothesis H_0: only noise in the data $\tilde{d} = M\tilde{n}$
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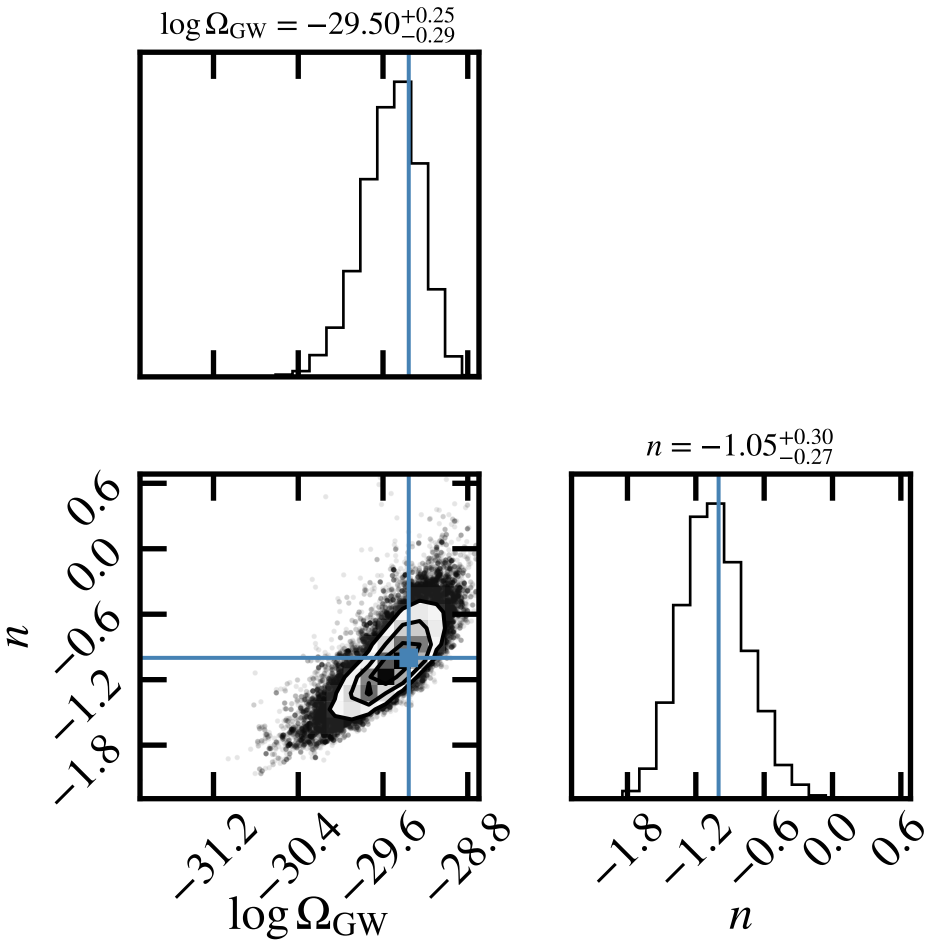
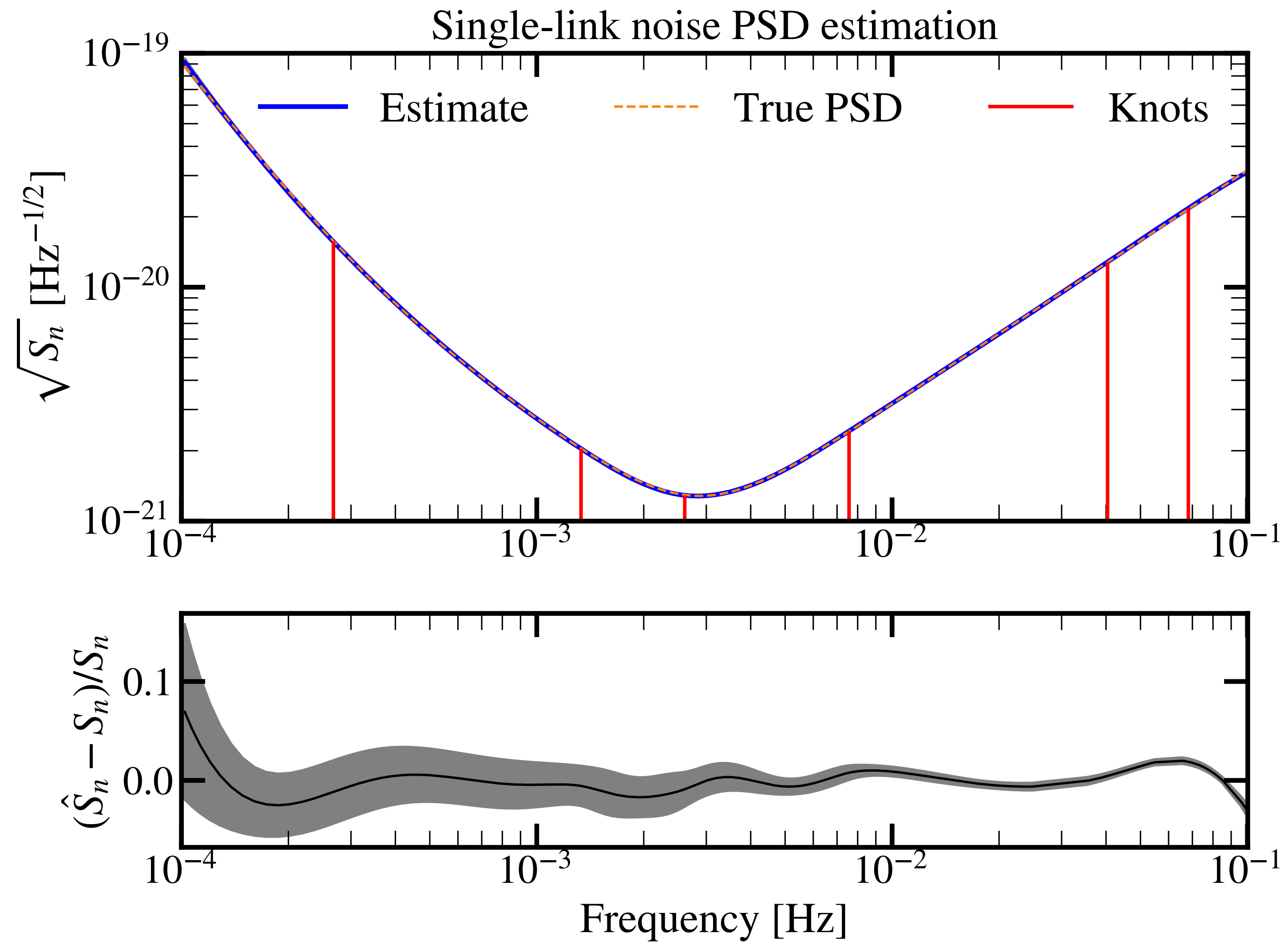
- Aim: compute the Bayes factors for a range of configurations (Ω_{m0}, n)

$$Z_i = \int_{\Theta} p(d\,|\,\theta, H_i)\, p(\theta)\, d\theta \qquad B_{10} = \frac{Z_1}{Z_0}$$

RESULTS: POSTERIORS

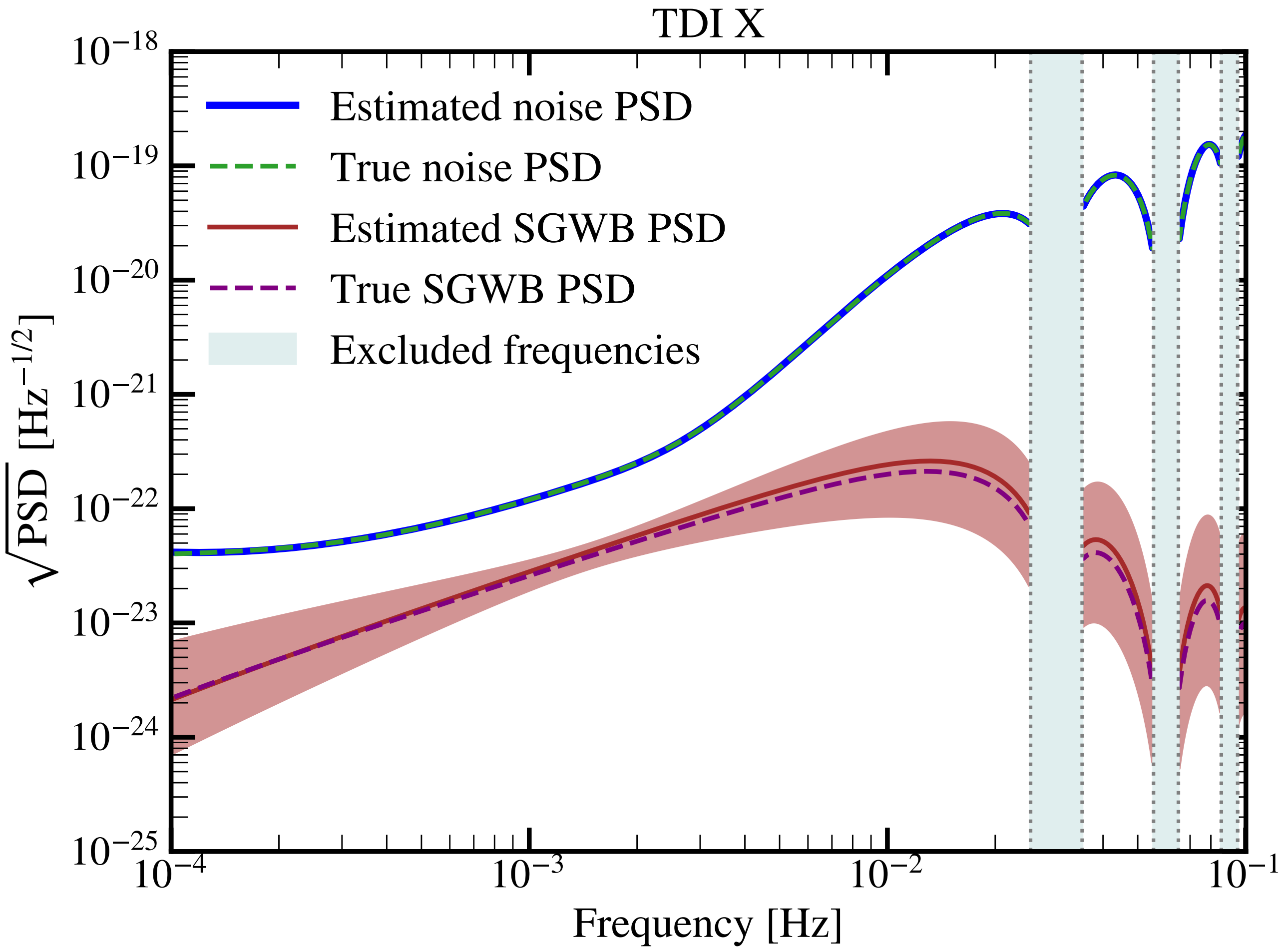
- Example with n = 0, Ω_0 = 1.8e-13
- Data contains noise + signal
- Model includes noise + signal (H1)





RESULTS: POSTERIORS

Under H1



RESULTS: POSTERIORS

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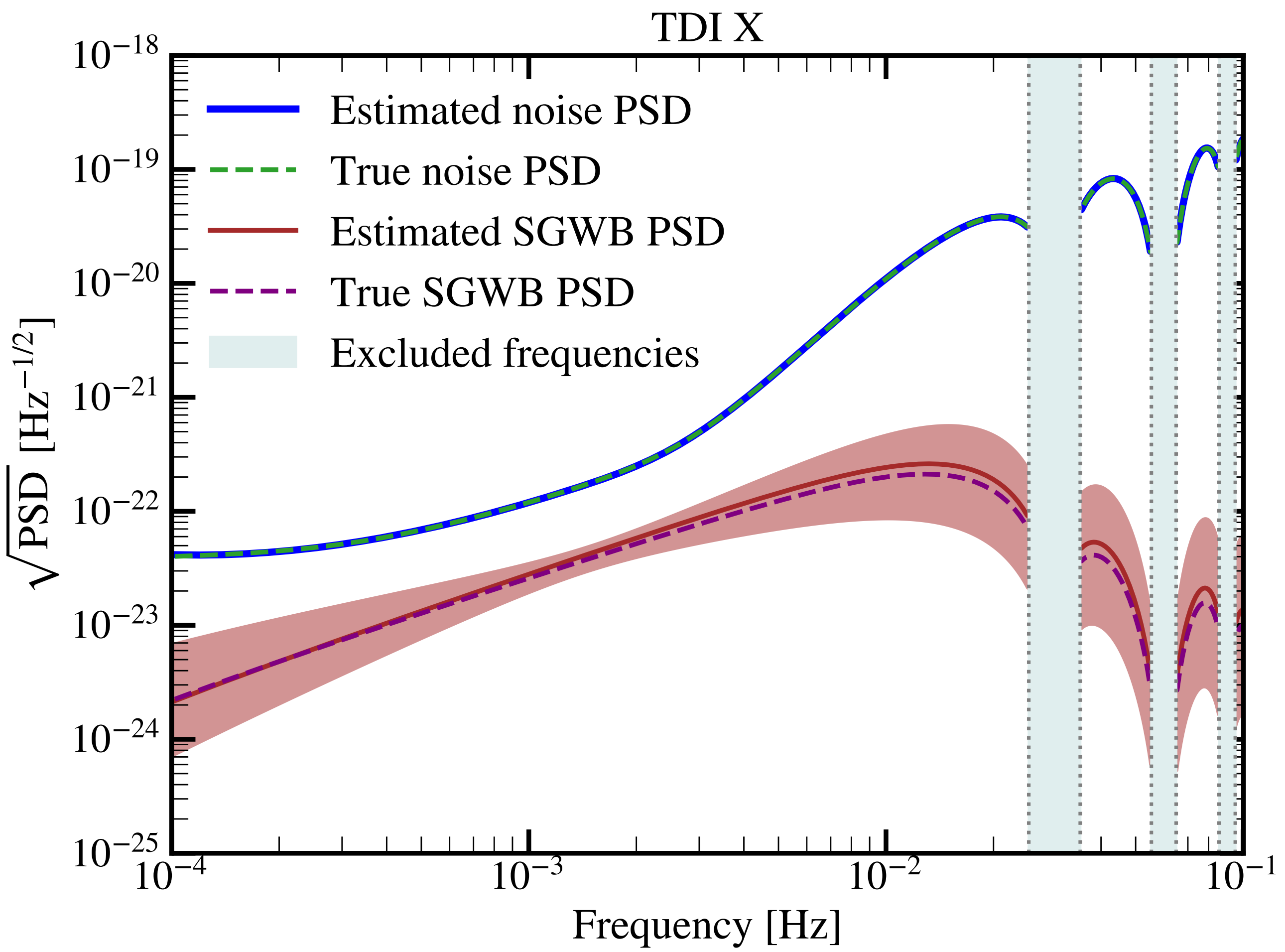
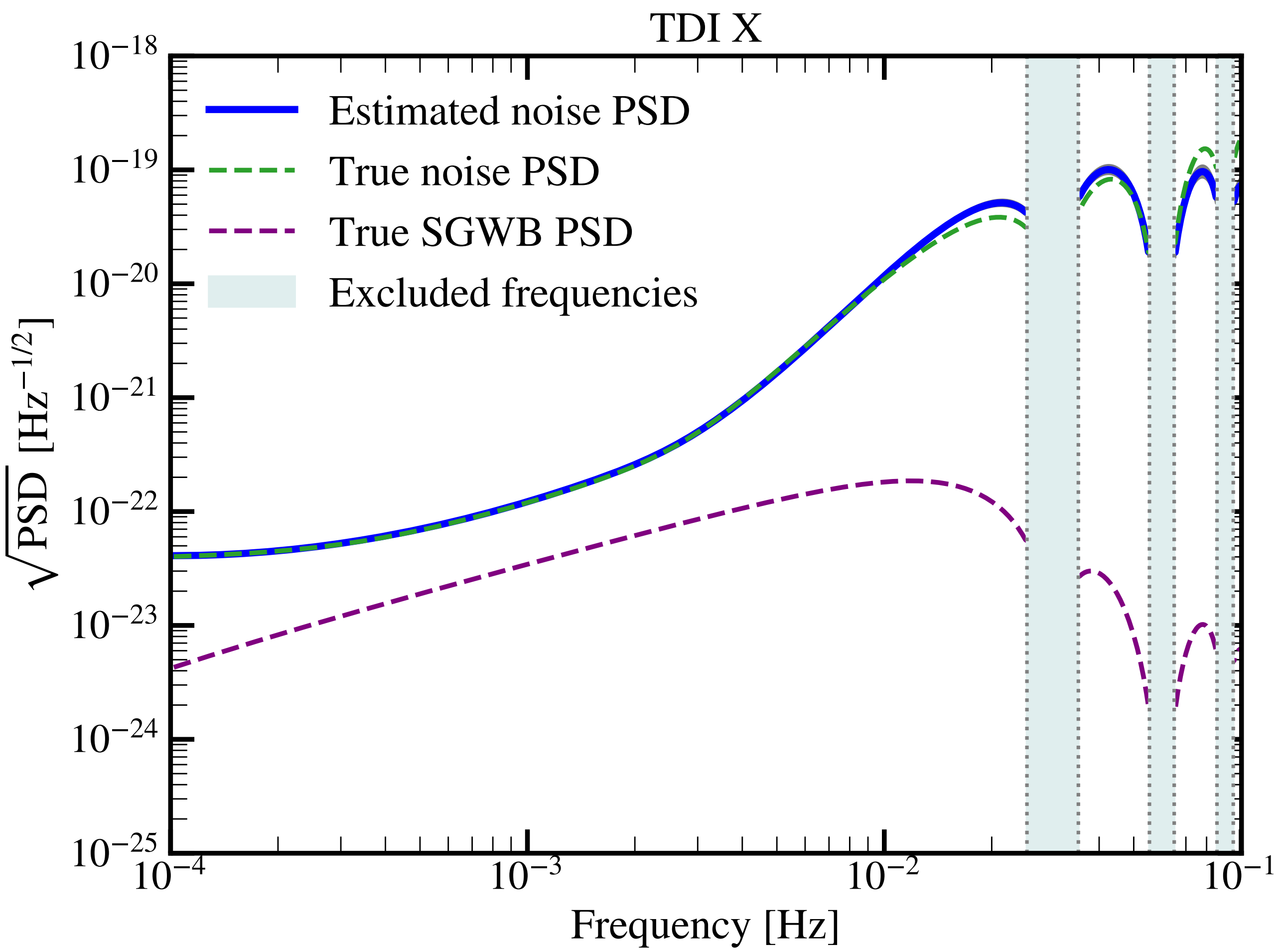


Under H0



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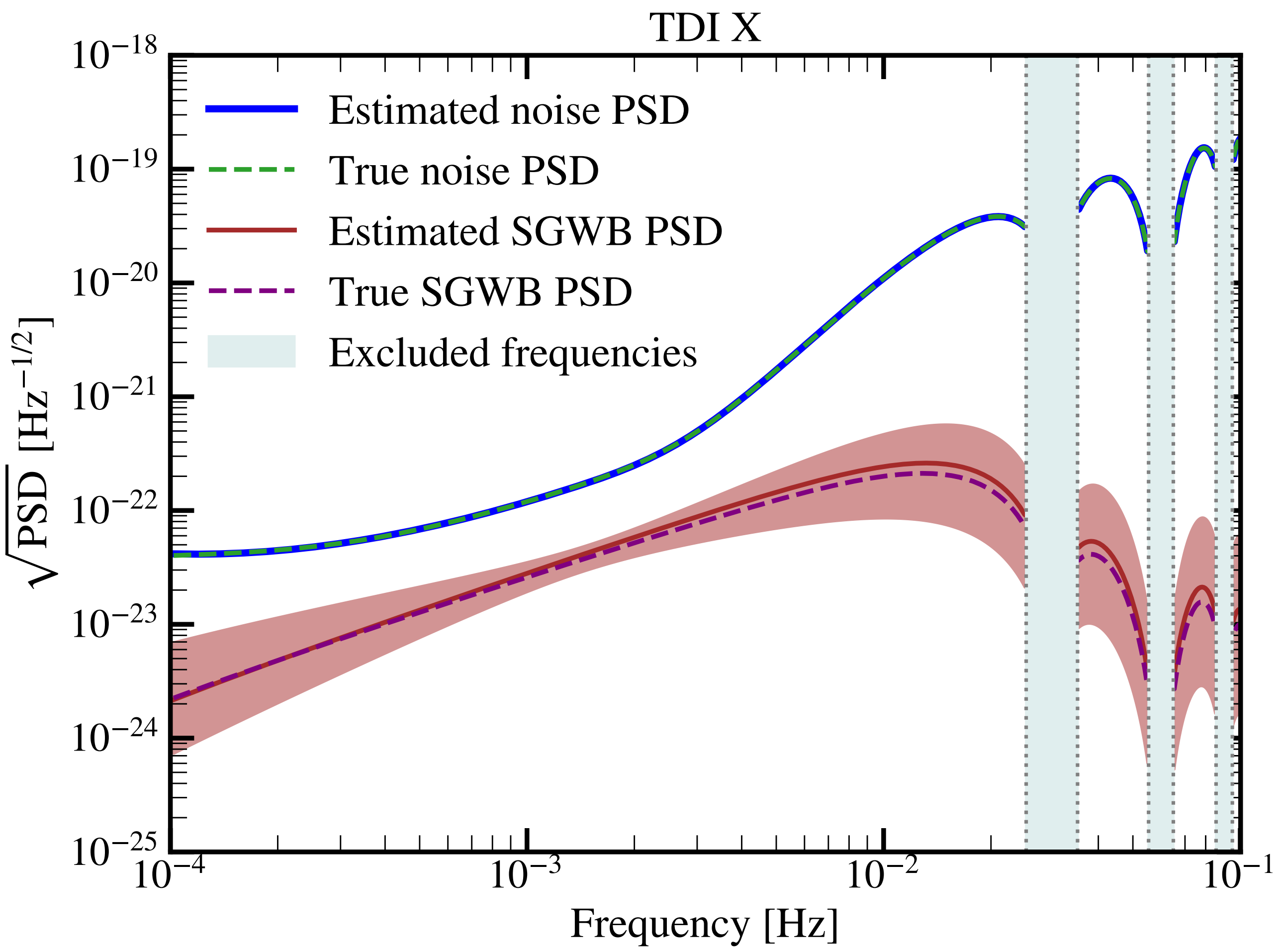

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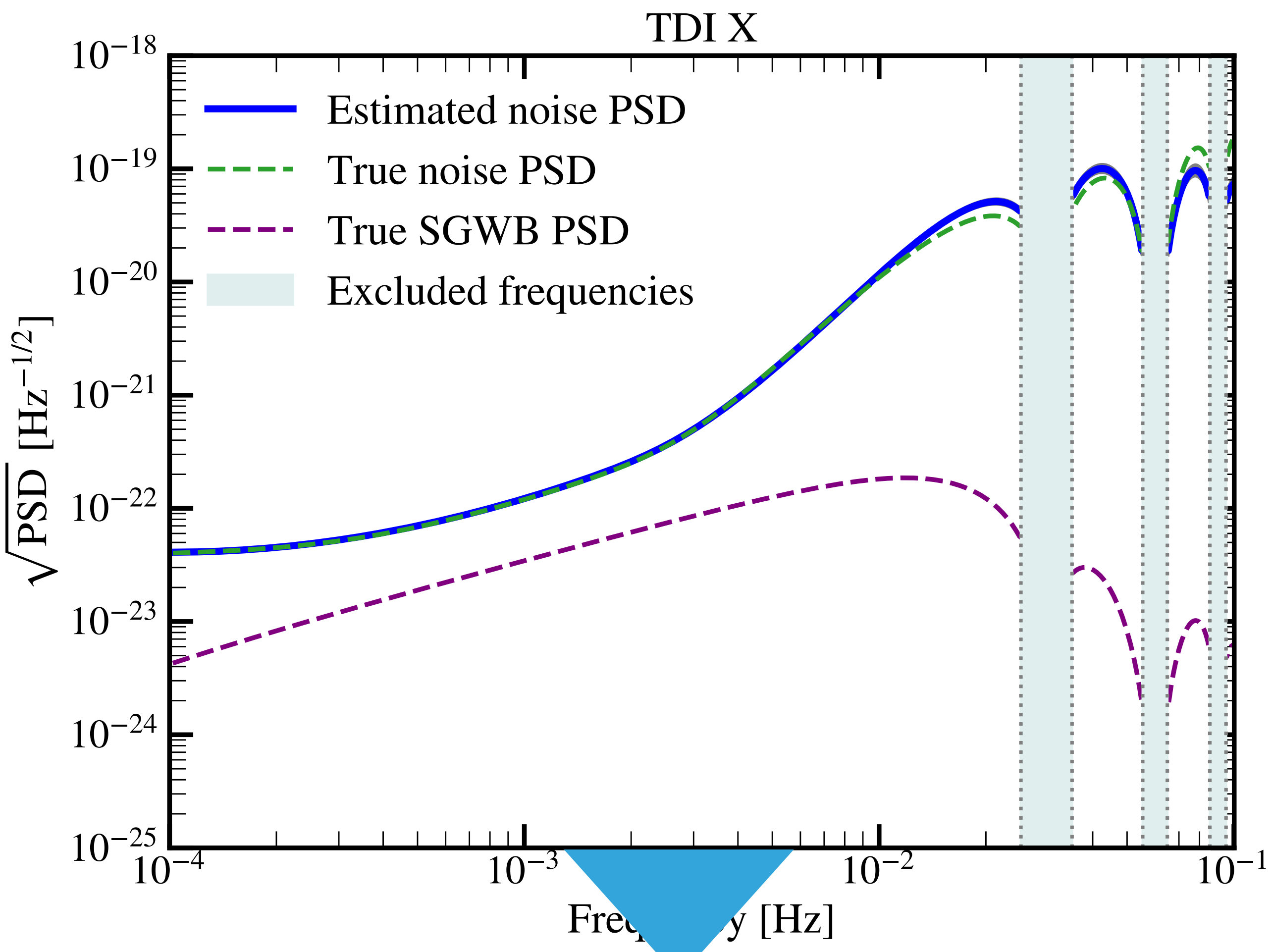

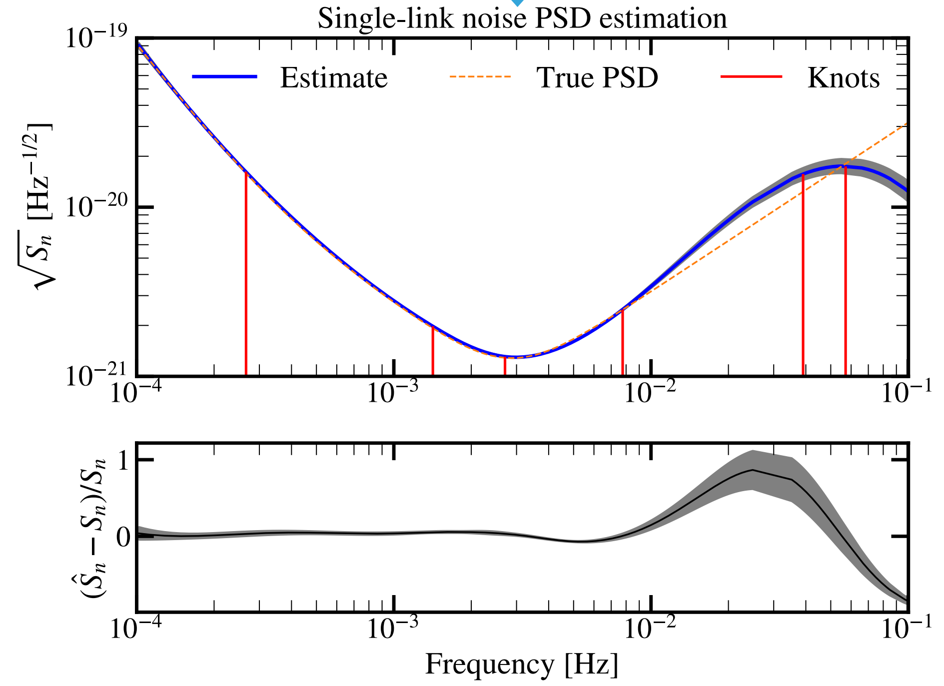

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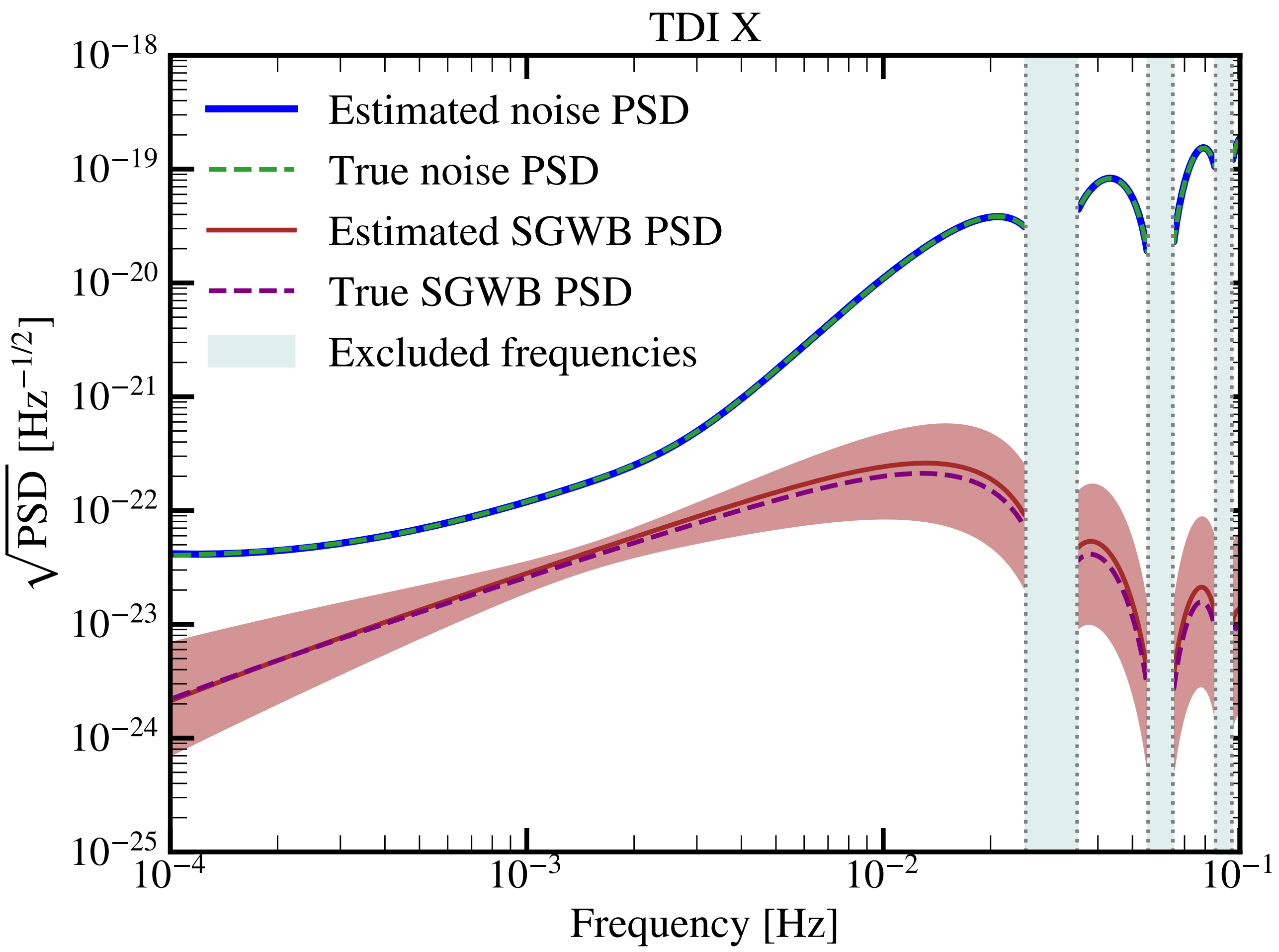
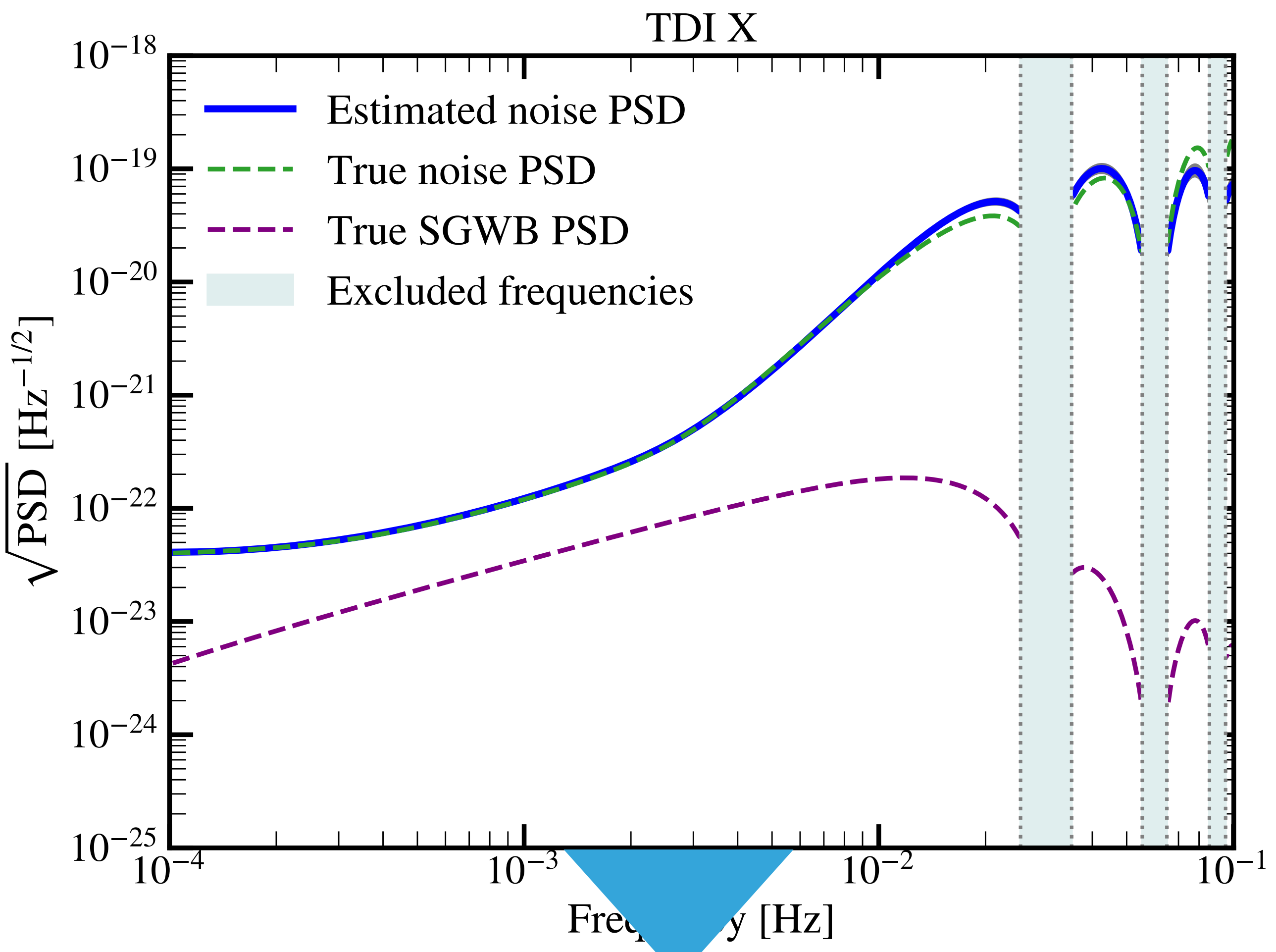
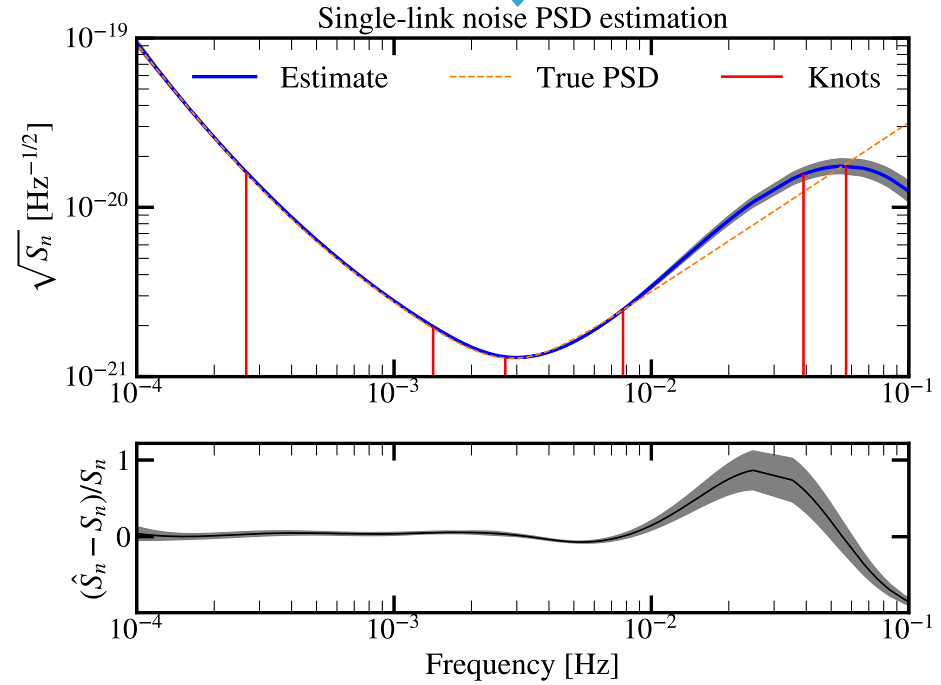
Under H1

Under H0

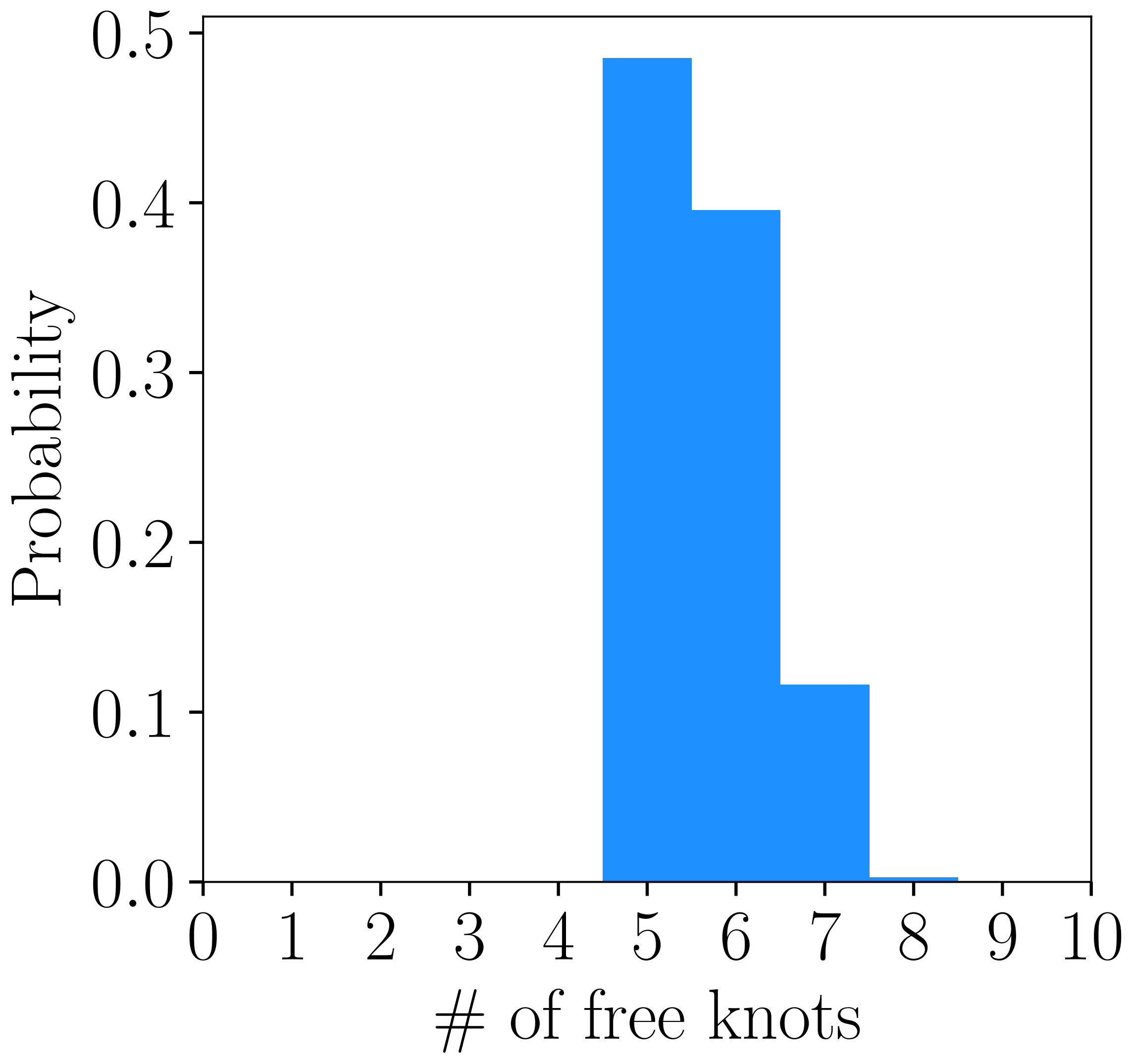


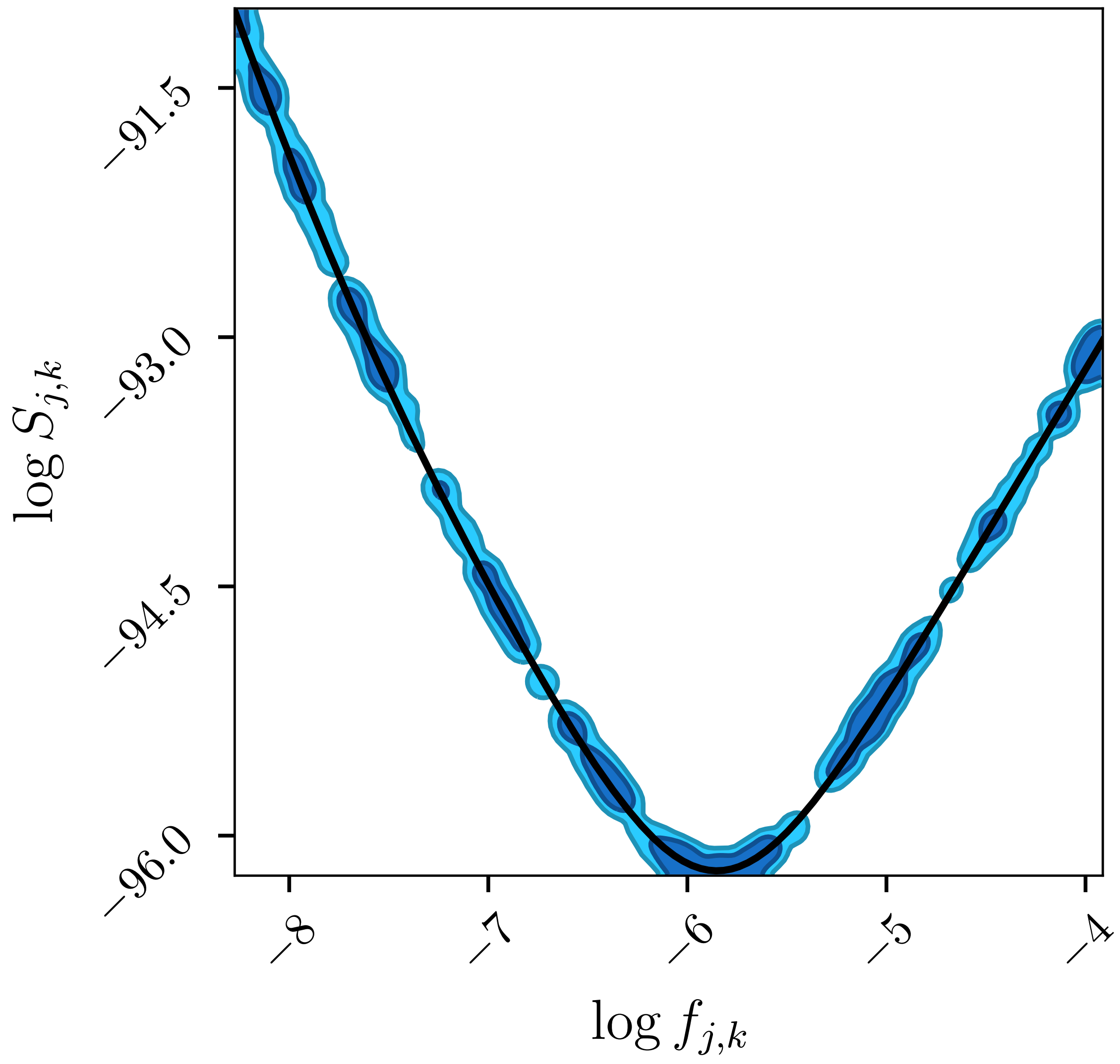




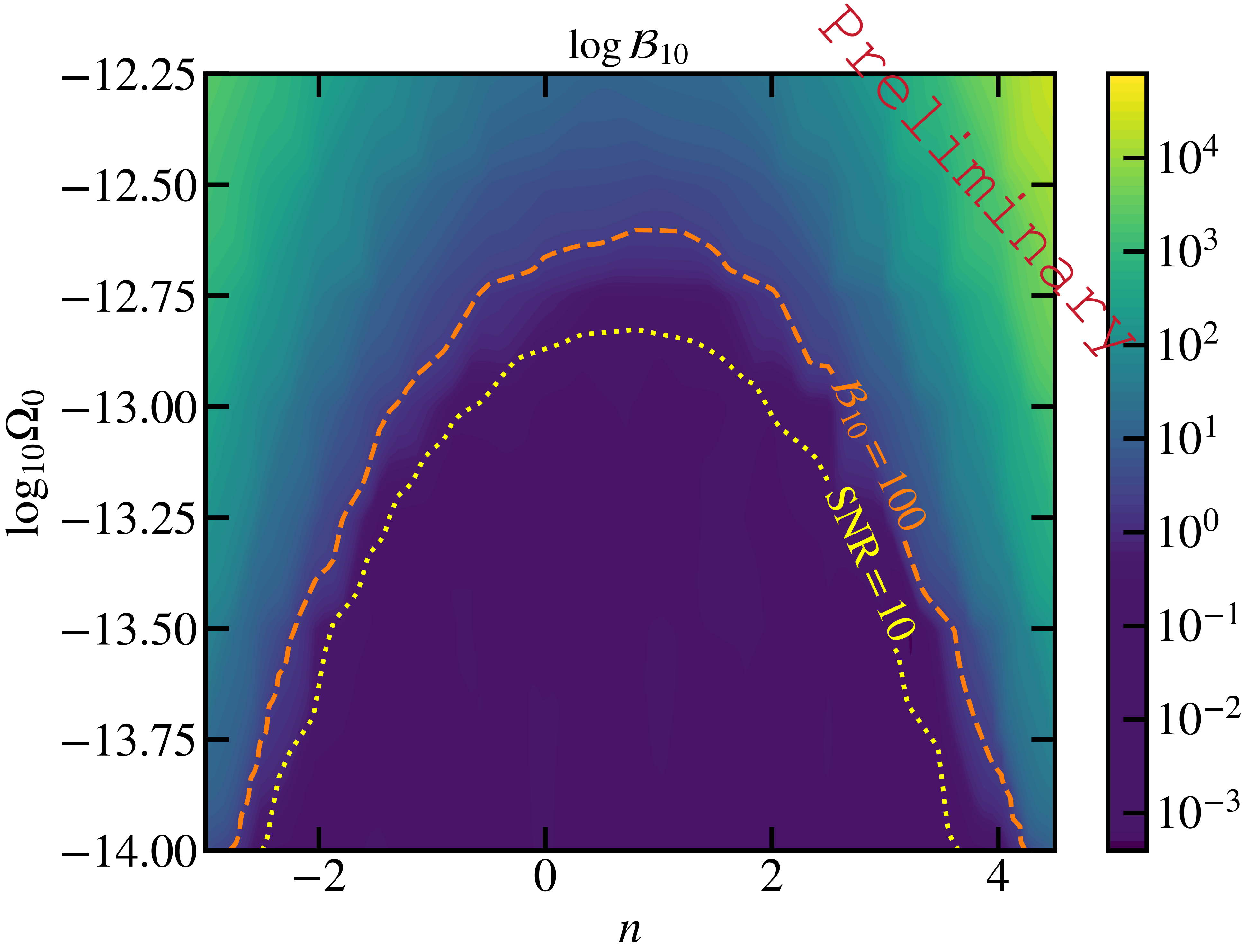
$\log_{10}\mathscr{B}_{10} = 3.8$ → Large evidence for H1

RESULTS: POSTERIORS

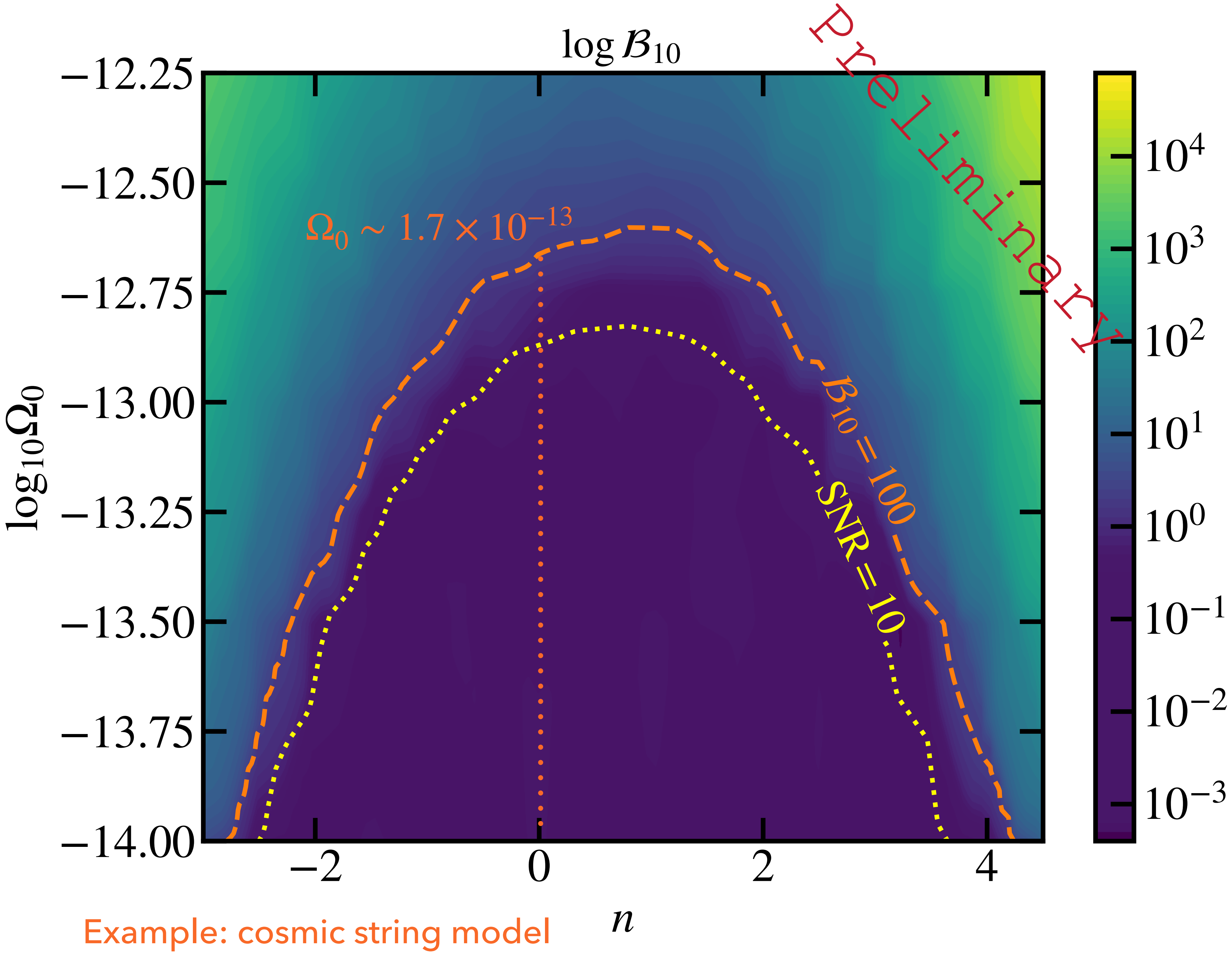




RESULTS: BAYES FACTORS



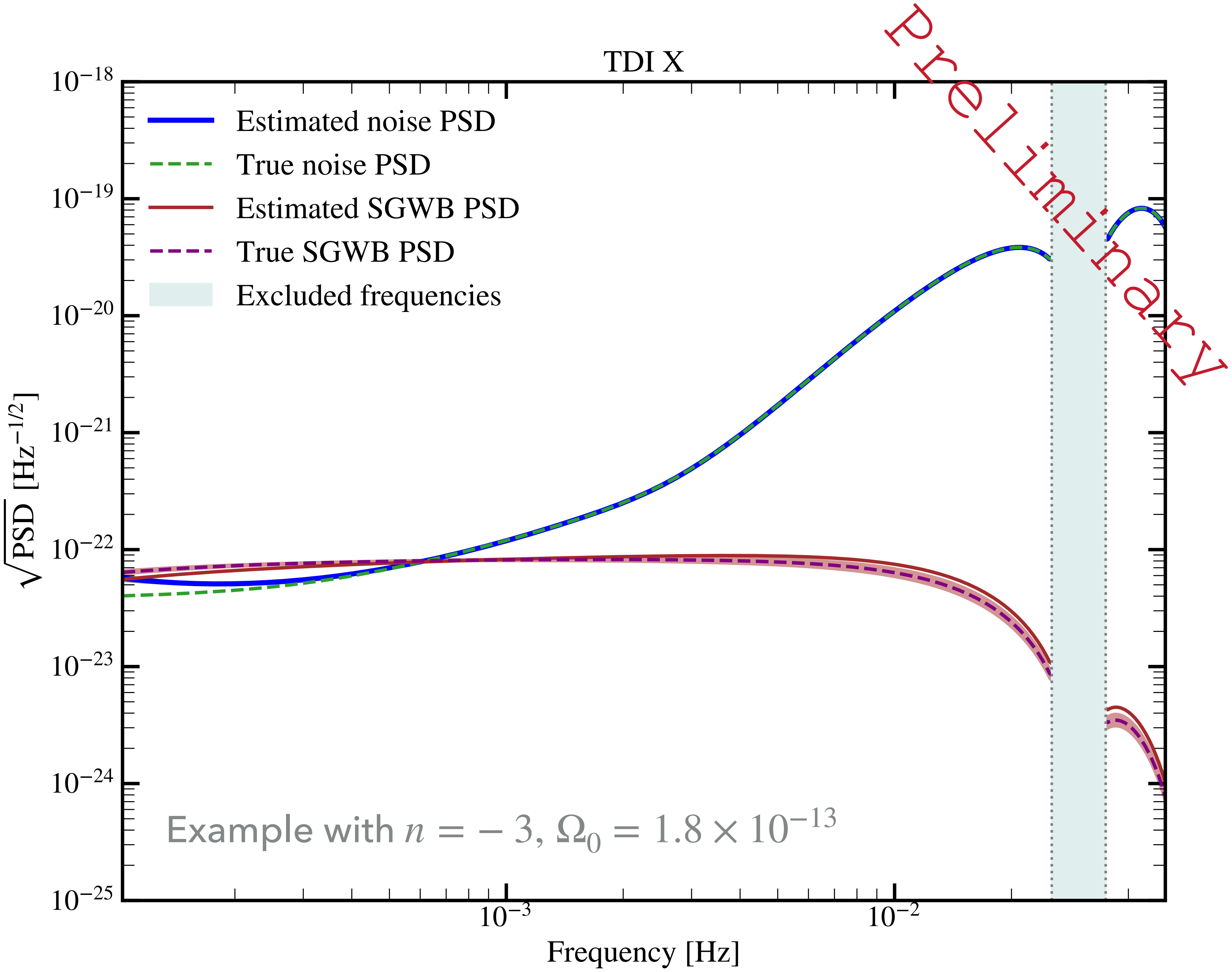
RESULTS: BAYES FACTORS



Example: cosmic string model

RESULTS

- But for some extreme values of n it's hard to converge to accurate posteriors (ambiguity)



Example with $n = -3$, $\Omega_0 = 1.8 \times 10^{-13}$

RESULTS

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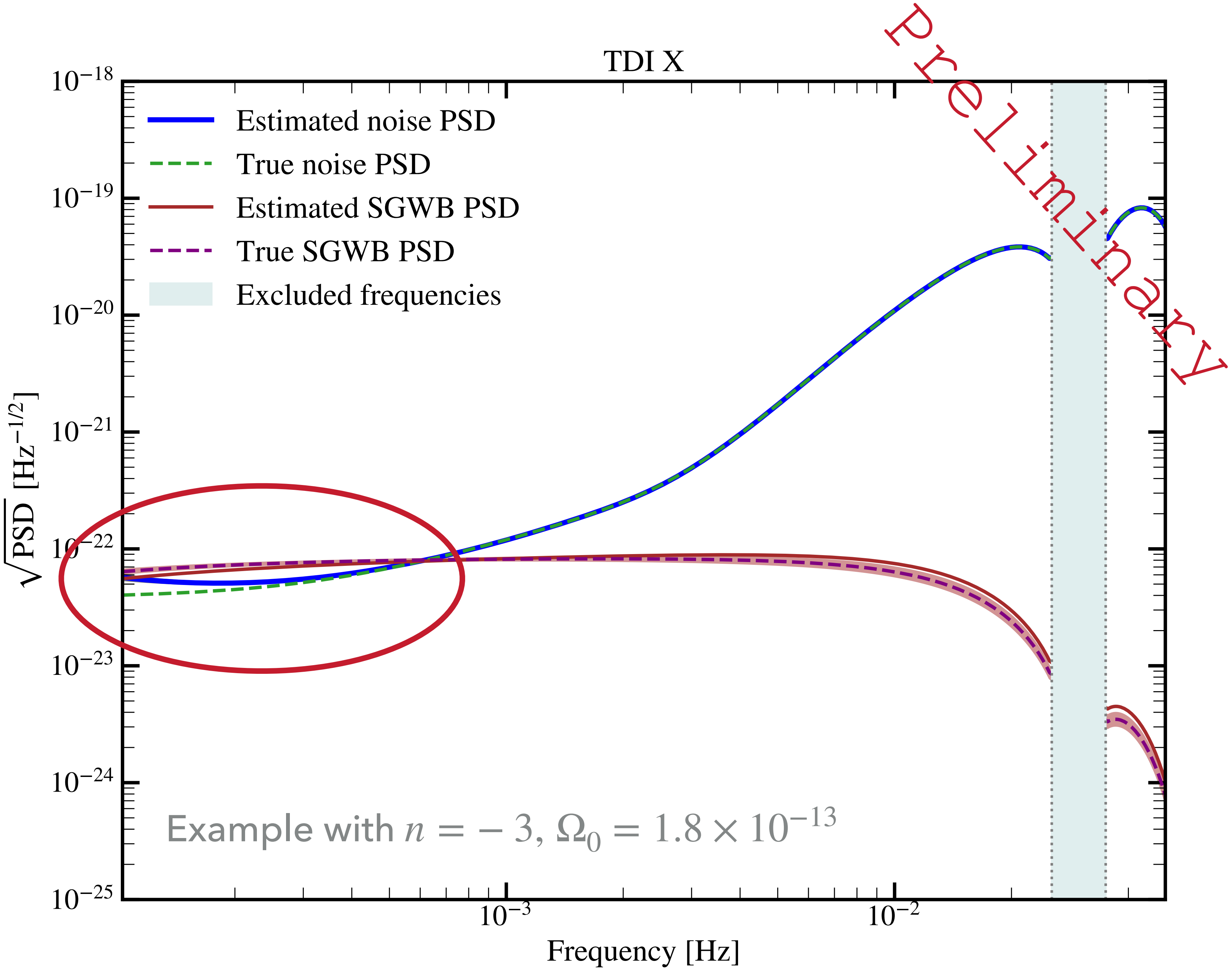

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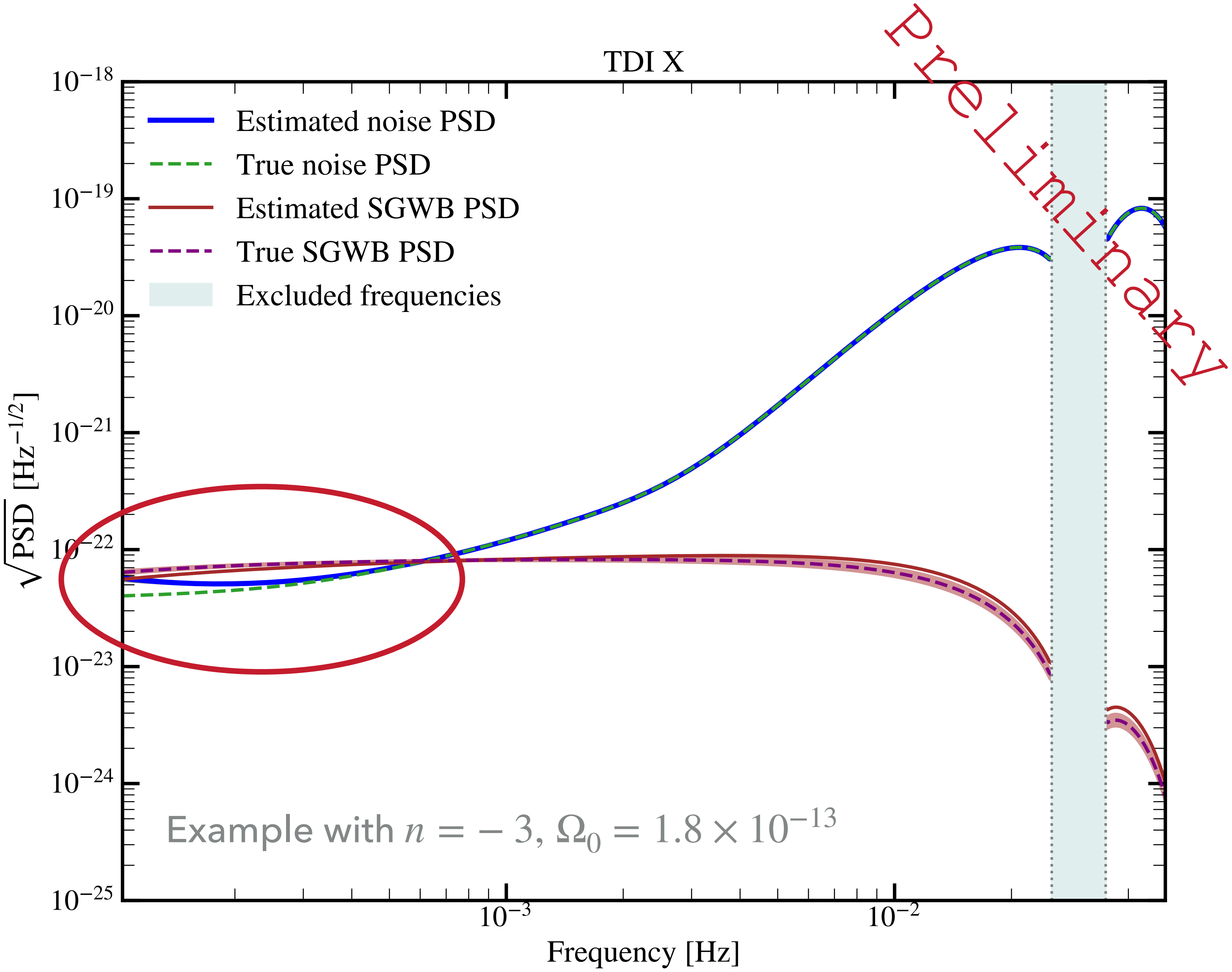

$\mathscr{B}_{10} \gg 100$ → Large evidence for H1

CONCLUSIONS

- We assumed no prior knowledge on the noise shape except smoothness
- Demonstrated that detection is possible
- Mainly depend on SNR for $-2 < n < 3$, but SGWB shape may impact parameter estimation accuracy
- Need to go beyond:
  - ✦ More realistic simulations: different noise levels in interferometers
  - ✦ Include other stochastic processes (Galaxy)
- The ultimate goal (if realisable): agnostic estimation of both signal and noise?

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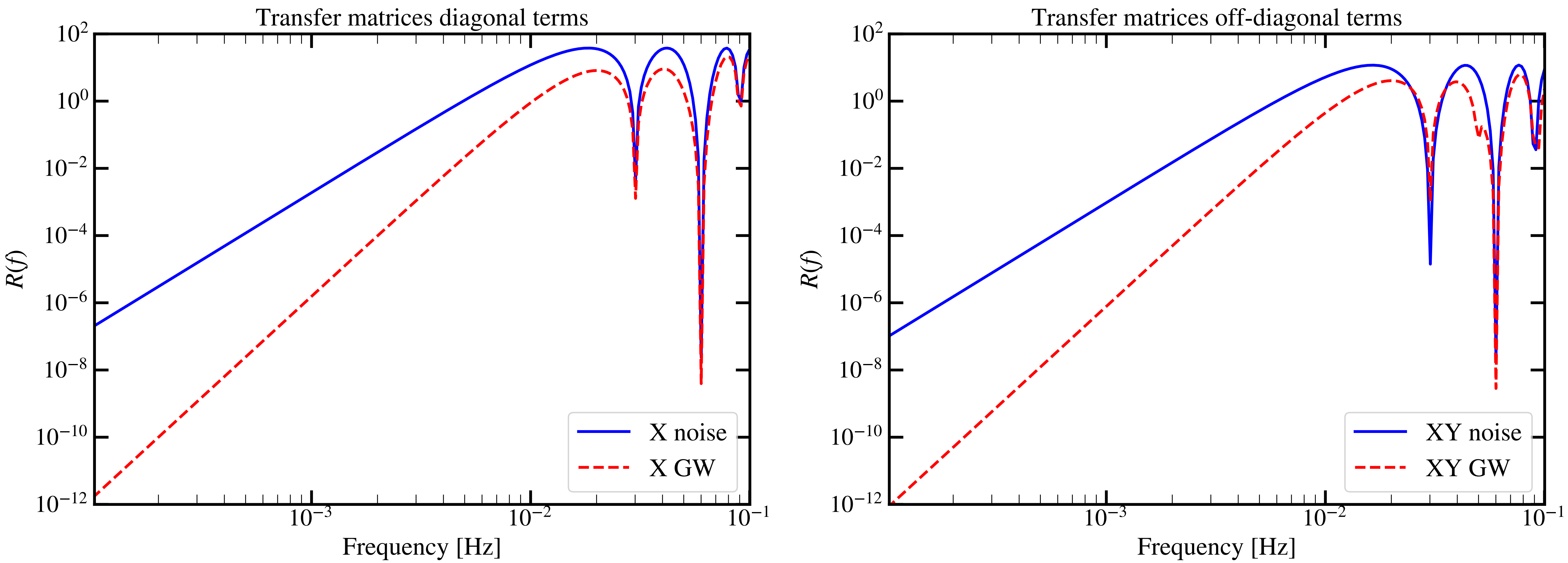
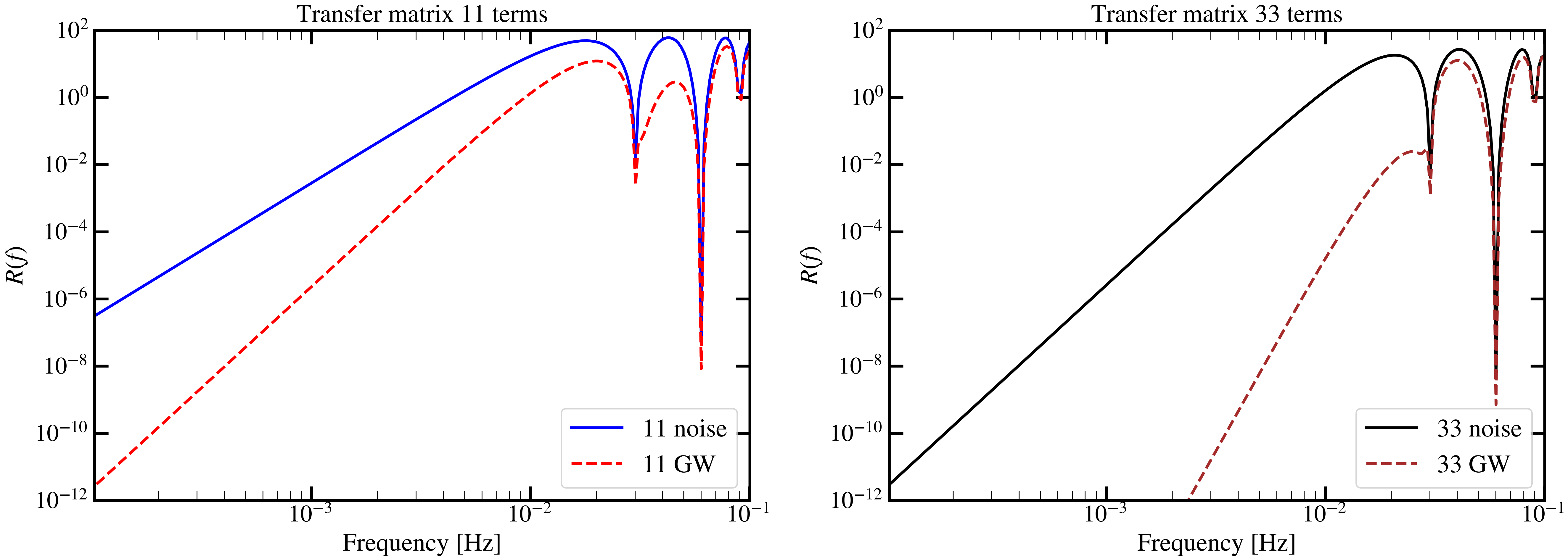
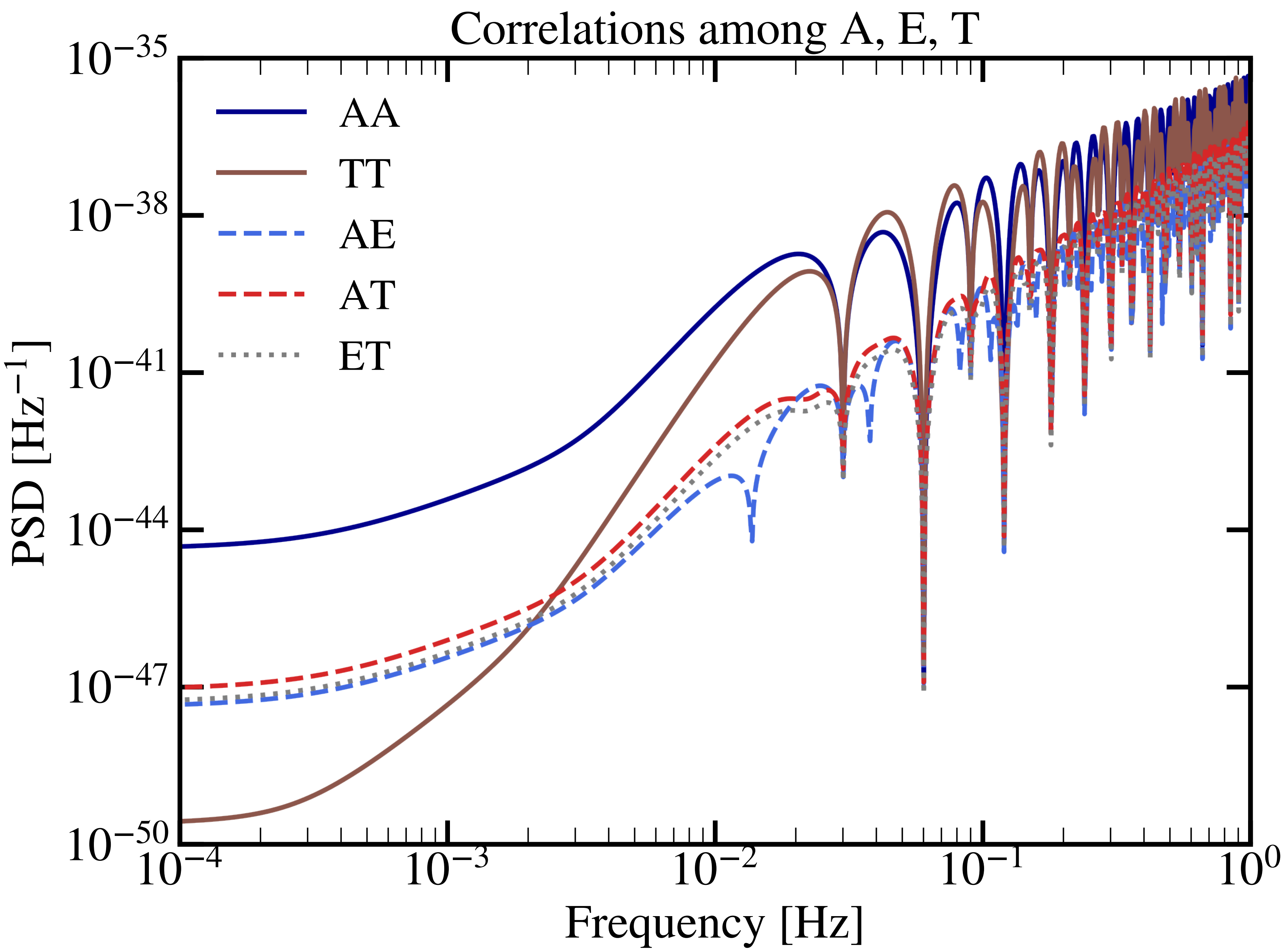
Thank you for your attention !

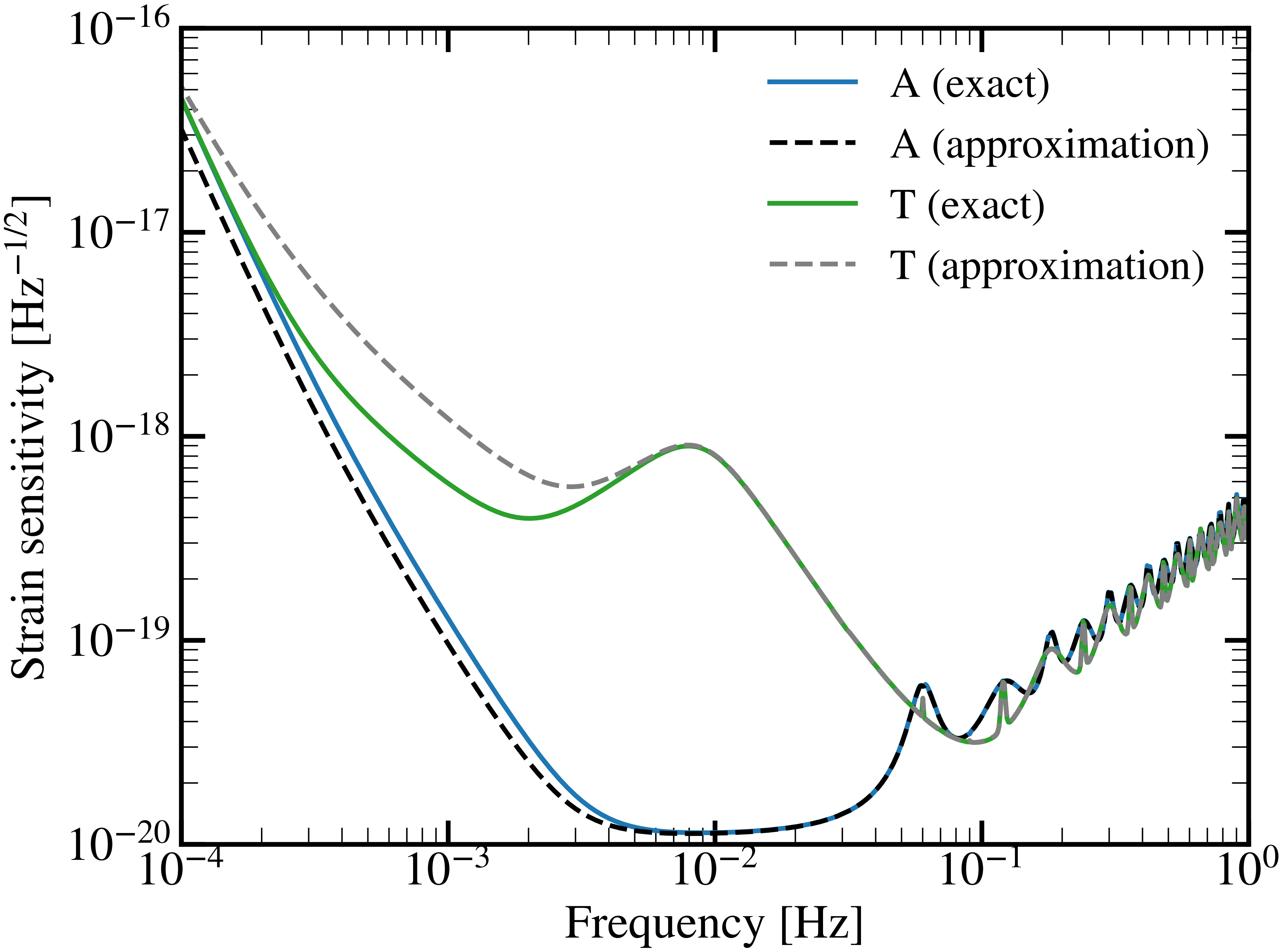
Michelson TDI covariance



Diagonalized TDI covariance




Correlations among A, E, T
AA
TT
AE
AT
ET
10^{-35}
10^{-38}
10^{-41}
10^{-44}
10^{-47}
10^{-50}
PSD [Hz^{-1}]
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
Frequency [Hz]


10^{-16}
10^{-17}
10^{-18}
10^{-19}
10^{-20}
Strain sensitivity [Hz$^{-1/2}$]
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
Frequency [Hz]
A (exact)
A (approximation)
T (exact)
T (approximation)